Low Dimensional Embeddings of Ultrametrics

Yair Bartal* Nathan Linial† Manor Mendel‡ Assaf Naor

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Abstract

In this note we show that every n-point ultrametric embeds with constant distortion in $\ell_p^{O(\log n)}$ for every $\infty \geq p \geq 1$. More precisely, we consider a special type of ultrametric with hierarchical structure called a k-hierarchically well-separated tree (k-HST). We show that any k-HST can be embedded with distortion at most $1 + O(1/k)$ in $\ell_p^{O(k^2 \log n)}$. These facts have implications to embeddings of finite metric spaces in low dimensional ℓ_p spaces in the context of metric Ramsey-type theorems.

1 Introduction

An *ultrametric* is a metric space (X, d) such that for every $x, y, z \in X$,

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

A more restricted class of finite metrics with an inherently hierarchical structure is that of *k-hierarchically well-separated trees*, defined as follows:

Definition 1. [1] For $k \geq 1$, a *k-hierarchically well-separated tree* (k-HST) is a metric space whose elements are the leaves of a rooted finite tree T. To each vertex $u \in T$ there is associated a label $\Delta(u) \geq 0$ such that $\Delta(u) = 0$ iff u is a leaf of T. It is required that if a vertex u is a child of a vertex v then $\Delta(u) \leq \Delta(v)/k$. The distance between two leaves $x, y \in T$ is defined as $\Delta(\mathrm{lca}(x, y))$, where $\mathrm{lca}(x, y)$ is the least common ancestor of x and y in T.

The notion of 1-HST coincides with that of an ultrametric. Any k-HST is also a 1-HST, i.e., an ultrametric. However, for every $k > 1$ the class of k-HST is a proper subclass of ultrametrics. Ultrametrics and k-HSTs have played a key role in recent work on embeddings of finite metric spaces [2, 3, 4, 5].

Let $f : X \to Y$ be an embedding of the metric space (X, d_X) into the metric space (Y, d_Y). We define the *distortion* of f by

$$\mathrm{dist}(f) = \sup_{\substack{x,y \in X \\ x \neq y}} \frac{d_Y(f(x), f(y))}{d_X(x, y)} \cdot \sup_{\substack{x,y \in X \\ x \neq y}} \frac{d_X(x, y)}{d_Y(f(x), f(y))}.$$

We denote by $c_Y(X)$ the least distortion with which X may be embedded in Y. When $c_Y(X) \leq \alpha$ we say that X α-embeds into Y. When there is a bijection f between two metric spaces X and Y with $\mathrm{dist(f)} \leq \alpha$ we say that X and Y are α-equivalent.

The following proposition provides a comparison between ultrametrics and k-HSTs.

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Proposition 1 ([2]). *For any $k > 1$, any ultrametric is k-equivalent to a k-HST.*

A basic property of ultrametrics is:

Proposition 2 ([8]). *Any ultrametric is isometrically embeddable in ℓ_2.*

Since ℓ_2 isometrically embeds into L_p for every $1 \le p \le \infty$, a similar result follows for embeddings in ℓ_p. Moreover, a careful analysis the proof of the above proposition yields an isometric embedding of any n-point HST into $\ell_p^{O(n)}$.

Here we consider the *dimension* for which ultrametrics and k-HST spaces embed with a given distortion in ℓ_p, $1 \le p \le \infty$. For ℓ_2 this is answered by the Johnson-Lindenstrauss dimension reduction lemma [7] which states that for every $\epsilon > 0$, any n-point metric space in ℓ_2 can be $(1+\epsilon)$-embedded in $\ell_2^{O(\log n/\epsilon^2)}$. Using [6], it follows that any set of n points in ℓ_2 can be embedded with constant distortion into $\ell_p^{O(\log n)}$ for $1 \le p \le 2$ and into $\ell_p^{O(\sqrt{p}(\log n)^{p/2})}$ for $p > 2$. The main result of this note improves the the upper bound on the dimension required to embed n-point ultrametrics into ℓ_p, $p > 2$, and gives additional structural information on the problem for embeddings into low dimensional ℓ_p spaces for $1 \le p \le 2$. Moreover, we show that any n-point k-HST can be embedded in ℓ_p with constant distortion and dimension logarithmic in n. Furthermore, the distortion approaches 1 as k grows.

