

KRIVINE SCHEMES ARE OPTIMAL

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ABSTRACT. It is shown that for every $k \in \mathbb{N}$ there exists a Borel probability measure μ on $\{-1, 1\}^{\mathbb{R}^k} \times \{-1, 1\}^{\mathbb{R}^k}$ such that for every $m, n \in \mathbb{N}$ and $x_1, \dots, x_m, y_1, \dots, y_n \in \mathbb{S}^{m+n-1}$ there exist $x'_1, \dots, x'_m, y'_1, \dots, y'_n \in \mathbb{S}^{m+n-1}$ such that if $G : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^k$ is a random $k \times (m+n)$ matrix whose entries are i.i.d. standard Gaussian random variables then for all $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$ we have

$$\mathbb{E}_G \left[\int_{\{-1,1\}^{\mathbb{R}^k} \times \{-1,1\}^{\mathbb{R}^k}} f(Gx'_i)g(Gy'_j)d\mu(f, g) \right] = \frac{\langle x_i, y_j \rangle}{(1 + C/k)K_G},$$

where K_G is the real Grothendieck constant and $C \in (0, \infty)$ is a universal constant. This establishes that Krivine's rounding method yields an arbitrarily good approximation of K_G .

1. INTRODUCTION

Grothendieck's inequality [4, 7] asserts that there exists a constant $K \in (0, \infty)$ such that for every $n, m \in \mathbb{N}$, every $m \times n$ matrix $A = (a_{ij})$ with real entries, and every choice of unit vectors $x_1, \dots, x_m, y_1, \dots, y_n \in \mathbb{S}^{m+n-1}$, there are signs $\varepsilon_1, \dots, \varepsilon_m, \delta_1, \dots, \delta_n \in \{-1, 1\}$ satisfying

$$\sum_{i=1}^m \sum_{j=1}^n a_{ij} \langle x_i, y_j \rangle \leq K \sum_{i=1}^m \sum_{j=1}^n a_{ij} \varepsilon_i \delta_j. \tag{1}$$

The infimum over those $K \in (0, \infty)$ for which (1) holds true is called the (real) Grothendieck constant and is denoted K_G . The exact value of K_G remains a long-standing mystery ever since Grothendieck's original 1953 work [4]. The best known bounds on K_G are due to [9, 2].

We call a Borel probability measure μ on $\{-1, 1\}^{\mathbb{R}^k} \times \{-1, 1\}^{\mathbb{R}^k}$ a k -dimensional *Krivine scheme* of quality $K \in (0, \infty)$ if for every $m, n \in \mathbb{N}$ and every choice of unit vectors $x_1, \dots, x_m, y_1, \dots, y_n \in \mathbb{S}^{m+n-1}$ there exist new unit vectors $x'_1, \dots, x'_m, y'_1, \dots, y'_n \in \mathbb{S}^{m+n-1}$ such that if $G : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^k$ is a random $k \times (m+n)$ matrix whose entries are i.i.d. standard Gaussian random variables then for all $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$ we have

$$\mathbb{E}_G \left[\int_{\{-1,1\}^{\mathbb{R}^k} \times \{-1,1\}^{\mathbb{R}^k}} f(Gx'_i)g(Gy'_j)d\mu(f, g) \right] = \frac{1}{K} \langle x_i, y_j \rangle. \tag{2}$$

For (2) to make sense we need to require that the function of G appearing under the expectation in (2) is Lebesgue measurable. The required measurability will be immediate for the Krivine schemes that we discuss below, but in general this assumption should be taken to be part of the definition of a Krivine scheme.

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The existence of a k -dimensional Krivine scheme of quality K implies that $K_G \leq K$. Indeed, since $f(Gx'_i), g(Gy'_j) \in \{-1, 1\}$ point-wise,

$$\begin{aligned} \sum_{i=1}^m \sum_{j=1}^n a_{ij} \langle x_i, y_j \rangle &= K \mathbb{E}_G \left[\int_{\{-1,1\}^{\mathbb{R}^k} \times \{-1,1\}^{\mathbb{R}^k}} \left(\sum_{i=1}^m \sum_{j=1}^n a_{ij} f(Gx'_i) g(Gy'_j) \right) d\mu(f, g) \right] \\ &\leq K \max_{\varepsilon_1, \dots, \varepsilon_m, \delta_1, \dots, \delta_n \in \{-1,1\}} \sum_{i=1}^m \sum_{j=1}^n a_{ij} \varepsilon_i \delta_j. \end{aligned} \quad (3)$$

