1. Let $p$ be a prime number and let $A \subset \mathbb{Z}_p$ be a set of $|A| < p^{2/3}$ residues modulo $p$. Show that there are elements $x, y \in \mathbb{Z}_p$ such that for $A + x = \{(a + x) \mod p : a \in A \}$ and $A + y = \{(a + y) \mod p : a \in A \}$, the three sets $A, A + x$ and $A + y$ do not have a common intersection, that is $A \cap (A + x) \cap (A + y) = \emptyset$.

2. The (multi-colored) Ramsey number $r_j(k)$ is the smallest integer $r$ so that in any coloring by $j$ colors of the edges of the complete graph on $r$ vertices there is a monochromatic copy of $K_k$.

Prove that if $\binom{n}{k}3^{k-1} - \binom{n}{k} < 1$ then it is possible to color the edges of the complete graph on $n$ vertices by 3 colors without a monochromatic copy of $K_k$ and conclude that for $k > 4$, $r_3(k) > 3^{k/2}$.

3. Let $\{(A_i, B_i)_{1 \leq i \leq h}\}$ be a collection of pairs of subsets of the integers so that $|A_i| + |B_i| = n$ and $A_i \cap B_i = \emptyset$ for all $1 \leq i \leq h$ and for every $1 \leq i < j \leq h$ $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$. Prove that $h \leq 2^n$.

4. Show that the vertices of any tournament $T = (V, E)$ in which all outdegrees are at least 10 can be colored by 2 colors so that every vertex has at least one outneighbor of each color.

5. Let $G = (V, E)$ be a directed graph with $n > 1$ vertices and $\lceil n \log_2 n \rceil$ directed edges. Prove that there is a tournament on $n$ vertices containing no subgraph isomorphic to $G$. 

MAT 577, Probabilistic Methods in Combinatorics:

Homework Assignment Number 1, due March 1