A DOUBLING SUBSET OF L_p FOR $p > 2$ THAT IS INHERENTLY INFINITE DIMENSIONAL

VINCENT LAFFORGUE AND ASSAF NAOR

ABSTRACT. It is shown that for every $p \in (2, \infty)$ there exists a doubling subset of L_p that does not admit a bi-Lipschitz embedding into $\mathbb{R}^k$ for any $k \in \mathbb{N}$.

1. INTRODUCTION

Given $K \in [1, \infty)$, a metric space X is said to be K-doubling if every ball in X can be covered by at most K balls of half its radius. X is said to be doubling if it is K-doubling for some $K \in (0, \infty)$.

Lang and Plaut [LP01] asked whether or not every doubling subset of Hilbert space admits a bi-Lipschitz embedding into $\mathbb{R}^k$ for some $k \in \mathbb{N}$. We refer to [NN12, Sec. 1.1] for further discussion on the ramifications of this question, as well as a construction of a doubling subset of Hilbert space that conceivably does not to admit a bi-Lipschitz embedding into $\mathbb{R}^k$ for any $k \in \mathbb{N}$. While the validity of this suggestion of [NN12] remains open, and hence also the Lang–Plaut question remains open, here we show that a variant of the method that was proposed in [NN12] can be used to prove that the analogue of the Lang–Plaut problem with Hilbert space replaced by L_p for $p \in (2, \infty)$ has a negative answer.

Theorem 1.1. *For every $p \in (2, \infty)$ there exists a doubling subset of L_p that does not admit a bi-Lipschitz embedding into $\mathbb{R}^k$ for any $k \in \mathbb{N}$.*

We thank Ofer Neiman for asking us the question that Theorem 1.1 answers. In [BGN13] Bartal, Gottlieb and Neiman concurrently found a construction that also yields Theorem 1.1; their (combinatorial) construction is entirely different from our (analytic) construction. The potential validity of Theorem 1.1 for $p \in (1, 2]$ remains an open question, while for $p = 1$ stronger results are known; see Remark 1.4 below.

Theorem 1.1 is a special case of the following result.

Theorem 1.2. *For every $p \in (2, \infty)$ there exists a doubling subset D_p of L_p that does not admit a bi-Lipschitz embedding into L_q for any $q \in (1, p)$. Furthermore, there exists $p_0 \in (2, \infty)$ such that D_p does not admit a bi-Lipschitz embedding into L_1 for every $p \in [p_0, \infty)$.*

Theorem 1.2 is a formal consequence of the following finitary result.

Theorem 1.3. *There exists a universal constant $K \in [1,\infty)$ and for every $n \in \mathbb{N}$ there exists an n-point metric space (X_n, d_{X_n}) with the following properties. For every $n \in \mathbb{N}$ and $p \in [2,\infty)$ there exists a mapping $f_{n,p} : X_n \to L_p$ such that $f_{n,p}(X_n) \subseteq L_p$ is K-doubling, and*

$$\forall\, x,y \in X_n, \quad d_{X_n}(x,y) \leqslant \|f_{n,p}(x) - f_{n,p}(y)\|_p \lesssim (\log n)^{1/p} d_{X_n}(x,y).$$

Moreover, for every $q \in (1,\infty)$, any embedding of X_n into L_q incurs distortion at least $c(q)(\log n)^{\min\{1/2,1/q\}}$, where $c(q) \in (0,\infty)$ may depend only on q. Any embedding of X_n into L_1 incurs distortion at least $(\log n)^c$ for some universal constant $c \in (0,1/2]$.

Here and in what follows, the notations $A \lesssim B$ and $B \gtrsim A$ mean that $A \leqslant CB$ for some universal constant $C \in (0,\infty)$. If we need to allow C to depend on parameters, we indicate this by subscripts, thus e.g. $A \lesssim_\beta B$ means that $A \leqslant C(\beta)B$ for some $C(\beta) \in (0,\infty)$ which is allowed to depend only on the parameter β. The notation $A \asymp B$ stands for $(A \lesssim B) \wedge (B \lesssim A)$.

The fact that Theorem 1.3 implies Theorem 1.2 is simple. Indeed, fix $p \in (2,\infty)$. By a standard "disjoint union" argument (see e.g. the beginning of Section 4 in [NN12]), there exists a doubling subset D_p of L_p that contains an isometric copy of a rescaling of $f_{n,p}(X_n)$ for every $n \in \mathbb{N}$. If $q \in (1,p)$ and $f_{n,p}(X_n)$ embeds with bi-Lipschitz distortion $M \in [1,\infty)$ into L_q, then by Theorem 1.3 we have

$$M \gtrsim_q (\log n)^{\min\left\{\frac12,\frac1q\right\}-\frac1p} \xrightarrow[n\to\infty]{} \infty.$$

Consequently, D_p does not admit a bi-Lipschitz embedding into L_q. For $q=1$ the same argument shows that D_p does not admit a bi-Lipschitz embedding into L_1 provided $p > 1/c$, where c is the (universal) constant from Theorem 1.3.

Remark 1.4. The above reasoning implies that for every $p \in (2,\infty)$ and every $n \in \mathbb{N}$ there exists an n-point $O(1)$-doubling subset S_n of L_p such that for every $k \in \mathbb{N}$, if S_n embeds with distortion $M \in [1,\infty)$ into ℓ_p^k then necessarily,

$$M \gtrsim \left(\frac{\log n}{k}\right)^{\frac12-\frac1p}.$$

This is true because ℓ_p^k embeds into Hilbert space with distortion $k^{\frac12-\frac1p}$. It is open whether or not a similar statement holds true for $p \in (1,2]$. For $p=1$ an even stronger lower bound was shown to hold true in [LMN05]: for every $n \in \mathbb{N}$ there exists an n-point $O(1)$-doubling subset A_n of L_1 such that for every $k \in \mathbb{N}$, if A_n embeds with distortion $M \in [1,\infty)$ into ℓ_1^k then necessarily

$$M \gtrsim \sqrt{\frac{\log n}{\log k}}. \tag{1}$$

The examples leading to (1) are the Laakso graphs [Laa00], which are doubling metric spaces that were shown to embed into L_1 in [GNRS04]. They yield a doubling subset of L_1 that does not admit a bi-Lipschitz embedding into Hilbert space [Laa00] (see also [LP01, Thm. 2.3]), any uniformly convex Banach space [MN08], or even any Banach space with the Radon–Nikodým property [Ost11].