Proposition 3. *Fix an integer $k > 5$. Then for any $1 \le p < \infty$, any k-HST can be $\left(\frac{k+1}{k-5}\right)$-embedded in ℓ_p^h with $h = \lceil C(1 + k/p)^2 \log D\rceil$, where D is the maximal out-degree of a vertex in the tree defining the k-HST, and $C > 0$ is a universal constant.*

Proposition 3 is proved in Section 2. Combining Proposition 1 and Proposition 3 we obtain:

Corollary 4. *For any $1 \le p \le \infty$, any n point ultrametric can be $O(1)$-embedded into $\ell_p^{O(\log n)}$.*

We also show how to apply this lemma to the metric Ramsey-type problems. A metric Ramsey-type theorem states that a given metric space contains a large subset which can be embedded with small distortion in some "well-structured" family of metric spaces (e.g. Euclidean metrics). This can be formulated using the following notion.

Definition 2. Let $\mathcal{M}$ be some class of metric spaces. Denote by $R_{\mathcal{M}}(\alpha, n)$ the largest integer m such that any n-point metric space has a subset of size m that α-embeds into a member of $\mathcal{M}$. When $\mathcal{M} = \{\ell_p\}$, we use R_p rather than R_{ℓ_p}.

In [4] it is shown that for every $1 \le p \le \infty$ and $\alpha > 2$, $R_p(\alpha, n) \ge n^{1-O\left(\frac{\log \alpha}{\alpha}\right)}$ and for every $0 < \varepsilon < 1$, $R_p(2 + \varepsilon, n) \ge n^{\Omega\left(\frac{\varepsilon}{\log(2/\varepsilon)}\right)}$. We refer to [4] and the references therein for a comprehensive description metric Ramsey problems and their history. Using Lemma 3, we prove the following variant of the result of [4] in which there is control on the dimension in the metric Ramsey problem for ℓ_p, $p \ge 1$. This application was our original motivation for studying low-dimensional embeddings of ultrametrics.

Theorem 1. *The following assertions hold:*

1. *There exist absolute constants $c, C > 0$ such that for all $1 \le p \le \infty$ and for every $\alpha > 2$,*

$$R_{\ell_p^d}(\alpha, n) \ge n^{1-C\frac{\log \alpha}{\alpha}}, \qquad \text{where} \quad d = \lceil c \log n\rceil .$$

2. *There are absolute constants $C, c > 0$ such that for every $0 < \epsilon < 1$, $1 \le p < \infty$ and every integer n,*

$$R_{\ell_p^d}(2+\epsilon, n) \ge n^{\frac{c\epsilon}{\log(2/\epsilon)}}, \qquad \textit{where} \quad d = \left\lceil C \frac{\varepsilon \lceil (\varepsilon p)^{-2} \rceil}{\log(2/\epsilon)} \log n \right\rceil.$$

2 Embedding HSTs in low dimensional ℓ_p spaces

We follow Definition 1, and associate with any k-HST, the tree T *defining the HST*. An internal vertex in T with out-degree 1 is said to be *degenerate*. If u is non-degenerate, then $\Delta(u)$ is the diameter of the sub-space induced on the subtree rooted by u. Degenerate nodes do not influence the metric on T's leaves, hence we may assume that all internal nodes are non-degenerate. In particular for an HST X, $\text{diam}(X) = \Delta(\text{root}(T))$, where T is the tree defining X.

We make use of the following standard construction of codes, the proof of which is included for the sake of completeness. In what follows, for $w, v \in \{0,1\}^h$, $w \oplus v$ denotes the point-wise addition modulo 2 of v and w.

Lemma 5. *For any $h \in \mathbb{N}$, and $\tau \in (0,1)$, there exists $K \subset \{0,1\}^h$ such that the Hamming distance between any two distinct elements of K is in the range $[(1-\tau)h/2, (1+\tau)h/2]$ and $|K| \ge \lfloor e^{h\tau^2/8} \rfloor$.*

