The spanning tree spectrum: improved bounds and simple proofs.

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Abstract

The number of spanning trees of a graph G, denoted $\tau(G)$, is a well studied graph parameter with numerous connections to other areas of mathematics. In a recent remarkable paper, answering a question of Sedláček from 1969, Chan, Kontorovich and Pak showed that $\tau(G)$ takes at least 1.1103^n different values across simple (and planar) n-vertex graphs G, for large enough n. We give a very short, purely combinatorial proof that at least 1.49^n values are attained. We also prove that exponential growth can be achieved with regular graphs, determining the growth rate in another problem first raised by Sedláček in the late 1960's. We further show that the following modular dual version of the result holds. For any integer N and any $u < N$ there exists a planar graph on $O(\log N)$ vertices whose number of spanning trees is u modulo N.

1 Introduction

Given a graph G, we denote by $\tau(G)$ the number of spanning trees of G. This simple quantity has numerous interpretations, with perhaps the simplest and most classical being its expression as the determinant of a certain type of symmetric matrices via Kirchoff's matrix tree theorem [13] from 1847. The notion has a long history dating back to 1860 and the first proof of Cayley's formula [4]. It can also be viewed as capturing a certain notion of "complexity" of G. Besides being extensively studied in combinatorics, it has many, often surprising, connections to other areas of mathematics and beyond, including Commutative algebra, Probability theory, Theory of Lie groups, Combinatorial Optimization and Electrical Networks.

In the present short paper we are interested in one of the most basic questions that can be asked about the function τ, namely how large is its range, when evaluated on, say, all n vertex graphs? This question was first raised by Sedláček in the 1960's [19, 22, 23] and has since been reiterated many times, the earliest of which being the book [17] of Moon from 1970. In fact, he raised this question for three natural families of graphs: $\mathcal{G}_n^{\text{all}}$, $\mathcal{G}_n^{\text{planar}}$, $\mathcal{G}_n^{k\text{-regular}}$ – the families of all, planar, and k-regular graphs on n vertices, respectively. With this in mind, given a family of simple[1] graphs $\mathcal{G}$, let us define its *tree spectrum* $T(\mathcal{G}) := \{\tau(G) \mid G \in \mathcal{G}\}$. It is immediate that $|T(\mathcal{G}_n^{\text{all}})| \geq |T(\mathcal{G}_n^{\text{planar}})|$, and somewhat remarkably, historically, the essentially best known lower bounds on $|T(\mathcal{G}_n^{\text{all}})|$ came from lower bounds on $|T(\mathcal{G}_n^{\text{planar}})|$, despite a seemingly vast amount of additional freedom afforded to graphs in $\mathcal{G}_n^{\text{all}}$ compared to $\mathcal{G}_n^{\text{planar}}$. In particular, for any $G \in \mathcal{G}_n^{\text{planar}}$, by exploiting degeneracy, it is easy to see that $\tau(G) \leq 6^n$, and the current best upper bound is $\tau(G) \leq 5.2852^n$ [8] (compared to n^{n-2} for $G \in \mathcal{G}_n^{\text{all}}$). This also gives the essentially best known upper bound on $|T(\mathcal{G}_n^{\text{planar}})|$.

The first lower bound on $|T(\mathcal{G}_n^{\text{planar}})|$ was given already in one of the original papers by Sedláček in 1969 [22], who proved that $|T(\mathcal{G}_n^{\text{planar}})| = \Omega(n^2)$. In [2], Azarija obtained a major improvement, by showing that $|T(\mathcal{G}_n^{\text{planar}})| \geq 2^{\Omega(\sqrt{n/\log n})}$. Subsequently, a result of Stong [25], who was interested in a certain dual question which we will discuss later, implies that $|T(\mathcal{G}_n^{\text{planar}})| \geq 2^{\Omega(n^{2/3})}$. In a remarkable recent paper, Chan, Kontorovich and Pak [9] showed that $|T(\mathcal{G}_n^{\text{planar}})| \geq 1.1103^n$, for large n. Their argument uses combinatorial ideas combined with ideas concerning the theory of continued fractions.

We give a purely combinatorial, short and simple proof with a significantly stronger exponential bound.