1.1. **The Heisenberg group.** The metric spaces $\{X_n\}_{n=1}^{\infty}$ of Theorem 1.3 arise from the discrete Heisenberg group. To explain this, recall that the discrete Heisenberg group, denoted $\mathbb{H}$, is the group generated by two elements $a, b \in \mathbb{H}$, with the relations asserting that the commutator $[a,b] = aba^{-1}b^{-1}$ is in the center of $\mathbb{H}$. Let $e_{\mathbb{H}}$ denote the identity element of $\mathbb{H}$. The left-invariant word metric on $\mathbb{H}$ induced by the symmetric generating set $\{a, b, a^{-1}, b^{-1}\}$ is denoted $d_W(\cdot,\cdot)$. For $r \in [1,\infty)$ let $B(r) = \{x \in \mathbb{H} :\ d_W(x, e_{\mathbb{H}}) \leqslant r\}$ denote the corresponding closed ball of radius r. Then $|B(r)| \asymp r^4$ (see e.g. [Bla03]). It follows that there exists $\eta_1, \eta_2 \in (0,\infty)$ such that for every large enough $n \in \mathbb{N}$ there exists $X_n \subseteq \mathbb{H}$ with $|X_n| = n$ and

$$B\left(\eta_1 \sqrt[4]{n}\right) \subseteq X_n \subseteq B\left(\eta_2 \sqrt[4]{n}\right). \tag{2}$$

By virtue of the leftmost inclusion in (2), the distortion lower bounds that are asserted in Theorem 1.3 follow from [CKN11] for $q = 1$, from [ANT10] for $q = 2$ and from [LN12] for $q \in (0,\infty) \smallsetminus \{2\}$. The remaining assertions of Theorem 1.3 follow from Theorem 1.5 below.

Theorem 1.5. *For every $\varepsilon \in (0, 1/2]$ and $p \in [2,\infty)$ there exists a mapping $F_{\varepsilon,p} : \mathbb{H} \to L_p$ such that $F_{\varepsilon,p}(\mathbb{H}) \subseteq L_p$ is 2^{16}-doubling and*

$$\forall\, x, y \in \mathbb{H}, \qquad d_W(x,y)^{1-\varepsilon} \leqslant \|F_{\varepsilon,p}(x) - F_{\varepsilon,p}(y)\|_p \lesssim \frac{d_W(x,y)^{1-\varepsilon}}{\varepsilon^{1/p}}.$$

The case $p = 2$ of Theorem 1.5 was previously proven in [LN06] relying on Hilbertian arguments, namely on Schoenberg's characterization [Sch38] of subsets of Hilbert space through positive definite kernels. Here we find a different approach that works also when $p \in (2,\infty)$. Note that [LN06] contains a stronger statement that is used crucially in the context of [LN06] and does not follow from our proof.

Theorem 1.5 implies Theorem 1.3 because in light of the rightmost inclusion in (2), all the nonzero distances in X_n are between 1 and $2\eta_2 \sqrt[4]{n}$.

Consequently, for $\varepsilon_n = 1/\log n$ we have $d_W(x,y)^{1-\varepsilon_n} \asymp d_W(x,y)$ for every $x,y \in X_n$. We can therefore take $f_{n,p} = F_{\varepsilon_n,p}$ in Theorem 1.3.

Remark 1.6. The dependence on ε in Theorem 1.5 is asymptotically sharp as $\varepsilon \to 0$. For $p=2$ this was proven in [NN12] using an inequality of [ANT10]. An analogous argument works for $p \in (2,\infty)$ using [LN12] instead of [ANT10]. Indeed, write $c = [a,b]$ (recall that a,b are the generators of $\mathbb{H}$). By [LN12] every $f : \mathbb{H} \to L_p$ satisfies

$$\sum_{k=1}^{n^2} \sum_{x\in B_n} \frac{\|f(xc^k)-f(x)\|_p^p}{k^{1+p/2}} \lesssim_p \sum_{x\in B_{21n}} \Big(\|f(xa)-f(x)\|_p^p + \|f(xb)-f(x)\|_p^p\Big). \quad (3)$$

Suppose that $M \in [1,\infty)$ satisfies

$$\forall\, x,y \in \mathbb{H}, \qquad d_W(x,y)^{1-\varepsilon} \leqslant \|f(x)-f(y)\|_p \leqslant M d_W(x,y)^{1-\varepsilon}. \quad (4)$$

Since $d_W(c^k, e_{\mathbb{H}}) \asymp \sqrt{k}$ and $|B(m)| \asymp m^4$ for every $k,m \in \mathbb{N}$ (see e.g. [Bla03]), by substituting (4) into (3) we see that

$$\sum_{k=1}^{n^2} n^4 \cdot \frac{k^{(1-\varepsilon)p/2}}{k^{1+p/2}} \lesssim_p M^p n^4.$$

Hence $M^p \gtrsim_p \sum_{k=1}^{\infty} \frac{1}{k^{1+\varepsilon p/2}} \gtrsim_p \frac{1}{\varepsilon}$, so $M \gtrsim_p 1/\varepsilon^{1/p}$.