Proof. Let $w, v \in \{0,1\}^h$ be random elements. Then by Chernoff bound, the probability that $w \oplus v$ has less than $(1-\delta)h/2$ 1's is at most $e^{-\delta^2 h/4}$. Similarly, the probability it has more than $(1+\delta)h/2$ 1's is also at most $e^{-\delta^2 h/4}$. Given m random elements $w_1, \ldots, w_m \in \{0,1\}^h$, the probability that the distance between any two of them isn't in the range $[(1-\delta)h/2, (1+\delta)h/2]$ is at most $\binom{m}{2} 2e^{-\delta^2 h/4} < m^2 e^{-\delta^2 h/4}$. Thus, choosing $m = \lfloor e^{\delta^2 h/8} \rfloor$ implies that with a positive probability the subset $K = \{w_1, \ldots, w_m\}$ has the required properties. □

Proof of Lemma 3. Let u be the root of the tree defining X and $X_1, \ldots, X_s$ be the leaf sets of subtrees rooted at the children of u. Note that $s \le D$. For $p < \infty$, let $\tau = (1 + k/p)^{-1}/6$. Set $h = \lceil 8\tau^{-2} \log s \rceil$, so that $e^{h\tau^2/8} \ge s$. By Lemma 5 there exists $K \subset \{0,1\}^h$ with all Hamming distances in the range $[(1-\tau)h/2, (1+\tau)h/2]$ and $|K| \ge s$. Choose s distinct $c_1, \ldots, c_s \in K$. By switching to $c_1 \oplus c_1, c_2 \oplus c_1, \ldots, c_s \oplus c_1$ we may assume that $c_1 = 0$, in which case for $1 \le i \le s$, $\|c_i\|_1 \le \frac{1+\tau}{2} h$.

Assume inductively that for each i we have an embedding $\phi_i : X_i \to \ell_p^h$, such that:

- For all $x, y \in X_i$, $\frac{k-5}{k+1} d_{X_i}(x,y) \le \|\phi_i(x) - \phi_i(y)\|_p \le d_{X_i}(x,y)$.
- For every $x \in X_i$, $\|\phi_i(x)\|_p \le \text{diam}(X_i)$.

Let $\lambda = \left(\frac{1+\tau}{2} h\right)^{-1/p} \frac{k-2}{k}$, and let $\Delta = \text{diam}(X)$. Define an embedding $\phi : X \to \ell_p^h$ of X as follows: for $x \in X_i$,

$$\phi(x) = \phi_i(x) + \lambda \Delta c_i.$$

Then:

$$\begin{aligned}
\|\phi(x)\|_p &\le \|\phi_i(x)\|_p + \lambda\Delta\|c_i\|_p \le \text{diam}(X_i) + \left(\frac{1+\tau}{2} h\right)^{-1/p} \frac{k-2}{k} \Delta \|c_i\|_1^{1/p} \\
&\le \frac{\Delta}{k} + \frac{k-2}{k}\Delta < \Delta.
\end{aligned}$$

For $x, y \in X_i$, $\|\phi(x) - \phi(y)\|_p = \|\phi_i(x) - \phi_i(y)\|_p$, so by the induction hypothesis:

$$\frac{k-5}{k+1} d_X(x, y) \leq \|\phi(x) - \phi(y)\|_p \leq d_X(x, y).$$

For $x \in X_i$, $y \in X_j$ and $i \neq j$, we have $d_X(x, y) = \Delta$. Now

$$\begin{aligned}
\|\phi(x) - \phi(y)\|_p &\leq \lambda\Delta\|c_i - c_j\|_p + \|\phi_i(x)\|_p + \|\phi_j(x)\|_p \\
&\leq \lambda\Delta\|c_i - c_j\|_1^{1/p} + \mathrm{diam}(X_i) + \mathrm{diam}(X_j) \\
&\leq \left(\frac{1+\tau}{2}h\right)^{-1/p} \frac{k-2}{k}\Delta \left(\frac{1+\tau}{2}h\right)^{1/p} + \frac{2}{k}\Delta = \Delta = d_X(x, y),
\end{aligned}$$

and

$$\begin{aligned}
\|\phi(x) - \phi(y)\|_p &\geq \lambda\Delta\|c_i - c_j\|_p - \|\phi_i(x)\|_p - \|\phi_j(x)\|_p \\
&\geq \lambda\Delta\|c_i - c_j\|_1^{1/p} - \mathrm{diam}(X_i) - \mathrm{diam}(X_j) \\
&\geq \left(\frac{1+\tau}{2}h\right)^{-1/p} \frac{k-2}{k}\Delta \left(\frac{1-\tau}{2}h\right)^{1/p} - \frac{2}{k}\Delta \\
&\geq \left(\left(\frac{1-\tau}{1+\tau}\right)^{1/p} \frac{k-2}{k} - \frac{2}{k}\right)\Delta \\
&\geq \left(\frac{k}{k+1} \cdot \frac{k-2}{k} - \frac{2}{k}\right)\Delta \geq \frac{k-5}{k+1} d_X(x, y).
\end{aligned}$$