A different way to describe the above reasoning is via the following procedure that, given unit vectors $x_1, \dots, x_m, y_1, \dots, y_n \in \mathbb{S}^{m+n-1}$, produces signs $\varepsilon_1, \dots, \varepsilon_m, \delta_1, \dots, \delta_n \in \{-1, 1\}$ so as to satisfy the desired inequality (1). Identify $f \in \{-1, 1\}^{\mathbb{R}^k}$ with the bipartition of \mathbb{R}^k given by $f^{-1}(-1), f^{-1}(1) \subseteq \mathbb{R}^k$. Thus μ could be thought of as a carefully chosen notion of a random pair $(\{A, \mathbb{R}^k \setminus A\}, \{B, \mathbb{R}^k \setminus B\})$ of bipartitions of \mathbb{R}^k . The new unit vectors $x'_1, \dots, x'_m, y'_1, \dots, y'_n \in \mathbb{S}^{m+n-1}$ should be thought of as the result of *preprocessing* the initial vectors $x_1, \dots, x_m, y_1, \dots, y_n \in \mathbb{S}^{m+n-1}$, in anticipation of the “random projection” step to come. In this step one chooses a random $k \times (m+n)$ Gaussian matrix G , and, independently of G , a pair of bipartitions $(\{A, \mathbb{R}^k \setminus A\}, \{B, \mathbb{R}^k \setminus B\})$ whose law is μ . One then defines *random signs* $\varepsilon_1, \dots, \varepsilon_m, \delta_1, \dots, \delta_n \in \{-1, 1\}$ by

$$\varepsilon_i \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } Gx'_i \in A, \\ -1 & \text{if } Gx'_i \in \mathbb{R}^k \setminus A, \end{cases} \quad \text{and} \quad \delta_j \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } Gy'_j \in B, \\ -1 & \text{if } Gy'_j \in \mathbb{R}^k \setminus B. \end{cases}$$

Inequality (3) means that for this choice of random signs the desired Grothendieck inequality (1) holds *in expectation*, with respect to the randomness of the matrix G and the pair of bipartitions $(\{A, \mathbb{R}^k \setminus A\}, \{B, \mathbb{R}^k \setminus B\})$.

The preprocessing step of a Krivine scheme has been somewhat suppressed in the above discussion because it is essentially determined by the measure μ itself. Indeed, by rotation invariance the left hand side of (2) depends only on $\langle x'_i, y'_j \rangle$; say, it equals $\Psi(\langle x'_i, y'_j \rangle)$ for some function $\Psi : [-1, 1] \rightarrow [-1, 1]$. If Ψ were invertible on $[-1/K, 1/K]$ then (2) would become $\langle x'_i, y'_j \rangle = \Psi^{-1}(\langle x_i, y_j \rangle / K)$. This means that the mutual angles between the new unit vectors (if they indeed exist) are determined by (2) and the initial configuration of vectors. Only these angles matter for the purpose of applying a Krivine scheme to bound K_G .

We note that in the known examples of Krivine schemes (including those described below) the preprocessing step that associates new unit vectors $x'_1, \dots, x'_m, y'_1, \dots, y'_n \in \mathbb{S}^{m+n-1}$ to the initial unit vectors $x_1, \dots, x_m, y_1, \dots, y_n \in \mathbb{S}^{m+n-1}$ is *oblivious* to the initial configuration in the following sense. There exist (nonlinear) mappings $S, T : \mathbb{S}^\infty \rightarrow \mathbb{S}^\infty$, where \mathbb{S}^∞ is the unit sphere of ℓ_2 , such that if we identify the span of $T(x_1), \dots, T(x_m), S(y_1), \dots, S(y_n)$ with a subspace of \mathbb{R}^{m+n} then (2) holds true with $x'_i = T(x_i)$ and $y'_j = S(y_j)$. We call such Krivine schemes *oblivious Krivine schemes*. It is possible to show that any Krivine scheme satisfying an additional continuity assumption is oblivious. For all purposes that we can imagine, one can take the definition of a Krivine scheme to include the assumption that it is oblivious.

