Degrees of Freedom Versus Dimension for Containment Orders

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ABSTRACT

Given a family of sets $\mathcal{S}$, where the sets in $\mathcal{S}$ admit k 'degrees of freedom', we prove that not all $(k+1)$-dimensional posets are containment posets of sets in $\mathcal{S}$. Our results depend on the following enumerative result of independent interest: Let $P(n,k)$ denote the number of partially ordered sets on n labeled elements of dimension k. We show that $\log P(n,k) \sim nk \log n$ where k is fixed and n is large.

KEY WORDS: partially ordered set, containment order, degrees of freedom, partial order dimension
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1. Introduction

Let $\mathcal{S}$ be a family of sets. We say that a partially ordered set P has an *$\mathcal{S}$-containment representation* provided there is a map $f : P \to \mathcal{S}$ such that $x < y$ iff $f(x) \subset f(y)$. In this case we say that P is an $\mathcal{S}$-order.

For example, *circle orders* are the containment orders of circles (actually disks) in the plane (see [8,9]). Similarly, angle orders are the containment orders of angles in the plane, where an angle includes its interior (see [3]). The containment orders of d-dimensional boxes are discussed in [4] where it was shown that this family of posets is exactly the set of $2d$-dimensional posets.

Note that circles admit three 'degrees of freedom': two center coordinates and a radius. An angle admits four degrees of freedom: the two coordinates of its vertex and the slopes of its rays. Further, it is known that not all 4-dimensional posets are circle orders [9] nor are all 5-dimensional posets angle orders [7]. These are confirming instances of the following intuitive notion:

If the sets in $\mathcal{S}$ admit k degrees of freedom, then not all $(k+1)$-dimensional posets are $\mathcal{S}$-orders.

Our main result is to prove (a precise version of) this intuitive principle.

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2. Enumeration of k-dimensional posets

Recall that the *dimension* of a partially ordered set P is the minimum number of linear extensions whose intersection is P. Alternatively, it is the smallest k so that the elements of P can be mapped to points in $\mathbf{R}^k$ so that $x < y$ iff each coordinate of x's point is less than or equal to the corresponding coordinate of y's point. (See [2,6].)

Denote by $P(n,k)$ the number of posets with element set $\{1,\ldots,n\}$ and dimension at most k.

By $f(n) \sim g(n)$ we mean that the limit $f(n)/g(n)$ as n tends to infinity is 1.

Theorem 1. *For k fixed and n large we have $\log P(n,k) \sim nk \log n$.*

Proof. First, one has $P(n,k) \le (n!)^k \le n^{nk}$ as there are $n!$ possible linear orders on $\{1,\ldots,n\}$ and we intersect k of them to form every possible k-dimensional poset. This gives $\log P(n,k) \le nk \log n$.

Second, let $m = n/\log n$. Let $\mathcal{B}_j$ denote the set of m boxes in $\mathbf{R}^k$ of the form

$$[0, m+1] \times \cdots \times [0, m+1] \times [i, i + \tfrac{1}{2}] \times [0, m+1] \times \cdots \times [0, m+1]$$

where there are k factors and $[i, i+\frac{1}{2}]$ is the j^{th} factor with $1 \le i \le m$. Each box has $2k$ corners and is uniquely determined by its two *extreme corners*: the one with the smallest values in each coordinate and the one with the largest. Also, notice that by choosing one box from each $\mathcal{B}_j$ one determines a cube (with side length $\frac{1}{2}$) which is the intersection of the k boxes. There are m^k such cubes.

We define a k-dimensional poset on $\{1,\ldots,n\}$ as follows: Let the first $2mk$ elements be assigned to the extreme corners of boxes in $\mathcal{B}_1,\ldots,\mathcal{B}_k$. Now to each element of $\{2mk+1,\ldots,n\}$, assign a point in one of the m^k small cubes. Note that each assignment of the remaining elements to cubes gives a different k-dimensional poset, as an element in the poset lies between the two elements corresponding to the extreme points of a box iff the point assigned to it lies in that box. Thus we have,

$$P(n,k) \ge \left(m^k\right)^{n-2km} = \left(\frac{n}{\log n}\right)^{nk - 2k^2 n/\log n} \qquad (*)$$

whence $\log P(n,k) \ge n(k - o(1)) \log n$.∎

Remark. Our construction uses the fact that km boxes in $\mathbf{R}^k$ can determine at least m^k cells. A more exact estimate for this problem appears in [5]. For our purposes here our estimate suffices.