Theorem 1.1. *$|T(\mathcal{G}_n^{all})| \geq |T(\mathcal{G}_n^{planar})| \geq 1.49^n$, for large enough n.*

In fact, we give an even simpler, one page proof of a slightly weaker bound $|T(\mathcal{G}_n^{\text{planar}})| \geq 2^{n/2-1}$, which holds for all n.

Turning to regular graphs, we note that the natural question of determining $|\mathcal{G}_n^{k\text{-regular}}|$ has also been originally raised by Sedláček [22] in 1969 and has since been reiterated several times over the years by various authors [2, 9, 17, 20, 21]. In the original paper, Sedláček proved for $k = 3$ that $|\mathcal{G}_n^{k\text{-regular}}|$ grows at least linearly in n (provided n is even[2]). In a subsequent paper [20], he extended this result to any fixed k. We improve these linear bounds to exponential ones.

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[1]In this paper, all graphs we consider are assumed to be simple, unless otherwise specified.

[2]Note that if k is odd there are no k-regular graphs on n vertices unless n is even.

Theorem 1.2. *For any fixed integer $k \geq 3$ there are at least $2^{\Omega(n)}$ different values of $\tau(G)$ among k-regular connected graphs G on n vertices, provided that kn is even.*

Note that the exponential growth rate can be easily seen to be tight since the number of spanning trees in a k-regular connected graph for a fixed k is always exponential (see e.g. [1, 16]).

We note that a natural approach to understanding the growth rate of the tree spectrum, already introduced in 1970 by Sedláček [19], is to consider a "dual" problem. Namely, given $t > 2$, what is the minimum number of vertices in a planar graph with exactly t spanning trees? Let us denote the answer by $\alpha(t)$. By considering a cycle on t vertices we obtain a trivial upper bound of $\alpha(t) \leq t$, observed by Sedláček [19] already in 1970. Following a number of subsequent improvements [2, 3, 11, 18] the current state of the art bound of $\alpha(t) \leq O(\log^{3/2}(n)/\log\log n)$ is due to a recent result of Stong [25] (a stronger bound in an appropriate sense is known for multigraphs, see [10]). A natural conjecture which remains open, raised explicitly in [9], is that $\alpha(t) \leq O(\log t)$. In [9] the authors show, relying on the Bourgain-Kontorovich [6] machinery developed towards proving Zaremba's conjecture, that this conjecture is true for a positive proportion of integers smaller than any large n. We prove a natural modular analogue of the full conjecture.

Theorem 1.3. *For any $u \in \mathbb{Z}_N$ there exists a planar graph G on $O(\log N)$ vertices with $\tau(G) \equiv u \bmod N$.*

In fact, our result is in several ways even stronger, see Theorem 3.1 for more details.

While we do not use the connection to continued fractions which was used to prove the exponential growth of $|T(\mathcal{G}_n^{\text{planar}})|$ in [9], we do obtain several interesting (simple) consequences in the opposite direction. Here, given a sequence of integers $a_1, \ldots, a_\ell \geq 1$, where $\ell \geq 0$, the corresponding *continued fraction* is defined as follows:

$$[a_1, \ldots, a_\ell] := \cfrac{1}{a_1 + \cfrac{1}{\ddots + \frac{1}{a_\ell}}}.$$

The classical Zaremba's conjecture [26] asserts that there exists $C > 0$ such that for any u there is a coprime $t < u$ such that $\frac{t}{u} = [a_1, \ldots, a_\ell]$ with $a_1, \ldots, a_\ell \leq C$. It is known that the conjecture is not true for $C = 4$ but Hensley [12] conjectured in 1996 that $C = 2$ is enough for large enough u. While the conjecture is still open in general, in a remarkable paper Bourgain and Kontorovich [6] showed that it holds for $(1 - o(1))n$ of the values of u smaller than n for any large enough n. We give a very simple proof of the following weaker result (ignoring the $a_i \in \{1, 2\}$ part).

Theorem 1.4. *For any N, there are at least $N^{1/4}/2 - 1$ values of $u < N$ such that there exists a coprime $t < u$, an $\ell \leq \frac{1}{2}\log N$ and $a_1, \ldots, a_\ell \in \{1, 2\}$, such that $\frac{t}{u} = [a_1, 1, \ldots, a_\ell, 1]$.*

Theorem 1.3 (in reality its strengthening Theorem 3.1) has as an essentially immediate consequence the following "modular" version of Zaremba's Conjecture. We note that a similar result is described in [14].

