On three zero-sum Ramsey-type problems

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Abstract

For a graph G whose number of edges is divisible by k, let $R(G, Z_k)$ denote the minimum integer r such that for every function $f : E(K_r) \mapsto Z_k$ there is a copy G' of G in K_r so that $\sum_{e \in E(G')} f(e) = 0$ (in Z_k). We prove that for every integer k, $R(K_n, Z_k) \leq n + O(k^3 \log k)$ provided n is sufficiently large as a function of k and k divides $\binom{n}{2}$. If, in addition, k is an odd prime-power then $R(K_n, Z_k) \leq n + 2k - 2$ and this is tight if k is a prime that divides n. A related result is obtained for hypergraphs. It is further shown that for every graph G on n vertices with an even number of edges $R(G, Z_2) \leq n + 2$. This estimate is sharp.

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1 Introduction

The starting point of almost all the recent combinatorial research on zero-sum problems is the following theorem.

Theorem 1.1 (Erdös, Ginzburg and Ziv [17]) *Let $m \geq k \geq 2$ be two integers and suppose $k|m$. Then any sequence of $m+k-1$ integers contains a non-empty subsequence of cardinality m the sum of whose elements is divisible by k.*

Inspired by this beautiful result, Bialostocki and Dierker ([6],[7]) raised several interesting variants of the classical Ramsey Theory problems. To describe these we need a few definitions.

All graphs considered here are finite, undirected and simple (i.e., have no loops and no parallel edges). Let Z_k denote the cyclic additive group of order k. A *Z_k-coloring* of the edges of a graph $G = (V(G), E(G))$ is a function $f : E(G) \mapsto Z_k$. If $\sum_{e \in E(G)} f(e) = 0$ (in Z_k), we say that G is a *zero-sum* graph (with respect to f). If k divides the number $e(G)$ of edges of G, then the *zero-sum Ramsey number* $R(G, Z_k)$ is the smallest integer r such that for every Z_k-coloring of the edges of K_r there is a zero-sum copy of G in K_r. (Note that this number is finite and is at most the classical Ramsey number $R(G, k)$ that guarantees a monochromatic copy of G when k colors are used. Note also that the assumption that $k|e(G)$ is necessary, since otherwise the constant Z_k-coloring $f \equiv 1$ would give no zero-sum copy of G. Finally, observe that for $k = e(G)$, $R(G, Z_k) \geq R(G, 2)$ as can be seen by considering Z_k-colorings f whose image is $\{0, 1\}$.)

There is a rapidly growing literature on zero-sum problems. See, e.g., [6], [7],[8], [16], [21], [22], [5], [11],[12], [13],[14],[15].

The first problem we consider here, in Section 2, is that of estimating $R(K_n, Z_k)$ for a fixed k as n tends to infinity. As shown in [15], for every inetger k there exists a constant $c(k)$ such that for all n satisfying $k|\binom{n}{2}$, $R(K_n, Z_k) \leq n + c(k)$. The proof relies on Ramsey Theorem, and supplies a huge upper bound for $c(k)$, roughly the Ramsey number $R(K_{3k-1}, k)$. By combining an algebraic approach, similar to the one used in [2], [3], [4], with Ramsey theorem we prove that if n is sufficiently large as a function of k then, in fact, $R(K_n, Z_k) \leq n + O(k^3 \log k)$, and if, in addition, k is an odd prime-power then $R(K_n, Z_k) \leq n + 2k - 2$. This last inequality holds as an equality in case k is a prime which divides n. Note that the assumption that n is large as a function of k cannot be omitted, since, e.g., $R(K_n, Z_{\binom{n}{2}}) \geq R(K_n, 2)$ which is known to be bigger than $2^{n/2}$.

The above result can be generalized to the case of hypergraphs. The interesting part of this generalization is given in Section 3 together with several applications. Finally, in Section 4 , we return to the case of graphs and show that for every graph G on n vertices with an even number of edges $R(G, Z_2) \leq n + 2$. This settles a conjecture of Bialostocki.

2 Zero-sum Ramsey numbers for complete graphs

In this section we prove the following theorem.

Theorem 2.1 *Let k be an integer and suppose $k | \binom{n}{2}$.*
(i) If k is an odd prime-power, and $n + k$ is at least the Ramsey number $R(K_{2k-1}, k)$, then $R(K_n, Z_k) \leq n + 2k - 2$. If, in addition, k is a prime that divides n then $R(K_n, Z_k) = n + 2k - 2$.
(ii) If $n \geq R(K_{3k-1}, k)$ then $R(K_n, Z_k) \leq n + k(k + 1)(k + 2) \log_2 k$.

A basic tool in the proof of the first part of the theorem is the following theorem of Baker and Schmidt [9].