Theorem 1.1 (Optimality of Krivine schemes). *For every $k \in \mathbb{N}$ there exists a k -dimensional oblivious Krivine scheme of quality $(1 + O(1/k))K_G$.*

One might initially believe that restricting attention to Krivine schemes in order to prove Grothendieck’s inequality is too restrictive: the role of random Gaussian matrices is not

mandated by the problem at hand, and moreover it might be beneficial to choose the signs $\varepsilon_1, \dots, \varepsilon_m, \delta_1, \dots, \delta_n \in \{-1, 1\}$ via a procedure that depends on the given matrix (a_{ij}) . Nevertheless, Theorem 1.1 shows that oblivious Krivine rounding schemes capture K_G exactly.

To date, the best known upper estimates on K_G were obtained via oblivious Krivine schemes. Krivine introduced [6] this approach to Grothendieck's inequality in 1977 in order to obtain a new upper bound on K_G . His argument used a one-dimensional Krivine scheme that is based on a *deterministic* partition of \mathbb{R} : in Krivine's original work the measure μ is supported on the pair $(\text{sign}(\cdot), \text{sign}(\cdot)) \in \{-1, 1\}^{\mathbb{R}} \times \{-1, 1\}^{\mathbb{R}}$. Krivine conjectured [6] that the bound thus obtained is actually the exact value of the Grothendieck constant. Clearly, if true, Krivine's conjecture would have implied that oblivious Krivine schemes yield an exact evaluation of K_G , and would have thus made Theorem 1.1 redundant.

However, in 2011 Krivine's conjecture was refuted [2], yielding the best known upper bound on K_G via a *two-dimensional* Krivine rounding scheme. It was this development that motivated us to investigate the possible validity of Theorem 1.1. While at present there is no compelling conjecture as to the exact value of K_G (though in [2] a candidate for this value is proposed based on a certain geometric conjecture that at this point seems quite speculative), Theorem 1.1 justifies the focus on oblivious Krivine schemes. In essence, Theorem 1.1 says that in order to understand the Grothendieck constant it suffices to focus on ways to partition Euclidean space so as to ensure the validity of (2) with K as small as possible.

2. PROOF OF THEOREM 1.1

Fix $k \in \mathbb{N}$. By a well-known duality argument (see [8, Thm. 3.4] and [1, Prop. 1.1]) there exists a Borel probability measure ν_k on $\{-1, 1\}^{\mathbb{S}^{k-1}} \times \{-1, 1\}^{\mathbb{S}^{k-1}}$ such that

$$\forall x, y \in \mathbb{S}^{k-1}, \quad \int_{\{-1, 1\}^{\mathbb{S}^{k-1}} \times \{-1, 1\}^{\mathbb{S}^{k-1}}} f(x)g(y)d\nu_k(f, g) = \frac{1}{K_G} \langle x, y \rangle. \quad (4)$$

By identifying $f \in \{-1, 1\}^{\mathbb{S}^{k-1}}$ with its radial extension $x \mapsto f(x/\|x\|_2)$ (with the convention that the radial extension equals 1 at $x = 0$), the measure ν_k induces a probability measure μ_k on $\{-1, 1\}^{\mathbb{R}^k} \times \{-1, 1\}^{\mathbb{R}^k}$ that satisfies

$$\forall x, y \in \mathbb{R}^k \setminus \{0\}, \quad \int_{\{-1, 1\}^{\mathbb{R}^k} \times \{-1, 1\}^{\mathbb{R}^k}} f(x)g(y)d\mu_k(f, g) = \frac{1}{K_G} \left\langle \frac{x}{\|x\|_2}, \frac{y}{\|y\|_2} \right\rangle. \quad (5)$$

Our goal will be to show that μ_k is the desired Krivine scheme (note that, due to (5), the measurability of the function of G appearing under the expectation in (2) is immediate). To this end it would be beneficial to rewrite the key requirement (2) using (5). Let $G_1, G_2 \in \mathbb{R}^k$ be i.i.d. standard Gaussian random vectors in \mathbb{R}^k . For $t \in [-1, 1]$ define

$$f_k(t) \stackrel{\text{def}}{=} \mathbb{E}_{G_1, G_2} \left[\left\langle \frac{G_1}{\|G_1\|_2}, \frac{tG_1 + \sqrt{1-t^2}G_2}{\|tG_1 + \sqrt{1-t^2}G_2\|_2} \right\rangle \right]. \quad (6)$$

Using this notation, combined with (5) and rotation invariance, the desired identity (2) becomes

$$\forall (i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}, \quad \frac{1}{K_G} f_k(\langle x'_i, y'_j \rangle) = \frac{1}{K} \langle x_i, y_j \rangle. \quad (7)$$