Theorem 1.5. *Let N be large enough. For any $u < N$ there exist $\ell \leq O(\log N)$ and $a_1, \ldots, a_\ell \in \{1, 2\}$ such that* [3] *$u \equiv [a_1, 1, \ldots, a_\ell, 1] \bmod N$.*

Notation. All our logarithms are in base two and all our graphs are simple unless otherwise specified.

2 Counting spanning trees

Given a graph G and its edge e we denote by $G \setminus e$ the (multi)-graph obtained by contracting e and by $G - e$ the graph obtained by deleting e. Note that $\tau(G - e)$ counts the number of spanning trees of G which do not use e, and $\tau(G \setminus e)$ counts the number of spanning trees using e. In particular, this establishes the classical recursive formula $\tau(G) = \tau(G \setminus e) + \tau(G - e)$ for computing the number of spanning trees.

We say that a vector $\begin{bmatrix} t \\ u \end{bmatrix}$ is *n-planar-feasible* if there exists a planar graph G on up to n vertices and an edge $e \in E(G)$ such that $t = \tau(G \setminus e)$ and $u = \tau(G - e)$. We call the edge e the *witness* for the n-planar-feasibility of $\begin{bmatrix} t \\ u \end{bmatrix}$. Note that if a

[3] We note here that one should first compute $[a_1, 1, \ldots, a_\ell, 1]$ over Q and only then evaluate it modulo N

vector is n-planar-feasible then it can also be realized by a planar graph with *exactly* n vertices, by attaching leaves (which does not change the number of spanning trees).

The next simple lemma shows that in order to get good lower bounds on $|\mathcal{T}(n)|$ it suffices to show that there are many distinct planar feasible vectors.

Lemma 2.1. *If there are at least N distinct n-planar-feasible vectors, then $|\mathcal{T}(n)| \geq \sqrt{N}$.*

Proof. Let $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \dots, \begin{bmatrix} x_N \\ y_N \end{bmatrix}$ be distinct n-planar-feasible vectors. Note that either there is some y such that $y_i = y$ for at least $\sqrt{N}$ of these vectors, or $|\{y_1, \dots, y_N\}| \geq \sqrt{N}$. In the latter case we are done since each y_i denotes the number of spanning trees of some $G - e$ for some n-vertex planar graph G and its edge e, which is also an n-vertex planar graph. In the former case, note that $|\{x_1 + y_1, \dots, x_N + y_N\}| \geq \sqrt{N}$ since all the (at least $\sqrt{N}$) vectors agreeing in the second coordinate must have a different sum of their coordinates (since they are different vectors). Since $x_i + y_i$ is the number of spanning trees of some n-vertex planar graph, we again get at least $\sqrt{N}$ distinct values taken by the spanning tree function. $\square$

The following lemmas establish a way of recursively generating planar-feasible vectors. Similar considerations were also used in [9].

Lemma 2.2. *If $\begin{bmatrix} t \\ u \end{bmatrix}$ is n-planar-feasible then $\begin{bmatrix} t+u \\ u \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} t \\ u \end{bmatrix}$ is $(n+1)$-planar-feasible.*

Proof. Take the witness edge e and subdivide it with a single vertex to create a new graph G'. Let f, f' be the two newly created edges. Then, $\tau(G' \setminus f)$ counts the number of spanning trees of G' using f, which equals the sum of the number of those using both f and f', plus the number of those using f but not f'. The former equals $\tau(G \setminus e) = t$, namely the number of spanning trees of G using e, and the latter equals $\tau(G - e) = u$, namely the number of spanning trees of G not using e. Hence, $\tau(G' \setminus f) = t + u$. On the other hand $\tau(G' - f)$ counts the number of spanning trees of G' not using f (forcing us to use f'), which equals $\tau(G - e) = u$. Thus, f is the witness for $\begin{bmatrix} t+u \\ u \end{bmatrix}$ being $(n+1)$-planar-feasible, as desired. $\square$

Lemma 2.3. *If $\begin{bmatrix} t \\ u \end{bmatrix}$ is n-planar-feasible then $\begin{bmatrix} 2t \\ t+2u \end{bmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{bmatrix} t \\ u \end{bmatrix}$ is $(n+1)$-planar-feasible.*