Theorem 2.2 (Baker and Schmidt [9]) *Let q be a prime-power. If $t \geq d(q - 1) + 1$ and $h_1(x_1, \ldots, x_t), h_2(x_1, \ldots, x_t), \ldots, h_l(x_1, \ldots, x_t) \in \mathbf{Z}[x_1, \ldots, x_t]$ satisfy $h_1(0) = \ldots = h_l(0) = 0$, and $\sum_{i=1}^{l} \deg h_i \leq d$, then there exists an $0 \neq \epsilon \in \{0, 1\}^t$ such that $h_1(\epsilon) \equiv \ldots \equiv h_l(\epsilon) \equiv 0 \pmod{q}$.*

When q is a prime this theorem is an easy consequence of the classical Chevalley-Warning Theorem (see, e.g., [10]). A short proof of it, following the method in [2], is given in [4] (only for the special case $l = 1$, but the proof can be easily extended to the general case). See also [1] for various related results.

Lemma 2.3 *Let q be a prime power and let r and t be integers satisfying $r = t + 2q - 2$ and $q | \binom{t}{2}$. Then for every Z_q-coloring $f : E(K_r) \mapsto Z_q$ there is an $m \geq t$ satisfying $m \equiv t (mod\ q)$ and a zero-sum copy of K_m in K_r.*

Proof Since $q | \binom{t}{2}$ it follows that either $t \equiv 0 (mod\ q)$ or $t \equiv 1 (mod\ q)$. Let m be the largest integer satisfying $m \equiv t (mod\ q)$ such that there exists a zero-sum copy of K_m in K_r. (Such an m clearly exists since we can take a copy of the trivial empty graph K_0 or the trivial one point graph K_1 corresponding to the value of $t(mod\ q)$.)

Let M be the vertex set of a zero-sum copy of K_m in K_r. If $m \geq t$ we have nothing to prove. Otherwise, there are at least $3q-2$ vertices outside M. Let $\{v_1, \ldots, v_{3q-2}\}$ be $3q-2$ of them. For each i, $1 \leq i \leq 3q-2$, let $b_i = \sum_{u \in M} f(v_i, u)$ be the sum of the f-values of the edges connecting v_i to M, computed in Z_q.

Consider the following two polynomial equations;

$$f_1 = \sum_{i=1}^{3q-2} x_i \equiv 0(mod\ q) \quad (\textit{the vertex counter equation}),$$

and

$$f_2 = \sum_{i=1}^{3q-2} b_i x_i + \sum_{1 \leq i < j \leq 3q-2} f(v_i, v_j) x_i x_j \equiv 0(mod\ q) \quad (\textit{the edge sum equation}).$$

Obviously, $x_1 = \ldots = x_{3q-2} = 0$ is a solution of these two equations and $3q-2 = 1+(q-1)(deg(f_1)+deg(f_2))$. Therefore, by Theorem 2.2 there is a nontrivial solution $x_i = \epsilon_i$ for this system, where $\epsilon_i \in \{0,1\}$ for each $1 \leq i \leq 3q-2$. Define $V = \{v_i : 1 \leq i \leq 3q-2 \ \ and \ \ \epsilon_i = 1\}$. It is easy to check that $M \cup V$ is the set of vertices of a zero-sum complete subgraph in K_r, contradicting the maximality of m. This completes the proof of the lemma. $\Box$

Proof of Theorem 2.1, part (i) Suppose $r = n + 2q - 2$, where q is an odd prime-power, and suppose $n + q \geq R(K_{2q-1}, q)$. Let $f : E(K_r) \mapsto Z_q$ be a Z_q-coloring. We must show that there is a zero-sum copy of K_n in K_r. By Lemma 2.3 there is a zero-sum copy of K_m, where $m \geq n$ and $m \equiv n(mod\ q)$. If $m = n$ this completes the proof. Otherwise $m = n + q \geq R(K_{2q-1}, q)$. Let M be the vertex set of a zero-sum copy of K_m. Since $|M| = m \geq R(K_{2q-1}, q)$ there is a set A of $2q-1$ vertices in M so that $f(a, a')$ is constant (say c) for all distinct $a, a' \in A$. For each $a \in A$, define $f_a = \sum_{u \in M \setminus A} f(a, u)$, where the sum is taken in Z_q. By Theorem 1.1 there is a subset I of cardinality q of A so that $\sum_{a \in I} f_a = 0$ (in Z_q). Define $N = M \setminus I$. Observe that $|N| = n$. Moreover, N is the set of vertices of a zero-sum complete graph K_N in K_r, since

$$\sum_{e \in E(K_N)} f(e)$$

$$= \sum_{e \in E(K_M)} f(e) - \sum_{e \in E(K_I)} f(e) - \sum_{a \in I} f_a - \sum_{i \in I, j \in A \setminus I} f(i,j)$$

$$\equiv 0 - c\binom{q}{2} - 0 - cq(q-1)(mod\ q) \equiv 0(mod\ q).$$

This completes the proof of the upper bound in Theorem 2.1, part (i).

We next show that this is tight, provided q is an odd prime that divides n. (It is certainly tight when $q = 2$ as shown in Lemma 4.3). Let D, C_1, C_2 be three disjoint sets of vertices, where $|D| = n - 1$ and $|C_1| = |C_2| = q - 1$. Let K be the complete graph whose set of vertices is $D \cup C_1 \cup C_2$. Clearly K has $n + 2q - 3$ vertices. Define a Z_q-coloring f of the edges of K as follows. Choose an element $b \in Z_q$ so that $b^2 - 1$ is a quadratic non-residue and define $f(e) = 1$ if e joins a vertex in D with a vertex in $C_1 \cup C_2$, $f(e) = 2b + 2$ if e has one end in C_1 and one end in C_2 and $f(e) = 0$ otherwise.