2. Proof of Theorem 1.5

For every $n \in \mathbb{N}$ and $x \in \mathbb{R}^{2n+1}$ let $\pi(x) \in \mathbb{R}^{2n}$ denote the canonical projection of x to $\mathbb{R}^{2n}$, i.e,

$$\pi(x_1,x_2,\ldots,x_{2n},x_{2n+1}) \stackrel{\mathrm{def}}{=} (x_1,\ldots,x_{2n}).$$

For $x,y \in \mathbb{R}^{2n}$ write

$$\langle x,y\rangle \stackrel{\mathrm{def}}{=} \sum_{j=1}^{2n} x_j y_j \qquad \text{and} \qquad [x,y] \stackrel{\mathrm{def}}{=} \sum_{j=1}^{n} (x_{2j-1}y_{2j} - x_{2j}y_{2j-1}).$$

We also write as usual $\|x\|_2 \stackrel{\mathrm{def}}{=} \sqrt{\langle x,x\rangle}$.

The Heisenberg group product on $\mathbb{R}^{2n+1}$ is defined as follows. For every $x,y \in \mathbb{R}^{2n+1}$ let $xy \in \mathbb{R}^{2n+1} \cong \mathbb{R}^{2n} \times \mathbb{R}$ be defined as

$$xy \stackrel{\mathrm{def}}{=} \left(\pi(x)+\pi(y), x_{2n+1}+y_{2n+1} - 2[\pi(x),\pi(y)]\right).$$

Under this product $\mathbb{R}^{2n+1}$ becomes a noncommutative group, called the nth (continuous) Heisenberg group and denoted $\mathbb{H}_n$, whose identity element is $0 \in \mathbb{R}^{2n+1}$ and the multiplicative inverse of $x \in \mathbb{R}^{2n+1}$ is given by $x^{-1} = -x$; see e.g. [Sem03]. The Lebesgue measure on $\mathbb{R}^{2n+1}$ is a Haar measure for $\mathbb{H}_n$.

The Korányi norm of $x \in \mathbb{R}^{2n+1}$ is defined by

$$N(x) \stackrel{\mathrm{def}}{=} \left(\|\pi(x)\|_2^4 + x_{2n+1}^2\right)^{1/4}.$$

As shown in e.g. [Cyg81], we have $N(xy) \leqslant N(x) + N(y)$ for every $x, y \in \mathbb{R}^{2n+1}$. Consequently, if we set

$$\forall\, x, y \in \mathbb{R}^{2n+1}, \qquad d_N(x,y) \stackrel{\mathrm{def}}{=} N(x^{-1}y),$$

then d_N is a left-invariant metric on $\mathbb{H}_n$. For every $\theta \in \mathbb{R}$ define $\delta_\theta : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ by

$$\forall\, x \in \mathbb{R}^{2n+1}, \qquad \delta_\theta(x) \stackrel{\mathrm{def}}{=} \left(\theta\pi(x), \theta^2 x_{2n+1}\right). \tag{5}$$

Then $d_N(\delta_\theta(x), \delta_\theta(y)) = |\theta| d_N(x,y)$ for every $x, y \in \mathbb{R}^{2n+1}$.

Fix $p \in [1,\infty)$ and $\varepsilon \in (0,1)$. Choose an integer $n \in \mathbb{N}$ such that

$$n \leqslant p < n+1, \tag{6}$$

and define

$$\alpha \stackrel{\mathrm{def}}{=} \frac{2n+2}{p} - 1 + \varepsilon. \tag{7}$$

Note that by the choice of n we have $\alpha \in [1+\varepsilon, 3+\varepsilon) \subseteq (1,4)$. For every $x, z \in \mathbb{R}^{2n+1} \setminus \{0\}$ define $T(x)(y) \in \mathbb{R}$ by

$$T(x)(z) \stackrel{\mathrm{def}}{=} \frac{1}{N(x^{-1}z)^\alpha} - \frac{1}{N(z)^\alpha}. \tag{8}$$

Lemma 2.1. *For every $R \in (0,\infty)$ we have*

$$\left(\int_{B_N(0,R)} \frac{dz}{N(z)^{\alpha p}}\right)^{1/p} \asymp \frac{R^{1-\varepsilon}}{p(1-\varepsilon)^{1/p}}.$$

Proof. This is a straightforward computation. First, since

$$B_N(0,R) \subseteq \left\{z \in \mathbb{R}^{2n+1} :\ \|\pi(z)\|_2 \leqslant R \ \wedge\ |z_{2n+1}| \leqslant R^2\right\}, \tag{9}$$

by integration in polar coordinates on $\mathbb{R}^{2n}$ we have,

$$\left(\int_{B_N(0,R)} \frac{dz}{N(z)^{\alpha p}}\right)^{1/p}$$

$$\leqslant \left(\int_0^R 2n v_{2n} r^{2n-1}\left(\int_{-R^2}^{R^2} \frac{dt}{(r^4+t^2)^{\alpha p/4}}\right) dr\right)^{1/p} \tag{10}$$

$$\lesssim \frac{1}{p}\left(\int_0^R \frac{dr}{r^{\alpha p - 2n - 1}}\right)^{1/p} \asymp \frac{R^{1-\varepsilon}}{p(1-\varepsilon)^{1/p}}, \tag{11}$$

where in (10) $v_{2n}=\pi^n/n!$ denotes the volume of the Euclidean unit ball in $\mathbb{R}^{2n}$. The penultimate inequality of (11) uses the fact that $v_{2n}^{1/p}\asymp 1/p$ (recall (6)). In the final step of (11) we used (7). Using the inclusion

$$B_N(0,R)\supseteq\left\{z\in\mathbb{R}^{2n+1}:\ \|\pi(z)\|_2\leqslant\frac{R}{\sqrt[4]{2}}\ \wedge\ |z_{2n+1}|\leqslant\frac{R^2}{\sqrt{2}}\right\}\tag{12}$$

in place of (9), the reverse inequality is proved analogously. □

Corollary 2.2. *For every $x\in\mathbb{R}^{2n+1}$ and $K\in[1/3,\infty)$ we have*

$$\frac{N(x)^{1-\varepsilon}}{p(1-\varepsilon)^{1/p}}\lesssim\left(\int_{B_N(0,KN(x))}|T(x)(z)|^pdz\right)^{1/p}\lesssim\frac{K^{1-\varepsilon}N(x)^{1-\varepsilon}}{p(1-\varepsilon)^{1/p}}.\tag{13}$$