Proof. Take the witness edge e, add an extra vertex v and join it to both endpoints of e to create a new graph G'. We claim that e is the witness for $(n+1)$-planar-feasibility of $\begin{bmatrix} 2t \\ t+2u \end{bmatrix}$. Indeed, $\tau(G' \setminus e) = 2\tau(G \setminus e) = 2t$ since we have to pick one of the two new edges to connect v as well as connect $G \setminus e$. For $\tau(G' - e)$, we can either use exactly one of the two new edges giving a contribution of $2\tau(G - e) = 2u$, or both new edges giving a contribution of $\tau(G \setminus e) = t$. Hence, $\tau(G' - e) = t + 2u$, as desired. $\square$

Lemma 2.4. *If $\begin{bmatrix} t \\ u \end{bmatrix}$ is n-planar-feasible then $\begin{bmatrix} 2t+u \\ t+u \end{bmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{bmatrix} t \\ u \end{bmatrix}$ is $(n+1)$-planar-feasible.*

Proof. Take the witness edge e, add an extra vertex v and join it to both endpoints of e to create a new graph G'. Denote the two new edges by f, f'. We claim that f is the witness for the $(n+1)$-planar-feasibility of $\begin{bmatrix} 2t \\ t+2u \end{bmatrix}$. Indeed, $\tau(G' \setminus f)$ is equal to the number of spanning trees of G with the edge e doubled (i.e., replaced with two parallel edges), which equals to $2\tau(G \setminus e) + \tau(G - e) = 2t + u$. Furthermore, $\tau(G' - f) = \tau(G) = t + u$, because every spanning tree of G' not using f is obtained from a spanning tree of G by adding f'. This proves the lemma. $\square$

Let us denote by

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad B := \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \qquad C := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad D := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Observe the relations $B^2 = 4D$ and $C = AD = \frac{1}{4}AB^2$, which we will use often.

We start with a weaker result which only makes use of the operations corresponding to the matrices A and C.

Theorem 2.5. *There are at least 2^n distinct $(n+2)$-planar-feasible vectors.*

Proof. Note that $\begin{bmatrix}1\\0\end{bmatrix}$ is 2-planar-feasible. This implies that for any choice of t and $a_1, \dots a_t \geq 0$, by repeated use of Lemmas 2.2 and 2.4, the vector

$$\vec{u}(a_1, \dots, a_t) := A^{a_1} C A^{a_2} C \cdots A^{a_t} C \begin{bmatrix}1\\0\end{bmatrix} = A^{a_1+1} D A^{a_2+1} D \cdots A^{a_t+1} D \begin{bmatrix}1\\0\end{bmatrix} \tag{1}$$

is $(n+2)$-planar-feasible for $n = a_1 + a_2 + \ldots + a_t + t$.

We further claim that all vectors $\vec{u} = \vec{u}(a_1, \dots, a_t)$ are distinct. To see this, we need the following simple fact: for every vector $v = \begin{bmatrix}x\\y\end{bmatrix}$ of positive numbers, $Av = \begin{bmatrix}x+y\\y\end{bmatrix}$ has its first coordinate larger than the second, whereas $Dv = \begin{bmatrix}x\\x+y\end{bmatrix}$ has its second coordinate larger than the first. Now, given $\vec{u}$ as in (1), we can determine a_1 by multiplying the vector by A^{-1} so long as the first coordinate is larger than the second. The number of times we did this is equal to a_1+1, because $DA^{a_2+1} D \cdots A^{a_t+1} D \begin{bmatrix}1\\0\end{bmatrix}$ has its second coordinate larger than the first. We then multiply by D^{-1}. Repeating allows us to decode $a_2, \dots, a_t$ in the same way, where the argument stops once the current vector equals $\begin{bmatrix}1\\0\end{bmatrix}$ and the number of iterations gives t.

Hence, for a given n, the number of vectors we can obtain is the number of solutions to $a_1 + a_2 + \ldots + a_t + t \leq n$ in non-negative integers (with variable t). By the usual "balls and bins" argument, this equals $\sum_{t=0}^{n} \binom{n}{t} = 2^n$, as desired. ☐

We note that the construction we used above is equal to the one used in [9]. We also note that the uniqueness of vectors $\vec{u}(a_1, \dots, a_t)$ can also be concluded from the classical fact that the matrices A, D generate a free semigroup [7] (although they do not generate a free group). The following corollary is an immediate consequence of Theorem 2.5 and Lemma 2.1.