3. Degrees of Freedom

We now make the intuitive notion of 'degrees of freedom' precise. Let $\mathcal{S}$ be a family of sets. We say the sets in $\mathcal{S}$ have k *degrees of freedom* provided:

1. Each set in $\mathcal{S}$ can be uniquely identified by a k-tuple of real numbers, i.e., there is an injection $f : \mathcal{S} \to \mathbf{R}^k$, and

2. There exists a finite list of polynomials $p_1, p_2, \ldots, p_t$ in $2k$ variables with the following property: If $S, T \in \mathcal{S}$ map to $(x_1, \ldots, x_k), (y_1, \ldots, y_k) \in \mathbf{R}^k$ respectively, then the containment $S \subset T$ can be determined based on the signs of the values $p_j(x_1, \ldots, x_k, y_1, \ldots, y_k)$ for $1 \le j \le t$.

For example, let us consider circles (disks) in the plane. Suppose we have two circles C_1 and C_2 with centers and radii given by x_i, y_i, r_i $(i = 1, 2)$. One checks that we have $C_1 \subset C_2$ iff both of the following hold:

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2 \le 0$$

$$r_1 - r_2 \le 0$$

Thus the family of circles in the plane admits three degrees of freedom. Similarly, the containment of one angle in another can be expressed in terms of a finite list of polynomial inequalities.

Our main result depends on the following result due (essentially) to Warren [10] (see also [1]): Let $p_1, \ldots, p_r$ be polynomials in ℓ variables. Let $s(p_1, \ldots, p_r)$ denote the number of sign patterns (pluses, minuses and zeroes) of the r polynomials have as their variables range over $\mathbf{R}^\ell$. That is,

$$s(p_1, \ldots, p_r) = \left| \left\{ \left(\operatorname{sgn}[p_1(\mathbf{x})], \ldots, \operatorname{sgn}[p_r(\mathbf{x})] \right) : \mathbf{x} \in \mathbf{R}^\ell \right\} \right|.$$

Theorem 2. *Let $p_1, \ldots, p_r$ be as above and suppose the degree of each polynomial is at most d. If $r \ge \ell$ then*

$$s(p_1, \ldots, p_r) \le \left[\frac{8edr}{\ell} \right]^\ell .$$

Proof. Warren [10] places an upper bound of $(4edr/\ell)^\ell$ on the number of sign patterns in which one counts only plus/minus sign patterns. One can extend this result to include zeroes by "doubling" each polynomial as follows:

Let S denote the set of all sign patterns (plus/minus/zero) for $p_1, \ldots, p_r$. Clearly S is finite; indeed $|S| \le 3^\ell$. Now let X denote a finite subset of $\mathbf{R}^\ell$ in which each sign pattern is represented exactly once. Put

$$\epsilon = \tfrac{1}{2} \min\left\{ |p_j(\mathbf{x})| : \mathbf{x} \in X,\ p_j(\mathbf{x}) \ne 0, \text{ and } 1 \le j \le r \right\}$$

and let $q_j^+ = p_j + \epsilon$ and $q_j^- = p_j - \epsilon$. Note that for each $\mathbf{x} \in X$, $q_j^+(\mathbf{x}) \ne 0$ and $q_j^-(\mathbf{x}) \ne 0$, and that the sign patterns of the q's attained at points in X are all distinct. The result now follows by applying Warren's Theorem to the q's.∎

We use this to achieve our main result:

Theorem 3. *Let $\mathcal{S}$ be a family of sets admitting k degrees of freedom. Then there exists a $(k+1)$-dimensional poset which is not an $\mathcal{S}$-containment order.*

Proof. Let $\mathcal{S}$ be a family of sets admitting k degrees of freedom. Let $\mathcal{S}_n$ denote the family of $\mathcal{S}$-orders on $\{1, \ldots, n\}$. For each n-tuple of sets in $\mathcal{S}$, $(S_1, \ldots, S_n)$, we have a (potentially) different poset depending on the sign pattern of $r = 2\binom{n}{2}t$ polynomials in $\ell = nk$ variables which have some maximum degree d (which is independent of n). Hence by Theorem 2:

$$|\mathcal{S}_n| \leq \left[\frac{16ed\binom{n}{2}t}{nk}\right]^{nk} = \big[O(1)n\big]^{nk}. \qquad (**)$$

Were every $(k+1)$-dimensional poset an $\mathcal{S}$-order we would have $\log P(n, k+1) \leq \log |\mathcal{S}_n|$, contradicting Theorem 1.∎

Remark. Our proof in Theorem 3 is nonconstructive. One can, however, give an explicit $(k+1)$-dimensional poset which is not an $\mathcal{S}$-containment order as follows. Choose n sufficiently large so that $P(n, k+1) > |\mathcal{S}_n|$; this can be done explicitly using inequalities $(*)$ and $(**)$. Let P be the partially ordered set which is the disjoint union of all $(k+1)$-dimensional posets on n elements. Necessarily, P is not an $\mathcal{S}$-containment order and $\dim P = k+1$.