In order to complete the proof it suffices to show that there is no zero-sum copy of K_n in K. Indeed, suppose this is false, and there is such a copy containing x vertices in C_1 and y in C_2 (and hence $n - x - y$ in D). Therefore

$$(2b + 2)xy + (x + y)(n - x - y) = 0 \quad (in\ \ Z_q),$$

and since q divides n we conclude that $(x+y)^2 - (2b+2)xy = 0$ (in Z_q), i.e., $(x - by)^2 = (b^2 - 1)y^2$. Since $b^2 - 1$ is a quadratic non-residue this is possible only if $x = y = 0$, but this is impossible since in this case D has n vertices in the zero-sum copy of K_n, contradicting the fact that $|D| = n - 1$. This completes the proof. □

Remark An easy modification of the proof works for the case $q = 2^r$ with a little worse bound. The details are left to the reader. (For $q = 2$ we observe in Section 4 that $R(K_n, Z_2) = n + 2$ for all n satisfying $2|\binom{n}{2}$.)

In order to prove the second part of Theorem 2.1 we need the following result of van Emde Boas and Kruyswijk [18], (see also [20] for a different proof). We note that we can also derive this second part, with a slightly worse estimate, from Theorem 6 in [9].

Theorem 2.4 ([18]) *Let Z_k^r denote the sum of r copies of the group Z_k (i.e., the abelian group of all vetors of length r over Z_k). Let $v_1, \ldots, v_p$ be a sequence of p (not necessarily distinct) members of Z_k^r. If $p > r(k-1)\log_2 k$ then there is a nonempty subset $I \subset \{1, 2, \ldots, p\}$ such that in Z_k^r $\sum_{i \in I} v_i = 0$.*

Proof of Theorem 2.1, part (ii) Suppose

$$k|\binom{n}{2} \quad where \quad n \geq R(K_{3k-1}, k) \tag{1}$$

and suppose $m \geq n + k(k+1)(k+2)\log_2 k$. Let $f : E(K_m) \mapsto Z_k$ be a Z_k-coloring of the edges of the complete graph K_m.

Let t be the maximum integer such that there is a subset $T \subset V(K_m)$, of cardinality $|T| = t$ satisfying

$$t \equiv n(mod\ 2k), \tag{2}$$

and

$$\sum_{e \in E(K_T)} f(e) = 0 \quad (in\ Z_k), \tag{3}$$

where K_T denotes the complete graph on the vertex set T.

Observe that since k divides $\binom{n}{2}$ and $t \equiv n(mod\ 2k)$, k divides $\binom{t}{2}$ as well. Note also that (1) and Ramsey Theorem imply the existence of a set T satisfying (2) and (3). The main part of the proof is the following.

Claim: $t \geq n$.

Proof Suppose this is false and $t < n$. Let $N = \{1, \ldots, m\}$ denote the vertex set of K_m and let $T \subset N$, $|T| = t$ be a set satisfying (2) and (3). Our objective is to show that there is a set $T' \subset N$ which strictly contains T and satisfies (2) and (3), contradicting the maximality of t. To do so we prove that there exist k non-empty pairwise disjoint subsets $A_1, \ldots, A_k$ of N, satisfying the following conditions.

$$(\cup_{i=1}^{k} A_i) \cap T = \emptyset \tag{4}$$

$$|A_i| \equiv 0(mod\ 2k) \quad for \quad all \quad 1 \leq i \leq k \tag{5}$$

$$\sum_{u \in A_i, v \in T} f(u,v) \equiv 0(mod\ k) \quad for \quad all \quad 1 \leq i \leq k \tag{6}$$

$$\sum_{u \in A_i, v \in A_j} f(u,v) \equiv 0(mod\ k) \quad for \quad all \quad 1 \leq i < j \leq k \tag{7}$$

The existence of the desired set T' that contradicts the maximality of t follows easily from the existence of the above sets. Indeed, simply define $a_i = \sum_{e \in E(K_{A_i})} f(e)$ and let I be a nonempty subset of $\{1, \ldots, k\}$ so that $\sum_{i \in I} a_i \equiv 0(mod\ k)$. Define $T' = T \cup (\cup_{i \in I} A_i)$ and observe that by (4), (5), (6) and (7), T' satisfies (2) and (3).