Corollary 2.6. *For any positive integer n we have $|T(\mathcal{G}_n^{planar})| \geq 2^{n/2-1}$.*

We next prove a stronger bound on the number of planar-feasible vectors using the operations behind all three matrices A, B, and C. We note that we did not choose the optimal value of the parameters in the following result for the sake of simplicity, since even after optimization the base of the exponent here is extremely unlikely to be best possible and only does slightly better than Theorem 2.5. The main purpose of the stronger theorem here is to showcase that one can substantially improve Theorem 2.5 by using additional operations and to showcase a route towards further improvement.

Theorem 2.7. *For large enough n, there are at least 2.23^n distinct $(n+2)$-planar-feasible vectors.*

Proof. We will first assume that $10 \mid n$ and prove a slightly stronger bound, which combined with monotonicity implies the claimed one for any n. We set $t = 2n/5$.

Note that $\begin{bmatrix}4\\4\end{bmatrix} = B^2 \begin{bmatrix}1\\0\end{bmatrix}$ is 4-planar-feasible. This implies that for any choice of t and any choice of $a_1, \dots a_t \geq 1$ and $b_1, \dots b_t \geq 2$, by repeated use of Lemmas 2.2 to 2.4, the vector

$$\vec{v}(a_1, \dots, a_t; b_1, \dots, b_{t-1}) := A^{a_1} B^{b_1} A^{a_2} B^{b_2} \cdots A^{a_t} B^{b_t} \begin{bmatrix}1\\0\end{bmatrix} = 4^t \cdot A^{a_1-1} C B^{b_1-2} A^{a_2-1} C B^{b_2-2} \cdots A^{a_t-1} C B^{b_t-2} \begin{bmatrix}1\\0\end{bmatrix}$$

is $(n+2)$-planar-feasible for $n = a_1 + a_2 + \ldots + a_t + b_1 + b_2 + \ldots + b_t - 2t$. We note here that the purpose of replacing AB^2 with $4C$ is that multiplying by C only "costs" us one vertex, whereas multiplying by B twice and then by A would "cost" three (we would produce different witness graphs for the same vector). Thus, the identity $4C = AB^2$ allows us to lower the number of vertices we use.

We will only restrict attention to vectors $\vec{v}(a_1, \dots, a_t; b_1, \dots, b_t)$ with $2 \mid b_i$ for all $i \leq t$. We claim that for all such choices of a_i's and b_j's the $\vec{v}$'s are distinct. Indeed, given $\vec{v}$, to determine a_1 one only needs to multiply the vector by A^{-1} so long as the first coordinate is larger than the second. The number of times we did this is equal to a_1 since $B^2 = 4D$ times any vector has second coordinate at least as large as the first, while A times any vector with positive coordinates has the first coordinate larger than the second. Similarly to determine b_1 we multiply by B^{-2} so long as the second coordinate is larger than the first.

Hence, the number of distinct vectors we can obtain is at least the number of solutions to $a'_1+a'_2+\ldots+a'_t+b'_1+\ldots+b'_t = n-t$ in non-negative integers where all b'_i are even. We will only count sequences where $a'_1+a'_2+\ldots+a'_t = 2n/5$ and $b'_1+\ldots+b'_t = n/5$. By the usual "balls and bins" argument there are $\binom{4n/5-1}{2n/5}$ such choices for $a'_1,\ldots,a'_t$ and $\binom{n/2-1}{2n/5}$ such choices for $b'_1,\ldots,b'_t$ (since they are required to be even) giving us in total at least

$$\binom{4n/5-1}{2n/5}\cdot\binom{n/2-1}{2n/5} \geq 2^{(0.8+0.36-o(1))n} \geq 2.234^n,$$

where the first inequality uses the entropy bound $\binom{n}{\alpha n} \geq 2^{(H(\alpha)-o(1))n}$, and the last one holds for large enough n. $\square$

We note that the construction we used above is different compared to the one used in [9] and Theorem 2.5, both of which would correspond to only using $b_1 = b_2 = \ldots = b_t = 2$. Note also that we made no particular effort to optimize the base here since our goal was mainly to show that one can improve upon the construction used in Theorem 2.5. We suspect that most of the vectors $\vec{u}(a_1,\ldots,a_t,b_1,\ldots,b_t)$ with $a_i \geq 1, b_j \geq 2$ are unique, which would lead to a slightly better bound of about 2.61^n. This is essentially the limit of what can be achieved by using matrices generated by A and B. The issue here is that A and B do not generate a free semigroup [7]. Similarly, as before, Lemma 2.1 combined with Theorem 2.7 gives Theorem 1.1.