Theorem 3 gives a common proof for the known results concerning circle and angle orders. We can also apply it to a prove a conjecture due to [9]:

Consider the family of p-gons in the plane. These are described by $2p$ real variables (the x, y coordinates of the corners) and the containment of one p-gon in another can be determined by a list of polynomial inequalities as follows:

First note that given four points $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$ and $D = (d_1, d_2)$ in general position, the line segment AB intersects the line segment CD iff

$$\det\left[\begin{pmatrix} 1 & a_1 & a_2 \\ 1 & b_1 & b_2 \\ 1 & c_1 & c_2 \end{pmatrix} \cdot \begin{pmatrix} 1 & a_1 & a_2 \\ 1 & b_1 & b_2 \\ 1 & d_1 & d_2 \end{pmatrix}\right] < 0$$

and

$$\det\left[\begin{pmatrix} 1 & c_1 & c_2 \\ 1 & d_1 & d_2 \\ 1 & a_1 & a_2 \end{pmatrix} \cdot \begin{pmatrix} 1 & c_1 & c_2 \\ 1 & d_1 & d_2 \\ 1 & b_1 & b_2 \end{pmatrix}\right] < 0.$$

Thus the intersection of two line segments can be determined by examining the signs of two quadratic polynomials.

Now suppose that we are given two p-gons V and W where the vertices of V (in order) are $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ where $\mathbf{v}_i = (v_{i1}, v_{i2})$. Likewise, the vertices of W are $\mathbf{w}_i = (w_{i1}, w_{i2})$ for $i = 1, \ldots, p$. Without loss of generality (and for ease of exposition) we may assume that the $2p$ points are in general position and that no two have the same y-coordinate. Furthermore, by rescaling, we may assume that the two polygons are contained in the rectangle $[-1, 1]^2$.

Now, to test if V is contained in W it suffices to show that the boundaries of the two p-gons do not intersect and that one of V's vertices is contained in the interior of W. To show that the boundaries do not intersect, one checks that for all $i, j \in \{1, \ldots, p\}$ the line segments $\mathbf{v}_i\mathbf{v}_{i+1}$ and $\mathbf{w}_j\mathbf{w}_{j+1}$ (subscript addition modulo p) do not intersect. This can be done by examining the signs of $2p^2$ polynomials. Next we test if $\mathbf{v}_1$ lies in the interior of W by counting the number of times a horizontal ray emerging from $\mathbf{v}_1$ intersects the boundary of W; this count is odd if and only if $\mathbf{v}_1$ is in the interior of W. Since all vertices are contained in $[-1, 1]^2$, we check if the line segment $(v_{11}, v_{12})(2, v_{12})$ intersects $\mathbf{w}_j\mathbf{w}_{j+1}$ for $j = 1, \ldots, p$. Hence by computing $2p$ further polynomials, we determine if $\mathbf{v}_1$ is contained in the interior of W.

The authors of [9] proposed the problem: Is there a $(2p + 1)$-dimensional order which is *not* a p-gon order? The existence of such an order is now readily verified using our Theorem 3.

Finally, it is known [3] that all 4-dimensional posets are angle orders and it is conjectured that all 3-dimensional posets are circle orders (see [8,9]). In [9] it is shown that all $2p$-dimensional posets are p-gon orders. One is tempted to conjecture: *If $\mathcal{S}$ admits k degrees of freedom (and no fewer) then every k-dimensional poset is an $\mathcal{S}$-containment order.* This, however, is false as the following simple example shows. Let $\mathcal{S}$ be the family of all horizontal rays in the plane which point in the positive x-direction. The $\mathcal{S}$-containment posets are exactly the disjoint unions of chains. One now checks that $\mathcal{S}$ admits two degrees of freedom, but $2^{\{1,2\}}$, the subsets of a 2-set poset, is not an $\mathcal{S}$-poset.

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