It thus remains to prove the existence of $A_1, \ldots, A_k$ as above. To this end we prove by induction on i, that for every i, $0 \leq i \leq k$, there exist non-empty and pairwise disjoint subsets $A_1, \ldots, A_i$ of $N \setminus T$ so that:

$$|A_j| \equiv 0(mod\ 2k) \quad for \quad all \quad 1 \leq j \leq i \tag{8}$$

$$|A_j| \leq 2(j+1)k \log_2 k \quad for \quad all \quad 1 \leq j \leq i \tag{9}$$

$$\sum_{u \in A_j, v \in T} f(u,v) \equiv 0(mod\ k) \quad for \quad all \quad 1 \leq j \leq i \tag{10}$$

$$\sum_{u \in A_j, v \in A_l} f(u,v) \equiv 0(mod\ k) \quad for \quad all \quad 1 \leq j < l \leq i. \tag{11}$$

The special case $i = k$ gives the required assertion of the claim. The induction conclusion holds trivially for $i = 0$. Assuming it holds for $i-1$ we prove it for i. Given T,$A_1, \ldots, A_{i-1}$ satisfying (8), (9), (10), (11), split arbitrarily the vertices of $N \setminus (T \cup (\cup_{j=1}^{i-1} A_j))$ into pairs (leaving one isolated vertex in the odd case). Let x, y be one of these pairs. Define a vector v_{xy} in Z_k^{i+1} as follows.

(i) For $1 \leq l \leq i-1$, the l-th component of v_{xy} is

$$\sum_{u \in A_l} f(x,u) + \sum_{u \in A_l} f(y,u),$$

(where the sum is taken in Z_k).

(ii) The i-th component of v_{xy} is

$$\sum_{u \in T} f(x,u) + \sum_{u \in T} f(y,u).$$

(iii) The $i+1$-st component of v_{xy} is simply 1.

By Theorem 2.4, any family of more than $(i+1)(k-1)\log_2 k$ such vectors contains a non-empty subfamily of the vectors whose sum is 0 (in Z_k^{i+1}). The set of all the vertices in the pairs corresponding to the vectors in such a subfamily can be taken as A_i. By (iii), (8) holds whereas the construction implies the validity of (9),(10) and (11). Since, by (1), $m \geq n + k(k+1)(k+2)\log_2 k$, the process of successively forming new subsets A_i as above can be performed until k sets are generated, completing the proof of the induction and that of the claim.

Returning to the proof of the second part of Theorem 2.1, we now observe that the rest of this proof is almost identical to that of the first part of the theorem. If $t = n$ there is nothing to prove, since K_T is the required zero-sum copy of K_n. Otherwise, there is a subset L of cardinality $3k-1$

of T, such that $f(e)$ is a constant on the edges of K_L. By Theorem 1.1 there is a subset F of cardinality $2k$ of L such that

$$\sum_{u \in F, v \in T \setminus F} f(u,v) \equiv 0 (mod\ k).$$

As before, this implies that $T_1 = T \setminus F$ is a zero-sum copy of K_{t-2k} in our complete graph. This process may be repeated and since in the beginning $t \geq n$, by the claim, and $t \equiv n (mod\ 2k)$ this proves the existence of a zero-sum copy of K_n in K_m, completing the proof of Theorem 2.1. □

3 Hypergraphs

The main result of the previous section can be generalized to the case of hypergraphs. Since such a generalization is somehwat lengthy we merely present part of it together with some consequences. By a *hypergraph* we mean here, as ususal, a pair $H = (V, E)$, where V is a finite set of vertices, and E is a finite multiset of subsets of V. (The same subset can appear several times). The *rank* of $H = (V, E)$ is the maximum number of vertices in an edge of it. If $U \subset V$, then the *induced* sub-hypergraph of H on U, denoted by $H(U)$, is the hypergraph whose set of vertices is U and whose set of edges is the set of all edges of H which are subsets of U. If $H = (V, E)$ is a hypergraph and $f : E \mapsto Z_k$ is a Z_k-coloring of the set of its edges, we call an induced subhypergraph $H(U)$ a *zero-sum subgraph* of H if $\sum_{e \in E(H(U))} f(e) = 0$, (where the sum is computed in Z_k).

Theorem 3.1 *Let q be a prime-power, let $H = (V, E)$ be a hypergraph of rank r and let $f : E \mapsto Z_q$ be a Z_q-coloring of E. Then there is a subset U of V, where $|U| \geq |V| - r(q-1)$ such that $H(U)$ is a zero-sum subgraph of H. The above lower bound for $|U|$ is sharp for all $r \geq 1$ and all prime-powers q.*

Proof Let m be the maximum inetger such that there exists a set M of m vertices of H so that $H(M)$ is a zero-sum subgraph of H. If $m \geq |V| - r(q-1)$ there is nothing to prove. Otherwise, there are more than $r(q-1)$ vertices in $V \setminus M$. Define $W = V \setminus M$ and let F be the set of all edges of H that contain at least one vertex of W. Associate each vertex w in W with a variable x_w and consider the following polynomial equation.

$$h = \sum_{e \in F} f(e) \prod_{w \in e} x_w \equiv 0 (mod\ q).$$

The trivial vector $x_w = 0$ for all $w \in W$ is clearly a solution of this equation. Since the degree of h is at most r, Theorem 2.2 implies that there is a nontrivial solution in which each variable x_w is either 0 or 1. Let W' be the set of all the vertices w for which $x_w = 1$ in this solution. It is easy to check that $H(M \cup W')$ is a zero-sum subgraph of H, contradicting the maximality of m. This proves the assertion of the theorem.