As written, our constructions do not produce regular graph witnesses. The following argument shows how we can regularize them to obtain Theorem 1.2.

Proof of Theorem 1.2. We first show that we can attain exponentially many values among planar graphs with maximum degree at most 3. To see this, suppose that we have a graph G with this property and note that if e is an edge of G with both endpoints having degree at most two, then by adding $i \geq 2$ vertices in such a way that together with e they make a C_{i+2}, we preserve the maximum degree property. Note also that since $i \geq 2$, one of the new edges will have both endpoints having degree equal to two, denote it by f and the new graph by G'. Then, we have $\begin{bmatrix}\tau(G'\setminus f)\\ \tau(G'-f)\end{bmatrix} = A^{i-1}C\begin{bmatrix}\tau(G\setminus e)\\ \tau(G-e)\end{bmatrix}$. This can be easily seen directly or by noticing that the construction used in Lemma 2.4 places a triangle on top of e and marks one of the new edges by f. We can then subdivide f for $i-1$ times using Lemma 2.2, giving us the stated product. Starting with an edge and repeatedly applying this operation, this implies that for every $t \geq 1$ and $a_1,\ldots,a_t \geq 1$, the vector

$$\vec{w}(a_1,\ldots,a_t) := A^{a_1}CA^{a_2}C\cdots A^{a_t}C\begin{bmatrix}1\\0\end{bmatrix}$$

is $(n+2)$-planar-feasible for $n = a_1+\ldots+a_t+t$ with a graph whose degrees are all equal to 2 or 3. As argued in the proof of Theorem 2.5, all these vectors are distinct, giving at least $\sum_{t=1}^{n/2-1}\binom{n-t-1}{t-1} \geq 2^{(2/3-o(1))n}$ different such vectors.

By the same argument as in the proof of Lemma 2.1, this implies there are at least $2^{(1/3-o(1))n/6}$ planar graphs G with distinct $\tau(G)$, each with at most $n/6$ vertices, and having all degrees equal to 2 or 3 (we can always remove vertices of degree one without changing the number of spanning trees). By a pigeonhole argument, we conclude that there is some $0 \leq i \leq n/6$ such that there are at least $2^{(1/3-o(1))n/6}/(n/6+1) \geq 2^{(1/3-o(1))n/6}$ such graphs with exactly i vertices of degree 2. For any such G we now append to each of its i vertices v of degree 2 a fixed connected graph H which has exactly one vertex of degree two which we connect to v, and with the remaining vertices of H having degree 3. Namely, we take H to be K_4 with one edge subdivided once. Note that the number of vertices in our new graph is at most n, that it remains planar, and that it is cubic. Moreover, note that the number of its spanning trees is equal to $\tau(G)\cdot(\tau(H))^i$. Since we fixed i, the term $(\tau(H))^i$ is independent of G from our family, and we obtain at least $2^{(1/18-o(1))n}$ different values taken by $\tau(G)$ on planar cubic graphs on at most n vertices. This completes the proof in the cubic case.

If $k > 3$, we can in our construction first add a triangle on top of our witness edge, while keeping the same witness edge, and then add another one where we move it to a new edge and then subdivide. Our first operation is captured by multiplying by matrix B (by Lemma 2.3) so each iteration corresponds to multiplication by $A^{i-1}CB$. By the same argument we still have $2^{\Omega(n/k)}$ of such graphs on up to $\Omega(n/k)$ vertices with the key difference being that the graphs we construct all have degrees 2 or 4. Now our pendant graph H will be a graph with exactly $k-2$ vertices of degree $k-1$ and the remaining vertices all of degree k, so that when appending to a degree two vertex, we simply join it to the $k-2$ vertices of degree $k-1$. For degree 4 vertices, we join them to $k-4$ vertices of their copy of H and simply join the remaining two vertices of degree $k-1$ in H by an edge. To see that such an H exists we may take K_{k+1}, remove a path of length $k-1$ and add a new vertex joined to all vertices on the removed path.