Proof. For the upper bound note that by the definition of T in (8),

$$\begin{aligned}&\left(\int_{B_N(0,KN(x))}|T(x)(z)|^pdz\right)^{1/p}\\ &\leqslant\left(\int_{B_N(0,KN(x))}\frac{dz}{N(z)^{\alpha p}}\right)^{1/p}+\left(\int_{B_N(0,KN(x))}\frac{dz}{N(x^{-1}z)^{\alpha p}}\right)^{1/p}\\ &=\left(\int_{B_N(0,KN(x))}\frac{dz}{N(z)^{\alpha p}}\right)^{1/p}+\left(\int_{B_N(x^{-1},KN(x))}\frac{dw}{N(w)^{\alpha p}}\right)^{1/p},\end{aligned}$$

where we used the fact that the Lebesgue measure on $\mathbb{R}^{2n+1}$ is a Haar measure of the Heisenberg group, and the left-invariance of the metric d_N. By the triangle inequality, $B_N(x^{-1},KN(x))\subseteq B_N(0,4KN(x))$, so the rightmost inequality in (13) follows from Lemma 2.1.

If $z\in B_N(0,N(x)/3)$ then $N(x^{-1}z)\geqslant N(x)-N(z)\geqslant 2N(z)$, whence $N(z)^{-\alpha}\geqslant 2N(x^{-1}z)^{-\alpha}$ (using $\alpha\geqslant 1$). So $|T(x)(z)|\geqslant N(z)^{-\alpha}$ for every $z\in B_N(0,N(x)/3)$. Since $K\geqslant 1/3$, the leftmost inequality in (13) now follows from Lemma 2.1. □

Lemma 2.3. *For every $x\in\mathbb{R}^{2n+1}$ we have $T(x)\in L_p(\mathbb{R}^{2n+1})$ and*

$$\|T(x)\|_{L_p(\mathbb{R}^{2n+1})}\lesssim\frac{N(x)^{1-\varepsilon}}{p}\left(\frac{1}{\varepsilon^{1/p}}+\frac{1}{(1-\varepsilon)^{1/p}}\right).\tag{14}$$

Proof. If $z\in\mathbb{R}^{2n+1}\smallsetminus B_N(0,2N(x))$ then by the triangle inequality for the Korányi norm we have $N(x^{-1}z)\leqslant N(x)+N(z)\leqslant 2N(z)$ and $N(x^{-1}z)\geqslant N(z)-N(x)\geqslant N(z)/2$. Consequently $N(x^{-1}z)\asymp N(z)$ for every $z\in\mathbb{R}^{2n+1}\smallsetminus B_N(0,2N(x))$, and it therefore follows that

$$|T(x)(z)|\lesssim\frac{|N(x^{-1}z)-N(z)|}{N(z)^{\alpha+1}}\leqslant\frac{N(x)}{N(z)^{\alpha+1}}.$$

We conclude that

$$\left(\int_{\mathbb{R}^{2n+1}\smallsetminus B_N(0,2N(x))}|T(x)(z)|^p dz\right)^{1/p}$$
$$\leqslant N(x)\left(\int_{\mathbb{R}^{2n+1}\smallsetminus B_N(0,2N(x))}\frac{dz}{N(z)^{(\alpha+1)p}}\right)^{1/p}$$
$$\leqslant N(x)\left(\int_{U_x}\frac{dz}{N(z)^{(\alpha+1)p}}\right)^{1/p}+N(x)\left(\int_{V_x}\frac{dz}{N(z)^{(\alpha+1)p}}\right)^{1/p}, \qquad (15)$$

where

$$U_x \stackrel{\mathrm{def}}{=} \left\{z\in\mathbb{R}^{2n+1}\smallsetminus B_N\left(0,2N(x)\right):\ \|\pi(z)\|_2\geqslant\sqrt{|z_{2n+1}|}\right\}$$

and

$$V_x \stackrel{\mathrm{def}}{=} \left\{z\in\mathbb{R}^{2n+1}\smallsetminus B_N\left(0,2N(x)\right):\ \|\pi(z)\|_2\leqslant\sqrt{|z_{2n+1}|}\right\}.$$

For $z\in U_x$ we have $\|\pi(z)\|_2\leqslant N(z)\leqslant 2\|\pi(z)\|_2$, and therefore

$$\left(\int_{U_x}\frac{dz}{N(z)^{(\alpha+1)p}}\right)^{1/p}\leqslant\left(\int_{\{w\in\mathbb{R}^{2n}:\ \|w\|_2\geqslant N(x)\}}\frac{2dw}{\|w\|_2^{(\alpha+1)p-2}}\right)^{1/p}$$
$$=\left(\int_{N(x)}^{\infty}\frac{4nv_{2n}dr}{r^{(\alpha+1)p-2n-1}}\right)^{1/p}\lesssim\frac{1}{p}\left(\int_{N(x)}^{\infty}\frac{dr}{r^{1+p\varepsilon}}\right)^{1/p}\lesssim\frac{1}{p\varepsilon^{1/p}N(x)^{\varepsilon}}. \qquad (16)$$

If $z\in V_x$ then $\sqrt{|z_{2n+1}|}\leqslant N(z)\leqslant 2\sqrt{|z_{2n+1}|}$, and therefore

$$\left(\int_{V_x}\frac{dz}{N(z)^{(\alpha+1)p}}\right)^{1/p}$$
$$\leqslant\left(\int_{N(x)^2}^{\infty}\mathrm{vol}_{2n}\left(\left\{w\in\mathbb{R}^{2n}:\ \|w\|_2\leqslant\sqrt{t}\right\}\right)\cdot\frac{2dt}{t^{\frac{(\alpha+1)p}{2}}}\right)^{1/p}$$
$$=\left(\int_{N(x)^2}^{\infty}\frac{2v_{2n}dt}{t^{\frac{(\alpha+1)p}{2}-n}}\right)^{1/p}\lesssim\frac{1}{p\varepsilon^{1/p}N(x)^{\varepsilon}}. \qquad (17)$$