We next show that this assertion is sharp. Given an integer r and an integer q (which is not necessarily a prime-power), let $H(r,q)$ denote the following hypergraph. Its set of vertices is the union of $q-1$ pairwise disjoint sets $V_1, \ldots, V_{q-1}$, each of cardinality r. Its set of edges is the set $E = \{e : e \neq \emptyset, \ \ and \ \ e \subset V_i \ \ for \ \ some \ \ 1 \leq i \leq q-1\}$. Define $f : E \mapsto Z_q$ by $f(e) = (-1)^{|e|}$ for all $e \in E$. It is easy to see that there is no nonempty zero-sum subgraph of $H(r,q)$, completing the proof of the theorem. $\Box$

Remarks

1). The above theorem can be extended to the case of non prime-powers q, by applying Theorem 6 in [9]. The estimate obtained for this case is not sharp.

2). A straightforward generalization of the proof of the last theorem shows that if $H_1 = (V, E_1)$,...,$H_l = (V, E_l)$ are l hypergraphs of ranks $r_1, \ldots, r_l$ respectively, and all have the same set of vertices V, then for every prime power q and every $f_i : E_i \mapsto Z_q$, there exists a subset $U \subset V$ so that $|U| \geq |V| - (q-1)\sum_{i=1}^{l} r_i$ and each $H_i(U)$ is a zero-sum subgraph of H_i. Observe that Lemma 2.3 (for $t \equiv 0(mod\ q)$) is a special case of this statement obtained by letting one of the hypergraphs be the trivial rank 1 hypergraph in which each singleton is mapped by the Z_q-coloring to 1. We omit the detailed proof.

3). The following proposition is an immediate application of Theorem 3.1, which we state as a representing example. Obviously, one can give various additional similar applications of this theorem and of the previous remark, and in particular, these obtained from the next proposition by replacing the cycle of length 6 by any other graph.

Proposition 3.2 *For any prime power q, any graph with n vertices contains an induced subgraph on at least $n - 6(q-1)$ vertices in which the number of cycles of length 6 is divisible by q.*

Proof Let $G = (V, E)$ be a graph on n vertices. Let H be the hypergraph whose set of vertices is V and whose set of edges F is the set of all subsets of cardinality 6 of V. Define $f : F \mapsto Z_q$ by letting $f(e)$ be the number of cycles of length 6 of G whose set of vertices is e, reduced modulo q. The result now clearly follows from Theorem 3.1. □

4 The binary case

The main result in this section is the following theorem which deals with zero-sum Ramsey numbers for graphs over the group Z_2.

Theorem 4.1 *Let G be a graph with n vertices and with an even number of edges. Then:*
(i) $R(G, Z_2) \leq n + 2$, and this inequality is sharp.
(ii) If $n \equiv 3(mod\ 4)$ is a prime-power then $R(G, Z_2) = n$.
(iii) If G is bipartite then $R(G, Z_2) = n$ unless all the degrees in G are odd, in which case $R(G, Z_2) = n + 1$.

The proof is rather lengthy. We start with the following definition.
Let $H_1, \ldots, H_r$ be a family of subgraphs of K_n. The *sum modulo* 2 of $H_1, \ldots, H_r$, denoted by $\oplus \sum_{i=1}^{r} H_i$, is the subgraph of K_n consisting of all edges of K_n that belong to an odd number of the graphs H_i.
Note that this is precisely the sum (in Z_2) of the characteristic vectors of the edge-sets of the graphs H_i, where the characteristic vector of H_i is simply the binary vector of length $\binom{n}{2}$ whose coordinates are indexed by the edges of K_n in which there is a 1 in each coordinate corresponding to an edge of H_i.
If $\oplus \sum_{i=1}^{r} H_i$ is the empty graph we write $\oplus \sum_{i=1}^{r} H_i = 0$ and say that $H_1, \ldots, H_r$ is a family with an *empty sum*.

The following lemma is very simple but useful. Since it will be used extensively in the rest of the section we name it, for future reference.

Lemma 4.2 (The parity Lemma) *Let G be a graph with an even number of edges. Then $R(G, Z_2)$ is the least integer n such that there is an odd family of copies of G in K_n with an empty sum. I.e., it is the least n such that there is a family of subgraphs $H_1, \ldots, H_m$ of K_n, where m is odd, each H_i is isomorphic to G, and $\oplus \sum_{i=1}^{m} H_i = 0$.*

Proof Let $I_n(G)$ denote the family of all copies of G in K_n. For each edge e of K_n, let x_e be a variable. Associate each member $H \in I_n(G)$ with the following linear equation over Z_2 in the variables x_e;

$$\sum_{e \in E(H)} x_e = 1.$$

Note that this equation asserts that H is not a zero-sum copy of G (for the coloring defined by $f(e) = x_e$). Therefore, the resulting system of $|I_n(G)|$ linear equations corresponding to all members of $I_n(G)$ does not have a solution over Z_2 iff for every Z_2-coloring of K_n there is a zero-sum copy of G, i.e., iff $n \geq R(G, Z_2)$. By standard linear algebra there is no solution iff there is a set of equations whose sum in Z_2 gives the contradiction $0 = 1$. Such a set is clearly the set of equations corresponding to an odd family of copies of G in K_n, whose sum is empty. Since $R(G, Z_2)$ is the smallest n for which there is no solution, the assertion of the lemma follows. $\Box$

Lemma 4.3 *If* $2|\binom{n}{2}$ *then* $R(K_n, Z_2) = n + 2$.