By direct computation (see [5, page 203] for $k = 2$ and [3, Lemma 2.1] for general $k \geq 2$) for every $t \in [-1, 1]$ we have $f_k(t) = \sum_{n=0}^{\infty} a_n(k)t^{2n+1}$, where

$$a_n(k) \stackrel{\text{def}}{=} \frac{1}{4^n \sqrt{\pi}} \binom{2n}{n} \frac{\Gamma((k+1)/2)^2 \Gamma(n+1/2)}{\Gamma(k/2) \Gamma(n+1+k/2)} = \frac{2}{k} \left(\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})} \right)^2 \prod_{j=1}^n \frac{(2j-1)^2}{2j(k+2j)}. \quad (8)$$

By Stirling's formula, for every integer $n \geq 2$ we have

$$a_n(k) \asymp \frac{k^{(k+1)/2}}{2^{k/2} \sqrt{n} (n+k/2)^{(k+1)/2} (1+k/(2n))^n} \lesssim \frac{k^{(k+1)/2}}{k^2 2^{k/2} (n+k/2)^{(k+1)/2}}. \quad (9)$$

Hence,

$$\sum_{n=2}^{\infty} a_n(k) \lesssim \frac{k^{(k+1)/2}}{k^2 2^{k/2}} \int_1^{\infty} \frac{dx}{(x+k/2)^{(k+1)/2}} \lesssim \frac{k^{(k+1)/2}}{k^3 2^{k/2} (k/2)^{(k-1)/2}} \lesssim \frac{1}{k^2}. \quad (10)$$

Also,

$$a_0(k) = 1 - \frac{1}{2k} + O\left(\frac{1}{k^2}\right) \quad \text{and} \quad a_1(k) = \frac{a_0(k)}{2(k+2)} \asymp \frac{1}{k}. \quad (11)$$

Write $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. By (10) for every $z \in \overline{\mathbb{D}}$ we have $f_k(z) = \sum_{n=0}^{\infty} a_n(k)z^{2n+1}$, and f_k is continuous on $\overline{\mathbb{D}}$ and analytic on \mathbb{D} . Moreover, by (10) and (11) we know that there exists a universal constant $C \in (1, \infty)$ such that $|f_k(z) - a_0(k)z| < C/k$ for every $z \in \partial\mathbb{D}$. We assume from now on that k is large enough so that $a_0(k) - 2C/k > 0$. Then $|a_0(k)z - f_k(z)| < |a_0(k)z - \zeta|$ for every $\zeta \in (a_0(k) - C/k)\mathbb{D}$ and $z \in \partial\mathbb{D}$. By Rouché's theorem it follows that $a_0(k)z - \zeta$ and $f_k(z) - \zeta$ have the same number of zeros in \mathbb{D} . Hence $f_k^{-1} : (a_0(k) - C/k)\mathbb{D} \rightarrow \mathbb{D}$ is well defined and analytic on $(a_0(k) - C/k)\mathbb{D}$.

For $w \in (a_0(k) - C/k)\mathbb{D}$ write $f_k^{-1}(w) = \sum_{n=0}^{\infty} b_n(k)w^{2n+1}$ for some $\{b_n(k)\}_{n=0}^{\infty} \subseteq \mathbb{R}$. Thus $b_0(k) = 1/a_0(k)$ and $b_1(k) = -a_1(k)/a_0(k)^4 = -1/(2(k+2)a_0(k)^3)$. Observe that for k large enough we have $1 - 4C/k \leq (a_0(k) - 2C/k)/a_0(k) \leq \sum_{n=0}^{\infty} |b_n(k)|(a_0(k) - 2C/k)^{2n+1} < \infty$, so that by continuity there exists $c_k \in (0, 1 - 2C/k]$ satisfying $\sum_{n=0}^{\infty} |b_n(k)|c_k^{2n+1} = 1 - 4C/k$.