The desired estimate (14) now follows from substituting (16) and (17) into (15), and using Corollary 2.2. □

Corollary 2.4. *Define* $S:\mathbb{R}^{2n+1}\to L_p(\mathbb{R}^{2n+1})$ *by*

$$S\stackrel{\mathrm{def}}{=}p(1-\varepsilon)^{1/p}T. \qquad (18)$$

Then for every $x,y\in\mathbb{R}^{2n+1}$ *we have*

$$d_N(x,y)^{1-\varepsilon}\lesssim\|S(x)-S(y)\|_{L_p(\mathbb{R}^{2n+1})}\lesssim\left(1+\frac{(1-\varepsilon)^{1/p}}{\varepsilon^{1/p}}\right)d_N(x,y)^{1-\varepsilon}.$$

Proof. Observe that since the Lebesgue measure is a Haar measure for the Heisenberg group,

$$\|S(x)-S(y)\|_{L_p(\mathbb{R}^{2n+1})}=p(1-\varepsilon)^{1/p}\|T(y^{-1}x)\|_{L_p(\mathbb{R}^{2n+1})}.$$

Hence the desired upper bound on $\|S(x)-S(y)\|_{L_p(\mathbb{R}^{2n+1})}$ follows from Lemma 2.3 and the desired lower bound on $\|S(x)-S(y)\|_{L_p(\mathbb{R}^{2n+1})}$ follows from the leftmost inequality in Corollary 2.2. □

Lemma 2.5. *Let S be defined as in* (18) *and let $\phi:\mathbb{R}^3\to\mathbb{R}^{2n+1}$ be the canonical embedding of the corresponding Heisenberg groups, i.e.,*

$$\forall (a,b,c)\in\mathbb{R}^3,\qquad \phi(a,b,c)\stackrel{\mathrm{def}}{=}(a,b,\underbrace{0,\ldots,0}_{2n-2\ \text{times}},c)\in\mathbb{R}^{2n+1}.$$

Then $S\circ\phi(\mathbb{R}^3)\subseteq L_p(\mathbb{R}^{2n+1})$ is $2^{8/(1-\varepsilon)}$-doubling.

Proof. For $\theta\in(0,\infty)$, recalling (5) we have

$$\begin{aligned}
&\|S(\delta_\theta(x))-S(\delta_\theta(y))\|_{L_p(\mathbb{R}^{2n+1})}\\
&=\left(\int_{\mathbb{R}^{2n+1}}\left|\frac{1}{N(\delta_\theta(x^{-1})z)^\alpha}-\frac{1}{N(\delta_\theta(y^{-1})z)^\alpha}\right|^p dz\right)^{1/p}\\
&=\left(\int_{\mathbb{R}^{2n+1}}\left|\frac{1}{N(\delta_\theta(x^{-1}w))^\alpha}-\frac{1}{N(\delta_\theta(y^{-1}w))^\alpha}\right|^p \theta^{2n+2}dw\right)^{1/p} \qquad (19)\\
&=\theta^{\frac{2n+2-\alpha p}{p}}\left(\int_{\mathbb{R}^{2n+1}}\left|\frac{1}{N(x^{-1}w)^\alpha}-\frac{1}{N(y^{-1}w)^\alpha}\right|^p \theta^{2n+2}dw\right)^{1/p} \qquad (20)\\
&=\theta^{1-\varepsilon}\|S(x)-S(y)\|_{L_p(\mathbb{R}^{2n+1})}, \qquad (21)
\end{aligned}$$

where (19) uses the change of variable $z=\delta_\theta(w)$ (so $dz=\theta^{2n+2}dw$), and (20) uses the fact that $N(\delta_\theta(u))=\theta N(u)$ for every $u\in\mathbb{R}^{2n+1}$. For (21), recall the definition of α in (7).

Let μ be the push-forward of the Lebesgue measure on $\mathbb{R}^3$ under the mapping $S\circ\phi$, i.e.,

$$\mu(A)\stackrel{\mathrm{def}}{=}\mathrm{vol}_3(\phi^{-1}(S^{-1}(A)))$$

for every Borel set $A\subseteq S\circ\phi(\mathbb{R}^3)\subseteq L_p(\mathbb{R}^{2n+1})$. For $f\in L_p(\mathbb{R}^{2n+1})$ and $r\in[0,\infty)$ let $B_p(f,r)$ denote the closed ball of radius r and center f in $L_p(\mathbb{R}^{2n+1})$, i.e., $B_p(f,s)=\{g\in L_p(\mathbb{R}^{2n+1}):\ \|f-g\|_{L_p(\mathbb{R}^{2n+1})}\leqslant r\}$. By (21) for every $0<r\leqslant R<\infty$ and every $f\in S\circ\phi(\mathbb{R}^3)$ we have

$$\phi^{-1}\circ S^{-1}\left(B_p(f,R)\right)=\phi^{-1}(S^{-1}(f))\delta_{(R/r)^{1/(1-\varepsilon)}}\left(\phi^{-1}\circ S^{-1}\left(B_p(0,r)\right)\right).$$

Consequently,

$$\mu\left(B_p(f,R)\right)=\left(\frac{R}{r}\right)^{\frac{4}{1-\varepsilon}}\mathrm{vol}_3\left(\phi^{-1}\circ S^{-1}\left(B_p(0,r)\right)\right)>0.$$

In particular,

$$\forall\, f \in S\circ\phi(\mathbb{R}^3),\ \forall\, r\in(0,\infty),\qquad \frac{\mu\left(B_p(f,2r)\right)}{\mu\left(B_p(f,r)\right)} = 2^{4/(1-\varepsilon)}.$$

By a standard packing argument (see e.g. [CW71, page 67]), this implies that $S\circ\phi(\mathbb{R}^3)$ is a $2^{8/(1-\varepsilon)}$-doubling subset of $L_p(\mathbb{R}^{2n+1})$. □