Proof Observe that $2|\binom{n}{2}$ iff either $n \equiv 0(mod\ 4)$ or $n \equiv 1(mod\ 4)$. In both cases the family of all $\binom{n+2}{2}$ copies of K_n in K_{n+2} is an odd family of copies of K_n with an empty sum. Hence, by the Parity Lemma, $R(K_n, Z_2) \leq n + 2$.

On the other hand, if $n \equiv 0(mod\ 4)$ the coloring of K_{n+1} which maps the edges of a Hamilton cycle C_{n+1} to 1 and every other edge to 0 contains no zero-sum copy of K_n. If $n \equiv 1(mod\ 4)$ then the coloring which maps the edges of a wheel of order $n + 1$ in K_{n+1} to 1 and every other edge to 0 contains no zero-sum copy of K_n. Thus $R(K_n, Z_2) > n + 1$ in both cases and the desired result follows. $\Box$

Lemma 4.4 *Let* G *be a bipartite graph with an odd number of vertices* n *and with an even number of edges. Then* $R(G, Z_2) = n$.

Proof We apply induction on n. For $n \leq 3$ the result is trivial. Assuming it holds for all odd values of $n' < n$, we prove it for n, $n \geq 5$. let G be a bipartite graph on an odd number of vertices $n \geq 5$, and suppose G has an even number of edges. Since G is bipartite it contains an independent set of at least three vertices and hence it has two non-adjacent vertices u and v such that $deg(u) + deg(v) \equiv 0(mod\ 2)$.

Put $H = G \setminus \{u, v\}$. By the induction hypothesis and by the Parity Lemma, there is an odd family $A = \{H_1, \ldots, H_m\}$ of copies of H in K_{n-2} whose sum is empty. Let V denote the set of $n-2$ vertices of the complete graph K_{n-2} containing the subgraphs H_i, and let x, y be two additional vertices. To complete the proof we construct an odd family of copies of G with an empty sum in the complete graph K on $V' = V \cup \{x, y\}$.

To do so, we first denote the vertices of V by $0, 1, \ldots, n-3$, and define, for each cyclic permutation π of V and for each member H_i of A, another copy of H, denoted by $\pi(H_i)$, obtained from H_i by mapping each of its vertices j to its image $\pi(j)$. Let A' be the family of all $(n-2)m$ copies of H in K_{n-2} obtained in this manner. Observe that this is an odd family of copies of H with an empty sum, and that each vertex of H appears as each vertex of V in exactly m members of this family.

We now "lift" each copy of H in A' to a copy of G in K by letting x play the role of u and y play the role of v. This gives a family B of $m(n-2)$ copies of G in K. Note that $m(n-2)$ is odd. It is not too difficult to check that the sum modulo 2 of the members of B is empty, if $deg(u) \equiv deg(v) \equiv 0 (mod\ 2)$. In this case, the desired result follows by the Parity Lemma. Otherwise, $deg(u) \equiv deg(v) \equiv 1 (mod\ 2)$, and this sum is the complete bipartite graph $K_{2,n-2}$ with classes of vertices $\{x, y\}$ and V. In this latter case, extend B to a family of $n|B|$ copies of G in K by replacing each member of B by its n cyclic shifts obtained by cyclically shifting the vertices in V', which we label $0, 1, \ldots n-1$, where $n-2, n-1$ are the labels of x and y respectively. A moment's reflection shows that the sum modulo 2 of the members of this extended family is precisely the sum modulo 2 of the n cyclic shifts of our complete bipartite graph $K_{2,n-2}$, which is empty, as can be easily checked. By the Parity Lemma this shows that $R(G, Z_2) = n$, completing the proof. $\Box$

The "lift and shift" technique used in the last proof and the Parity Lemma are the main tools in the proofs of the first and third part of Theorem 4.1. We start with the proof of the third part, which is simpler.

Proof of Theorem 4.1, part (iii) Let G be a bipartite graph with n vertices and with an even number of edges. If n is odd then, by Lemma 4.4, $R(G, Z_2) = n$, as needed. Therefore, we may assume that n is even. Consider, first, the case that G has a vertex v with an even degree. Define $H = G \setminus \{v\}$. Observe that H is bipartite, and has an odd number of vertices and an even number of edges. Therefore, by Lemma 4.4, $R(H, Z_2) = n - 1$. By the Parity Lemma there is an odd family $A = \{H_1, \ldots, H_m\}$ of copies of H in K_{n-1} with an empty sum. Let V be the vertex set of a complete graph K_{n-1} containing these subgraphs, and let A' be the family of all nm cyclic shifts of the members of A. This is again an odd family of copies of H with an empty sum. Let x be an additional vertex and lift each copy of H in A' to a copy of G in the complete graph on $V \cup \{x\}$ by letting x play the role of v. The resulting family is an odd family of copies of G and one can easily check that since the degree of v is even its sum modulo 2 is empty. Therefore, by the Parity Lemma $R(G, Z_2) = n$, as claimed in Theorem 4.1,(iii).