Lemma 2.1. *We have $c_k \geq 1 - O(1/k)$.*

Assuming the validity of Lemma 2.1 for the moment, we conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. This is a slight variant of Krivine's original argument [6]. Define two mappings $S, T : \mathbb{S}^{\infty} \rightarrow \left(\bigoplus_{n=0}^{\infty} (\ell_2)^{\otimes(2n+1)}\right) \oplus \mathbb{R}^2 \stackrel{\text{def}}{=} \mathcal{H} \cong \ell_2$ by

$$S(x) \stackrel{\text{def}}{=} \left(\left(|b_n(k)|^{1/2} c_k^{(2n+1)/2} x^{\otimes(2n+1)} \right)_{n=0}^{\infty}, \left(\sqrt{\frac{4C}{k}}, 0 \right) \right),$$

and

$$T(x) \stackrel{\text{def}}{=} \left(\left(\text{sign}(b_n(k)) |b_n(k)|^{1/2} c_k^{(2n+1)/2} x^{\otimes(2n+1)} \right)_{n=0}^{\infty}, \left(0, \sqrt{\frac{4C}{k}} \right) \right).$$

For every $x \in \mathbb{S}^{\infty}$ we have $\|S(x)\|_{\mathcal{H}}^2 = \|T(x)\|_{\mathcal{H}}^2 = \sum_{n=0}^{\infty} |b_n(k)|c_k^{2n+1} + 4C/k = 1$. Moreover, $f_k(\langle S(x), T(y) \rangle_{\mathcal{H}}) = f_k\left(\sum_{n=0}^{\infty} b_n(k)c_k^{2n+1} \langle x, y \rangle^{2n+1}\right) = f_k(f_k^{-1}(c_k \langle x, y \rangle)) = c_k \langle x, y \rangle$. Thus, recalling the formulation of the desired identity appearing in (7), μ_k is an oblivious k -dimensional Krivine scheme of quality $K_G/c_k = (1 + O(1/k))K_G$, as desired. \square

It remains to prove Lemma 2.1.

Proof of Lemma 2.1. For every $w \in (a_0(k) - C/k)\mathbb{D}$ write $z = f_k^{-1}(w) \in \mathbb{D}$. Then using (11),

$$\begin{aligned} f_k^{-1}(w) - b_0(k)w - b_1(k)w^3 &= f_k^{-1}(w) - \frac{w}{a_0(k)} + \frac{w^3}{2(k+2)a_0(k)^3} \\ &= -\frac{f_k(z) - a_0(k)z - a_1(k)z^3}{a_0(k)} - \frac{(a_0(k)z - f_k(z))(a_0(k)^2z^2 + a_0(k)zf_k(z) + f_k(z)^2)}{2(k+2)a_0(k)^3}. \end{aligned} \quad (12)$$

By (10) and (11) we have $|f_k(z) - a_0(k)z - a_1(k)z^3| \lesssim 1/k^2$ and $|f_k(z) - a_0(k)z| \lesssim 1/k$. Moreover, due to (11) we have $a_0(k) \asymp 1$ and therefore $|a_0(k)^2z^2 + a_0(k)zf_k(z) + f_k(z)^2| \lesssim 1$. Consequently, it follows from (12) that

$$|f_k^{-1}(w) - b_0(k)w - b_1(k)w^3| \lesssim \frac{1}{k^2}.$$

Hence, by Cauchy's integral formula for every integer $n \geq 2$ and every $r \in (0, a_0(k) - C/k)$,

$$|b_n(k)| = \left| \frac{1}{2\pi i} \oint_{r\partial\mathbb{D}} \frac{f_k^{-1}(w) - b_0(k)w - b_1(k)w^3}{w^{2n+2}} dw \right| \lesssim \frac{1}{k^2 r^{2n+1}}.$$

Thus there is a universal constant $A \in (0, \infty)$ such that $|b_n(k)| \leq Ak^{-2}(a_0(k) - C/k)^{-(2n+1)}$ for $n \geq 2$. If $c_k \geq a_0(k) - C/k$ then we are done. Assume therefore that $c_k < a_0(k) - C/k$, in which case, recalling that $b_0(k) = 1/a_0(k)$ and $b_1(k) = -1/(2(k+2)a_0(k)^3)$, we have

$$\begin{aligned} 1 = \sum_{n=0}^{\infty} |b_n(k)|c_k^{2n+1} &\leq \frac{c_k}{a_0(k)} + \frac{c_k^3}{2(k+2)a_0(k)^2} + \frac{A}{k^2} \sum_{n=2}^{\infty} \left(\frac{c_k}{a_0(k) - C/k} \right)^{2n+1} \\ &= \frac{c_k}{a_0(k)} + \frac{c_k^3}{2(k+2)a_0(k)^2} + \frac{A}{k^2} \cdot \frac{(c_k/(a_0(k) - C/k))^5}{1 - (c_k/(a_0(k) - C/k))^2}. \end{aligned} \quad (13)$$

By (11), a straightforward computation shows that (13) implies that $c_k \geq 1 - O(1/k)$. \square

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