Proof of Theorem 1.5. The discrete Heisenberg group $\mathbb{H}$ embeds into the continuous Heisenebrg group $\mathbb{H}_1$ as a co-compact discrete subgroup. Hence (see e.g. [BBI01, Thm. 8.3.19]) the metric space $(\mathbb{H}, d_W)$ is bi-Lipschitz to a subset of $(\mathbb{R}^3, d_N)$. By taking the mapping $F_{\varepsilon,p}$ of Theorem 1.5 to be the restriction of $S\circ\phi$ to $\mathbb{H}$, the assertions of Theorem 1.5 follow from Lemma 2.5 and Corollary 2.4 (observe that ϕ is an isometric embedding of $(\mathbb{R}^3, d_N)$ into $(\mathbb{R}^{2n+1}, d_N)$). □

3. A representation theoretic proof of Theorem 1.5

Here we present a different proof of Theorem 1.5 which uses the Schrödinger representation of the Heisenberg group $\mathbb{H}_n$. In what follows it will be notationally convenient to consider the Heisenberg group $\mathbb{H}_n$ as being $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$, equipped with the group product given by

$$(u,v,w)(u',v',w') \stackrel{\mathrm{def}}{=} \left(u+u', v+v', w+w'-2\langle u,v'\rangle + 2\langle v,u'\rangle\right),$$

for every $(u,v,w),(u',v',w')\in\mathbb{H}_n$. The corresponding Korànyi norm is then given by

$$\forall (u,v,w)\in\mathbb{H}_n,\qquad N(u,v,w) = \left(\left(\|u\|_2^2+\|v\|_2^2\right)^2 + w^2\right)^{1/4},$$

and for $\theta\in(0,\infty)$ the Heisenberg dilation $\delta_\theta:\mathbb{H}_n\to\mathbb{H}_n$ is given by

$$\forall (u,v,w)\in\mathbb{H}_n,\qquad \delta_\theta(u,v,w) = \left(\theta u, \theta v, \theta^2 w\right).$$

The Schrödinger representation of $\mathbb{H}_n$ corresponding to $\lambda\in(0,\infty)$ is defined as follows. For every $(u,v,w)\in\mathbb{H}_n$ and $h:\mathbb{R}^n\to\mathbb{C}$ define $\sigma_\lambda(u,v,w)h:\mathbb{R}^n\to\mathbb{C}$ by

$$\forall\, x\in\mathbb{R}^n,\qquad \sigma_\lambda(u,v,w)h(x) \stackrel{\mathrm{def}}{=} e^{i\lambda(w-2\langle u,v\rangle)+2i\sqrt{\lambda}\langle v,x\rangle} h\left(x-2\sqrt{\lambda}u\right).$$

One checks that this defines a unitary representation of $\mathbb{H}_n$ on $L_2(\mathbb{R}^n)$, i.e., that for every $h\in L_2(\mathbb{R}^n)$ we have $\|\sigma(x)h\|_{L_2(\mathbb{R}^n)} = \|h\|_{L_2(\mathbb{R}^n)}$ and $\sigma_\lambda(xy)h = \sigma_\lambda(x)\sigma_\lambda(y)h$ for every $x,y\in\mathbb{H}_n$.

Define $g:\mathbb{R}^n\to\mathbb{R}$ to be

$$g(x) \stackrel{\mathrm{def}}{=} e^{-\frac12\|x\|_2^2},$$

so that $\|g\|_{L_2(\mathbb{R}^n)}^2=\int_{\mathbb{R}^n} e^{-\|x\|_2^2}dx=\pi^{n/2}$. Then for every $(u,v,w)\in\mathbb{H}_n$ and $x\in\mathbb{R}^n$ we have

$$\sigma_\lambda(u,v,w)g(x)=e^{i\lambda(w-2\langle u,v\rangle)+2i\sqrt{\lambda}\langle v,x\rangle-\frac12\|x\|_2^2+2\sqrt{\lambda}\langle u,x\rangle-2\lambda\|u\|_2^2}.$$

Consequently,

$$\begin{aligned}
&\|g-\sigma_\lambda(u,v,w)g\|_{L_2(\mathbb{R}^n)}^2\\
&=2\|g\|_{L_2(\mathbb{R}^n)}^2-2\Re\left(\int_{\mathbb{R}^n} g(x)\sigma_\lambda(u,v,w)g(x)dx\right)\\
&=2\pi^{n/2}-2\Re\left(\int_{\mathbb{R}^n} e^{i\lambda(w-2\langle u,v\rangle)+2i\sqrt{\lambda}\langle v,x\rangle-\|x\|_2^2+2\sqrt{\lambda}\langle u,x\rangle-2\lambda\|u\|_2^2}dx\right)\\
&=2\pi^{n/2}-2\Re\left(e^{i\lambda w-\lambda\left(\|u\|_2^2+\|v\|_2^2\right)}\int_{\mathbb{R}^n} e^{-\|x-i\sqrt{\lambda}v-\sqrt{\lambda}u\|_2^2}dx\right)\\
&=2\pi^{n/2}\left(1-e^{-\lambda\left(\|u\|_2^2+\|v\|_2^2\right)}\cos(\lambda w)\right). \qquad (22)
\end{aligned}$$

Let $L_p((0,\infty),L_2(\mathbb{R}^n))$ denote as usual the space of all measurable mappings $F:(0,\infty)\to L_2(\mathbb{R}^n)$ that satisfy

$$\|F\|_{L_p((0,\infty),L_2(\mathbb{R}^n))}\stackrel{\mathrm{def}}{=}\left(\int_0^\infty \|F(\lambda)\|_{L_2(\mathbb{R}^n)}^p\right)^{1/p}<\infty.$$