To complete the proof of this part of the theorem it remains to show that if n is even and all the degrees in G are odd then $R(G, Z_2) = n+1$. The Z_2-coloring of K_n in which all the $n-1$ edges incident with a fixed vertex are colored 1 and all the other edges are colored 0 contains no zero-sum copy of G, showing that $R(G, Z_2) \geq n + 1$. On the other hand, the graph G' obtained from G by adding an isolated vertex is bipartite and has an even number of edges and an odd number of vertices. Hence, by Lemma 4.4, $R(G, Z_2) \leq R(G', Z_2) = n + 1$, completing the proof. $\square$

Proof of Theorem 4.1, part (i) Observe first that by Lemma 4.3 the estimate, if true, is sharp.

To prove the upper bound for $R(G, Z_2)$ we apply induction on n. The result is trivial for $n \leq 3$. Assuming it holds for all $n' < n$, we prove it for n. Let G be a graph on n vertices with an even number of edges. We consider two possible cases.

Case 1: G contains two non-adjacent vertices u and v such that $deg(u) + deg(v) \equiv 0 (mod\ 2)$. Let s be the smallest odd integer which is at least $n-2$ (i.e., s is $n-2$ if n is odd and $n-1$ if n is even), and define $t = 3 + s$ $(\leq n + 2)$. Let U and W be two disjoint sets of vertices, where $|U| = 3$ and $|W| = s$ and let K denote the complete graph on $U \cup W$. Fix a copy G' of G in K in which u, v are mapped into two of the three vertices in U and the rest of the vertices of G are mapped into vertices in W. Let A be the family of all $3s$ copies of G in K obtained from G' by applying, in all possible ways, a cyclic shift to the vertices in U and a cyclic shift to these in W. Observe that A is an odd family, and let $H = \oplus \sum_{F \in A} F$ denote its sum modulo 2. Since the sum of the degrees of u and v in G is even and these two vertices are not adjacent, it follows that H has only edges whose two endpoints are in W. Moreover, H clearly has an even number of edges, since it is the sum modulo 2 of graphs each of which has an even number of edges and the set of all such graphs forms a linear subspace of the set of all graphs with respect to addition in Z_2.

Therefore, by the induction hypothesis and the Parity Lemma, there is an odd family E of copies of H in K_{n+1} (and hence of course in K_{n+2}) with an empty sum. Let B be the family of copies of G in K_{n+2} obtained by replacing each member H' of E by an odd family of copies of G in K_{n+2} whose sum modulo 2 is H'. (This is possible since by the definition of H it is obtained as the sum of such a family, and hence the same holds for each of its copies.) The resulting family B is an odd family of copies of G in K_{n+2} with an empty sum. Therefore, by the Parity Lemma, in this case $R(G, Z_2) \leq n + 2$, as needed.

Case 2: If $deg(u) + deg(v) \equiv 0(mod\ 2)$ in G, then u and v are adjacent. In this case the set of all vertices of odd degree in G, which we denote by O, forms a clique and the set of all vertices of even degree, which we denote by E forms a clique. Clearly, $|O| \equiv 0(mod\ 2)$ and hence every member of O has an even number of neighbors in E. It follows that the number of edges between O and E is even, and since the total number of edges of G is even this implies that

$$\binom{|O|}{2} + \binom{|E|}{2} \equiv 0(mod\ 2),$$

and hence, since $|O|$ is even, it follows that

$$\binom{n}{2} = \binom{|E| + |O|}{2} \equiv 0(mod\ 2).$$

Therefore, either $n \equiv 0(mod\ 4)$ or $n \equiv 1(mod\ 4)$. We consider each of these two possibilities separately.

Subcase 2a: $n \equiv 1(mod\ 4)$. Given a Z_2-coloring $f : E(K_{n+2}) \mapsto Z_2$ of the edges of the complete graph on $n + 2$ vertices, there exists, by Lema 4.3, a zero-sum copy K of K_n in it. Let G^c denote the complement of our graph G. This is a bipartite graph with an odd number of vertices and an even number of edges. Therefore, by Lemma 4.4 there is a zero-sum copy G' of G^c in K. The complement of G' in K is a copy G'' of G and

$$\sum_{e \in E(G'')} f(e) = \sum_{e \in E(K)} f(e) - \sum_{e \in E(G')} f(e) \equiv 0(mod\ 2).$$

Therefore, in this subcase, $R(G, Z_2) \leq n + 2$, as needed.

Subcase 2b: $n \equiv 0(mod\ 4)$. If G contains no vertex of even degree then $G = K_n$ and by Lemma 4.3 $R(G, Z_2) = n + 2$, as needed. Otherwise, let v be a vertex of even degree in G and put $H = G \setminus \{v\}$. Since H has an even number of edges the induction hypothesis implies that

$R(H, Z_2) \leq n+1$. Therefore, by the Parity Lemma there is an odd family D of copies of H in the complete graph K on a set V of $n+1$ vertices so that $\oplus \sum_{F \in D} F = 0$. Let D' be the family of $(n+1)|D|$ copies of H obtained by taking all the cyclic shifts of each member of D. This is again an odd family of copies of H with an empty sum, and each vertex of H appears as each vertex of V in precisely $|D|$ members of this family.