This shows that for every $k \geq 3$, we can find $2^{\Omega(n/k)}$ k-regular graphs on up to n vertices. In order to get graphs on exactly n vertex, we simply pigeonhole again, pay a factor of up to n/k to make sure we have the same number of vertices m in remaining graphs, and then modify in each of them one of the copies of H we use to start from an arbitrary k-regular graph on $k+1+n-m$ vertices, remove a path of length $k-1$ and add a new vertex joined to all the vertices of the path. This only multiplies the tree counts of every graph in our family by the same amount, leaving them all distinct. $\square$

3 Hitting any value modulo N

In this section we prove our modular version of the dual question asking how large a (planar) graph G do we need to take in order to attain a given value of $\tau(G)$. The following is our stronger variant of Theorem 1.3.

Theorem 3.1. *Let N be sufficiently large. For any coprime a, b, there exists $t \leq O(\log N)$ and $i_1, \dots, i_t \in \{1, 2\}$ such that*

$$\begin{bmatrix} a \\ b \end{bmatrix} \equiv A^{i_1} D A^{i_2} D \dots A^{i_t} D \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bmod N.$$

In particular, there exists a $O(\log N)$-planar-feasible vector $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $x \equiv a \bmod N$ and $y \equiv b \bmod N$.

Before turning to the proof, we will need a few definitions and a well known fact about the expansion in Cayley graphs of $\mathrm{SL}_2(\mathbb{Z}_N)$. We say that an n-vertex graph is a *c-expander* if any subset U on up to $n/2$ vertices has its external neighborhood $N(U)$ of size at least $c|U|$. Given a group G and a subset of its elements S, the Cayley digraph of G generated by S, denoted $Cay(G, S)$, is the graph whose vertex-set is G and where an edge from a to b exists if and only if $a^{-1}b \in S$. In case S is symmetric (i.e., $a \in S \implies a^{-1} \in S$), we have an edge from a to b if and only if we have an edge from b to a, so it is customary to drop the directions and refer to this as the Cayley graph of G generated by S.

Selberg's Theorem [24] implies that $\mathrm{Cay}((\mathrm{SL}_2(\mathbb{Z}_N), S \cup S^{-1})$ is an expander provided $S \cup S^{-1}$ generates $\mathrm{SL}_2(\mathbb{Z})$ (in fact whenever it generates a subgroup of finite index). It is well known that for $S = \{A, D\}$ we have that $S \cup S^{-1}$ generates $\mathrm{SL}_2(\mathbb{Z})$. If we set $X = AD, Y = A^2 D$ we have $A = YX^{-1}$ and $D = XY^{-1}X$ so X, Y, X^{-1}, Y^{-1} generate $\mathrm{SL}_2(\mathbb{Z})$. This implies the following proposition, see also [5, 15] for more details on the expansion of Cayley graphs in $\mathrm{SL}_2(\mathbb{Z}_N)$.

Proposition 3.2. *There exists $c > 0$ so that for any large enough N the Cayley graph $\mathrm{Cay}(\mathrm{SL}_2(\mathbb{Z}_N), S \cup S^{-1})$, with $S = \{AD, A^2 D\}$, is a c-expander.*

Another ingredient is an observation that having expansion in the Cayley graph $\mathrm{Cay}(\mathrm{SL}_2(\mathbb{Z}_N), S \cup S^{-1})$ implies expansion in the Cayley digraph $\mathrm{Cay}(\mathrm{SL}_2(\mathbb{Z}_N), S)$. Here, we say that an n-vertex digraph is a *c-out-expander* if any subset U on up to $n/2$ vertices has its external out-neighborhood $N^+(U)$ of size at least $c|U|$.