The last inequality holds for $k > 5$ and the preceding derivation follows from the definition of τ:

$$\left(\frac{1-\tau}{1+\tau}\right)^{1/p} \geq (1+3\tau)^{-1/p} \geq (1+6\tau/p)^{-1} = (1+(1+k/p)^{-1}/p)^{-1} \geq (1+1/k)^{-1}.$$

□

3 Implications

Denote by UM the class of all ultrametrics. We will need the following theorem:

Theorem 2 ([4]). *The following assertions hold for every integer n:*

1. *There exists an absolute constant $C' > 0$ such that for every $\alpha > 2$,*

$$R_{\mathrm{UM}}(\alpha, n) \geq n^{1 - C' \frac{\log \alpha}{\alpha}}.$$

2. *There is an absolute constant $c > 0$ such that for any $k \geq 1$ and $0 < \epsilon < 1$, for any integer n:*

$$R_{k\text{-HST}}(2+\epsilon, n) \geq n^{\frac{c\epsilon}{\log(2k/\epsilon)}}.$$

Proposition 2 implies similar bounds for $R_2(\alpha, n)$. We next show how to extend those results for embedding into $\ell_p^{O(\log n)}$ by using Lemma 3.

Proof of Theorem 1. We begin with the first claim of the theorem. Let $C' > 0$ be the constant at the first assertion in Theorem 2, and let β be a universal constant such that any n-point ultrametric β embeds in $\ell_p^{O(\log n)}$ (Corollary 4). We choose $C = \beta C'$, so that $C\frac{\log\alpha}{\alpha} \geq C'\frac{\log(\alpha/\beta)}{\alpha/\beta}$. From Theorem 2 we deduce that

$$R_{\mathrm{UM}}(\alpha/\beta, n) \geq n^{1-C'\frac{\log(\alpha/\beta)}{\alpha/\beta}} \geq n^{1-C\frac{\log\alpha}{\alpha}}.$$

The subset described by this statement is (α/β)-equivalent to an ultrametric and so, by Corollary 4, it is α-embeddable in $\ell_p^{O(\log n)}$.

We next consider the second statement in the theorem. Let $\delta = \epsilon/4$ and $k = \lfloor 5 + 6/\delta \rfloor$, then by Theorem 2, there exists $c' > 0$ such that $R_{k\text{-HST}}(2+\delta, n) \geq n^{\frac{c'\delta}{\log(2/\delta)}}$. Let M be an arbitrary metric space. For an appropriate choice of c this means that M contains a subset Y of size $m = \lceil n^{\frac{c\epsilon}{\log(2/\epsilon)}} \rceil$ that is $(2+\delta)$-equivalent to some k-HST X. By Proposition 3 and our choice of k, there exists some constant $C' > 0$ such that X can be $(1+\delta)$-embedded in ℓ_p^d, where

$$d = \lceil C' \lceil (\delta p)^{-2} \rceil \log m \rceil = \left\lceil C \frac{\epsilon \lceil (\epsilon p)^{-2} \rceil}{\log(2/\epsilon)} \log n \right\rceil,$$

for an appropriate choice of C. Therefore Y is $(2+\delta)(1+\delta) \leq (2+\epsilon)$-embedded in ℓ_p^d. □

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Yair Bartal, Institute of Computer Science, Hebrew University, Jerusalem 91904, Israel.
yair@cs.huji.ac.il

Nathan Linial, Institute of Computer Science, Hebrew University, Jerusalem 91904, Israel.
nati@cs.huji.ac.il

Manor Mendel, Institute of Computer Science, Hebrew University, Jerusalem 91904, Israel.
mendelma@cs.huji.ac.il

Assaf Naor, Theory Group, Microsoft Research, One Microsoft Way 113/2131, Redmond WA 98052-6399, USA.
anaor@microsoft.com