Let x be a new vertex and let E be the family of all copies of G in the complete graph on $V \cup \{x\}$ obtained by lifting each member of D' to a copy of G by letting x play the role of v. It is not too difficult to check that E is an odd family of copies of G in the complete graph on $n+2$ vertices, with an empty sum. Thus, by the Parity Lemma, $R(G, Z_2) \leq n+2$ in this subcase as well, completing the proof. □

Proof of Theorem 4.1, part (ii) Suppose $n \equiv 3(mod\ 4)$ is a prime-power, and let G be a graph on n vertices with an even number of edges. In order to prove that $R(G, Z_2) = n$ it suffices to construct in K_n an odd family of copies of G with an empty sum. This can be done by mapping a fixed copy of G according to the members of a 2-set transitive permutation group of odd order, as we show next.

Let F be the finite field with n elements and let $A \subset F \setminus \{0\}$ be a set of cardinality $(n-1)/2$ such that for every $x \in F \setminus \{0\}$ either $x \in A$ or $-x \in A$ (but not both). For each $a \in A$ and $b \in F$, let $\pi_{a,b}$ denote the permutation of the elements of F defined by $\pi_{a,b}(x) = ax + b$ for all $x \in F$, (where the addition and the multiplication are, of course, performed in F). Let P denote the set of all these $n(n-1)/2$ permutations. Observe that P is a group of permutations. We claim that it is 2-set transitive, i.e., for every two distinct $x_1, x_2 \in F$ and for every two distinct $y_1, y_2 \in F$ there is a unique member $\pi_{a,b} \in P$ mapping the set $\{x_1, x_2\}$ onto the set $\{y_1, y_2\}$. This is because if $(y_1 - y_2)/(x_1 - x_2) \in A$ then there is a unique $b \in F$ and $a = (y_1 - y_2)/(x_1 - x_2) \in A$ such that $\pi_{a,b}(x_i) = y_i$ for $i = 1, 2$, and there is no $\pi_{a,b} \in P$ mapping x_1 to y_2 and x_2 to y_2. On the other hand, if $(y_1 - y_2)/(x_1 - x_2) \notin A$ then its inverse $(y_2 - y_1)/(x_1 - x_2) \in A$ and there is a unique member of P mapping x_1 to y_2 and x_2 to y_1, and no member of P mapping x_i to y_i for $i = 1, 2$. This establishes the assertion of the claim.

Let K_n be a complete graph on n vertices labelled by the elements of F and fix a copy G' of G in K_n. For each permutaion $\pi_{a,b} \in P$ let $\pi_{a,b}(G')$ denote the copy of G in K_n obtained from G' by applying the permutation $\pi_{a,b}$ to each of its vertices. Let E denote the family of $n(n-1)/2$ copies of G obtained in this manner. Note that since $n \equiv 3(mod\ 4)$, E is an odd family. Moreover, the claim above implies that each edge of K_n appears as each edge of G exactly in one of the copies, and hence it belongs to precisely $|E(G)| \equiv 0(mod\ 2)$ members of E. Thus E is an odd family of copies of G with an empty sum, and hence, by the Parity Lemma, $R(G, Z_2) = n$, completing the proof of Theorem 4.1. □

Remarks

1). In the last proof we used 2-set transitive permutation groups of odd order. These groups are fully characterised in [19], and in particular they are known to exist if and only if the number of elements permuted is a prime power congruent to 3 modulo 4.

2). Let C_n denote the cycle on n vertices and let tK_2 denote the graph consisting of t isolated edges. By Theorem 4.1, part(iii),

$$R(C_{4m}, Z_2) = 4m < 4m + 1 = R(2mK_2, Z_2).$$

Note that $2mK_2$ is a subgraph of C_{4m} showing that unlike the usual Ramsey numbers, the zero-sum numbers do not share the monotonicity property.

3). The Parity Lemma holds, of course, not only for graphs but for other structures as well. As a simple example we mention the *zero-sum Van der Waerden numbers* over Z_2, denoted $W(n, Z_2)$, and defined as follows. For every even integer n, $W(n, Z_2)$ is the smallest integer t such that for every Z_2-coloring $f : \{1, 2, \dots, t\} \mapsto Z_2$ there is a zero-sum arithmetic progresssion of length n, i.e., an arithmetic progression $A \subset \{1, \dots t\}$ of length n satisfying $\sum_{a \in A} f(a) = 0$ (in Z_2).

Proposition 4.5 *For every even n, $W(n, Z_2) = 2n - 1$.*

Proof The Z_2-coloring of $1, \dots, 2n-2$ which maps $n-1$ to 1 and every other element to 0 shows that $W(n, Z_2) > 2n - 2$. On the other hand, the family of all $n + 1$ arithmetic progressions of length n in $\{1, 2, \dots, 2n - 1\}$ is an odd family of length-n arithmetic progressions with an empty sum modulo 2, showing that $W(n, Z_2) \leq 2n - 1$ and hence that $W(n, Z_2) = 2n - 1$. □

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