MAT 577, Probabilistic Methods in Combinatorics:

Homework Assignment Number 4, due April 26, this deadline is strict

1. Show that for any \( \epsilon > 0 \) there is a \( C = C(\epsilon) \) such that every set \( S \) of at least \( \epsilon 4^n \) vectors in \( \mathbb{Z}_4^n \) contains four vectors so that the Hamming distance between any pair of them is at least \( n - C\sqrt{n} \).

   Hint: use an appropriate martingale to show that more than \( 3/4 \) of the vectors are within distance \( C\sqrt{n}/2 \) of \( S \).

2. Let \( H \) be a graph with \( m \) edges and maximum degree at most 10. Let \( U \) be a random set of vertices of \( H \) obtained by picking each vertex, randomly and independently, with probability \( p = \frac{1}{\log m} \). Show that the probability that \( U \) is independent is \( (1 - p^2)^{m}\cdot(1 - o(1)) \) (where the \( o(1) \) term tends to zero as \( m \) tends to infinity.)

3. Find a threshold function for the following property of a graph \( G \) on \( n \) vertices: every set of at least \( n/2 \) vertices of \( G \) contains a (not necessarily induced) cycle of length 5. (Recall that, by definition, \( t(n) \) is such a threshold function if when \( p(n) = o(t(n)) \), then with high probability, \( G(n, p(n)) \) does not satisfy the property, and if \( t(n) = o(p(n)) \) then with high probability \( G(n, p(n)) \) satisfies it.)