Note that since $L_2(\mathbb{R}^n)$ embeds isometrically into L_p (see e.g. [Woj91]), also $L_p((0,\infty),L_2(\mathbb{R}^n))$ embeds isometrically into L_p. For every measurable mapping $F:(0,\infty)\to L_2(\mathbb{R}^n)$ and every $(u,v,w)\in\mathbb{H}_n$ define $\sigma(u,v,w)F:(0,\infty)\to L_2(\mathbb{R}^n)$ by

$$\forall\,\lambda\in(0,\infty),\qquad \sigma(u,v,w)F(\lambda)\stackrel{\mathrm{def}}{=}\sigma_\lambda(u,v,w)F.$$

Thus σ is an action of $\mathbb{H}_n$ on $L_p((0,\infty),L_2(\mathbb{R}^n))$ by isometries. Next, define $G:(0,\infty)\to L_2(\mathbb{R}^n)$ by

$$\forall\,\lambda\in(0,\infty),\qquad G(\lambda)\stackrel{\mathrm{def}}{=}\frac{g}{\sqrt{2}\pi^{\frac{n}{4}}\cdot\lambda^{\frac1p+\frac{1-\varepsilon}{2}}}. \qquad (23)$$

Lemma 3.1. *We have $G-\sigma(u,v,w)G\in L_p((0,\infty),L_2(\mathbb{R}^n))$ for every $(u,v,w)\in\mathbb{H}_n$. Moreover,*

$$\begin{aligned}
&\|G-\sigma(u,v,w)G\|_{L_p((0,\infty),L_2(\mathbb{R}^n))}\\
&=\left(\int_0^\infty\left(1-e^{-\lambda\left(\|u\|_2^2+\|v\|_2^2\right)}\cos(\lambda w)\right)^{p/2}\frac{d\lambda}{\lambda^{1+(1-\varepsilon)p/2}}\right)^{1/p} \qquad (24)\\
&\asymp\left(\frac{1}{\varepsilon^{1/p}}+\frac{1}{(1-\varepsilon)^{1/p}}\right)\left(\|u\|_2^2+\|v\|_2^2\right)^{(1-\varepsilon)/2}+\frac{|w|^{(1-\varepsilon)/2}}{(1-\varepsilon)^{1/p}}. \qquad (25)
\end{aligned}$$

Proof. The identity (24) is a substitution of (23) into (22). Note that

$$(a,b)\in(0,1)\times(-1,1) \implies \frac{1}{3}\leqslant \frac{1-ab}{(1-a)+(1-b)}\leqslant 1. \tag{26}$$

Indeed, the leftmost inequality in (26) is equivalent to the inequality $(1+b)(1-a)+2a(1-b)\geqslant 0$, and the rightmost inequality in (26) is equivalent to the inequality $(1-a)(1-b)\geqslant 0$. It follows from (26) that for every $\lambda\in(0,\infty)$ we have

$$1-e^{-\lambda\left(\|u\|_2^2+\|v\|_2^2\right)}\cos(\lambda w)\asymp\left(1-e^{-\lambda\left(\|u\|_2^2+\|v\|_2^2\right)}\right)+\left(1-\cos(\lambda w)\right).$$

Hence,

$$\begin{aligned}
&\left(\int_0^\infty\left(1-e^{-\lambda\left(\|u\|_2^2+\|v\|_2^2\right)}\cos(\lambda w)\right)^{p/2}\frac{d\lambda}{\lambda^{1+(1-\varepsilon)p/2}}\right)^{1/p}\\
&\asymp\left(\int_0^\infty\left(1-e^{-\lambda\left(\|u\|_2^2+\|v\|_2^2\right)}\right)^{p/2}\frac{d\lambda}{\lambda^{1+(1-\varepsilon)p/2}}\right)^{1/p}\\
&\qquad+\left(\int_0^\infty\left(1-\cos(\lambda w)\right)^{p/2}\frac{d\lambda}{\lambda^{1+(1-\varepsilon)p/2}}\right)^{1/p}\\
&=\left(\|u\|_2^2+\|v\|_2^2\right)^{(1-\varepsilon)/2}\left(\int_0^\infty\frac{(1-e^{-t})^{p/2}}{t^{1+(1-\varepsilon)p/2}}dt\right)^{1/p}\\
&\qquad+|w|^{(1-\varepsilon)/2}\left(\int_0^\infty\frac{(1-\cos t)^{p/2}}{t^{1+(1-\varepsilon)p/2}}dt\right)^{1/p}.
\end{aligned}\tag{27}$$

Note that

$$\begin{aligned}
\left(\int_0^\infty\frac{(1-e^{-t})^{p/2}}{t^{1+(1-\varepsilon)p/2}}dt\right)^{1/p}&\asymp\left(\int_0^1\frac{dt}{t^{1-\varepsilon p/2}}+\int_1^\infty\frac{dt}{t^{1+(1-\varepsilon)p/2}}\right)^{1/p}\\
&\asymp\frac{1}{\varepsilon^{1/p}}+\frac{1}{(1-\varepsilon)^{1/p}},
\end{aligned}\tag{28}$$

and

$$\begin{aligned}
\left(\int_0^\infty\frac{(1-\cos t)^{p/2}}{t^{1+(1-\varepsilon)p/2}}dt\right)^{1/p}&\asymp\left(\int_0^1 t^{(1+\varepsilon)p/2-1}dt+\int_1^\infty\frac{dt}{t^{1+(1-\varepsilon)p/2}}\right)^{1/p}\\
&\asymp\frac{1}{(1-\varepsilon)^{1/p}}
\end{aligned}\tag{29}$$

By combining (27) with (28) and (29) we obtain (25). □

Second proof of Theorem 1.5. Fix an arbitrary isometric embedding J of $L_p((0,\infty),L_2(\mathbb{R}^n))$ into L_p and define $Q:H_n\to L_p$ by

$$\forall, x\in \mathbb{H}_n,\qquad Q(x)\stackrel{\mathrm{def}}{=}(1-\varepsilon)^{1/p}J\left(G-\sigma(x)G\right).$$

By Lemma 3.1, for every $\theta\in(0,\infty)$ and every $(u,v,w)\in\mathbb{H}_n$,

$$\begin{aligned}\|Q(\delta_\theta(u,v,w))\|_p^p&=\int_0^\infty \frac{\left(1-e^{-\lambda\left(\theta^2\|u\|_2^2+\theta^2\|v\|_2^2\right)}\cos(\lambda\theta^2w)\right)^{p/2}}{\lambda^{1+(1-\varepsilon)p/2}}d\lambda\\&=\theta^{(1-\varepsilon)p}\int_0^\infty \frac{\left(1-e^{-s\left(\|u\|_2^2+\|v\|_2^2\right)}\cos(\lambda s)\right)^{p/2}}{s^{1+(1-\varepsilon)p/2}}ds\\&=\theta^{(1-\varepsilon)p}\|Q(u,v,w)\|_p^p.\end{aligned}$$

Since σ is an action of $\mathbb{H}_n$ on $L_p((0,\infty),L_2(\mathbb{R}^n))$ by isometries, for every $x,y\in\mathbb{H}_n$ we have $\|Q(x)-Q(y)\|_p=\|Q(x^{-1}y)\|_p$, and it therefore follows that $\|Q(\delta_\theta(x))-Q(\delta_\theta(y))\|_p=\theta^{1-\varepsilon}\|Q(x)-Q(y)\|_p$ for every $x,y\in\mathbb{H}_n$ and $\theta\in(0,\infty)$. Arguing exactly as in the proof of Lemma 2.5, it follows that $Q(\mathbb{H}_1)\subseteq L_p$ is a $2^{8/(1-\varepsilon)}$-doubling subset of L_p. It remains to note that by Lemma 3.1, for every $x,y\in\mathbb{H}_n$ we have

$$d_N(x,y)^{1-\varepsilon}\lesssim\|Q(x)-Q(y)\|_p\lesssim\left(1+\frac{(1-\varepsilon)^{1/p}}{\varepsilon^{1/p}}\right)d_N(x,y)^{1-\varepsilon}.\qquad\square$$

Acknowledgements. V. L. was supported by ANR grant KInd. A. N. was supported by NSF grant CCF-0832795, BSF grant 2010021, the Packard Foundation and the Simons Foundation. Part of this work was completed while A. N. was a Visiting Fellow at Princeton University.

References

[ANT10] T. Austin, A. Naor, and R. Tessera. Sharp quantitative nonembeddability of the Heisenberg group into superreflexive Banach spaces, 2010. To appear in *Groups Geom. Dyn.*, preprint available at `http://arxiv.org/abs/1007.4238`.

[BBI01] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[BGN13] Y. Bartal, A. Gottlieb, and O. Neiman. On the impossibility of dimension reduction for doubling subsets of ℓ_p, $p>2$, 2013. Preprint.

[Bla03] S. Blachère. Word distance on the discrete Heisenberg group. *Colloq. Math.*, 95(1):21–36, 2003.

[CKN11] J. Cheeger, B. Kleiner, and A. Naor. Compression bounds for Lipschitz maps from the Heisenberg group to L_1. *Acta Math.*, 207(2):291–373, 2011.

[CW71] R. R. Coifman and G. Weiss. *Analyse harmonique non-commutative sur certains espaces homogènes.* Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin, 1971. Étude de certaines intégrales singulières.

[Cyg81] J. Cygan. Subadditivity of homogeneous norms on certain nilpotent Lie groups. *Proc. Amer. Math. Soc.*, 83(1):69–70, 1981.

[GNRS04] A. Gupta, I. Newman, Y. Rabinovich, and A. Sinclair. Cuts, trees and l_1-embeddings of graphs. *Combinatorica*, 24(2):233–269, 2004.

[Laa00] T. J. Laakso. Ahlfors Q-regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality. *Geom. Funct. Anal.*, 10(1):111–123, 2000.

[LMN05] J. R. Lee, M. Mendel, and A. Naor. Metric structures in L_1: dimension, snowflakes, and average distortion. *European J. Combin.*, 26(8):1180–1190, 2005.

[LN06] J. R. Lee and A. Naor. L_p metrics on the Heisenberg group and the Goemans-Linial conjecture. In *Proceedings of 47th Annual IEEE Symposium on Foundations of Computer Science*, pages 99–108. 2006. Available at `http://www.cims.nyu.edu/~naor/homepage%20files/L_pHGL.pdf`.

[LN12] V. Lafforgue and A. Naor. Vertical versus horizontal Poincaré inequalities on the Heisenberg group, 2012. To appear in *Israel J. Math.*, preprint available at `http://arxiv.org/abs/1212.2107`.

[LP01] U. Lang and C. Plaut. Bilipschitz embeddings of metric spaces into space forms. *Geom. Dedicata*, 87(1-3):285–307, 2001.

[MN08] M. Mendel and A. Naor. Markov convexity and local rigidity of distorted metrics [extended abstract]. In *Computational geometry (SCG'08)*, pages 49–58. ACM, New York, 2008.

[NN12] A. Naor and O. Neiman. Assouad's theorem with dimension independent of the snowflaking. *Rev. Mat. Iberoam.*, 28(4):1123–1142, 2012.

[Ost11] M. I. Ostrovskii. On metric characterizations of some classes of Banach spaces. *C. R. Acad. Bulgare Sci.*, 64(6):775–784, 2011.

[Sch38] I. J. Schoenberg. Metric spaces and positive definite functions. *Trans. Amer. Math. Soc.*, 44(3):522–536, 1938.

[Sem03] S. Semmes. An introduction to Heisenberg groups in analysis and geometry. *Notices Amer. Math. Soc.*, 50(6):640–646, 2003.

[Woj91] P. Wojtaszczyk. *Banach spaces for analysts*, volume 25 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1991.

Laboratoire de Mathématiques - Analyse, Probabilités, Modélisation - Orléans (MAPMO) UMR CNRS 6628, Université d'Orléans Rue de Chartres, B.P. 6759 - 45067 Orléans cedex 2

E-mail address: `vlafforg@math.jussieu.fr`

Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York NY 10012, USA

E-mail address: `naor@cims.nyu.edu`