Proposition 3.3. *Let G be a finite group and $S \subset G \setminus e$ (where e is the identity element). If $\mathrm{Cay}(G, S \cup S^{-1})$ is a c-expander for some $c > 0$, then $\mathrm{Cay}(G, S)$ is a $\frac{c}{2|S|}$-out-expander.*

Proof. Let $n = |G|$ and let $U \subseteq G$ have size $u = |U| \leq n/2$. By the c-expansion of $H = \mathrm{Cay}(G, S \cup S^{-1})$, H has at least $c|U|$ edges between U and $G \setminus U$. Since $D = \mathrm{Cay}(G, S)$ is Eulerian, we have

$$0 = \sum_{v \in U} (d_D^+(v) - d_D^-(v)) = \sum_{v \in U} (d_D^+(v, G \setminus U) - d_D^-(v, G \setminus U)) + \cancelto{0}{\sum_{v \in U} (d_{D[U]}^+(v) - d_{D[U]}^-(v))} \implies$$

$$\sum_{v \in U} d_D^+(v, G \setminus U) = \sum_{v \in U} d_D^-(v, G \setminus U) = \frac{1}{2} \sum_{v \in U} d_H(v, G \setminus U) \geq \frac{c}{2} \cdot |U|.$$

Since the maximum indegree in D is $|S|$, we obtain that there are at least $\frac{c}{2|S|} \cdot |U|$ vertices in $N^+(U)$ in D, as desired. $\square$

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $S = \{AD, A^2D\}$. By Proposition 3.2 $\mathrm{Cay}(\mathrm{SL}_2(\mathbb{Z}_N), S \cup S^{-1})$ is a c-expander for some $c > 0$. By Proposition 3.3, both $\mathrm{Cay}(\mathrm{SL}_2(\mathbb{Z}_N), S)$ and $\mathrm{Cay}(\mathrm{SL}_2(\mathbb{Z}_N), S^{-1})$ are $\frac{c}{4}$-out-expanders. Starting from an arbitrary vertex of $\mathrm{Cay}(\mathrm{SL}_2(\mathbb{Z}_N), S)$, by repeatedly expanding we reach at least $(1 + c/4)^i$ vertices after i steps and in particular, we reach more than half of the vertex set in at most $\log_{1+c/4}(N^3)$ steps. Similarly, starting from any vertex, by repeatedly expanding $\mathrm{Cay}(\mathrm{SL}_2(\mathbb{Z}_N), S^{-1})$ we reach more than half the vertices in at most $\log_{1+c/4}(N^3)$ steps. This shows that starting from, say, the identity matrix, we can reach any other matrix in $\mathrm{SL}_2(\mathbb{Z}_N)$ in at most $2\log_{1+c/4}(N^3) \le O(\log N)$ multiplications by elements of S.

Now, given coprime a, b we choose $e, f \in \mathbb{Z}$ such that $ae - bf = 1$, so that $M := \begin{pmatrix} a & e \\ b & f \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_N)$. By the above, we can write M as the product of $O(\log N)$ elements of S. The elements of S, i.e. $AD = C$ and $A^2D = AC$, both correspond to graph operations (via Lemmas 2.2 and 2.4). Hence, $M \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ is a $O(\log N)$-feasible vector, as required. □

We note that Theorem 3.1 gives another proof of the exponential growth of the tree spectrum function.

4 An application to continued fractions

The connection to the theory of continued fractions goes via the following, well-known association between continued fractions and matrix products. We note that the second equivalence was observed in [9] and involves precisely the matrices A and D we encounterd in the previous sections. Given $0 \le t < u$ such that $\gcd(t, u) = 1$ we get:

$$\begin{aligned}
\frac{t}{u} = [a_1, b_1, \dots, a_\ell, b_\ell] \quad &\Leftrightarrow \quad \begin{bmatrix} u \\ t \end{bmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_\ell & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_\ell & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&\Leftrightarrow \quad \begin{bmatrix} u \\ t \end{bmatrix} = \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a_\ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_\ell & 1 \end{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&\Leftrightarrow \quad \begin{bmatrix} u \\ t \end{bmatrix} = A^{a_1} D^{b_1} \cdots A^{a_\ell} D^{b_\ell} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \qquad (2)
\end{aligned}$$

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let $n = \left\lfloor \frac{\log N}{2} \right\rfloor$. As argued in the proof of Theorem 2.5, if we pick $\ell \le n$ and $1 \le a_2, \dots a_\ell \le 2$ the vectors

$$\vec{v}(a_2, \dots, a_\ell) := ADA^{a_2}D \cdots A^{a_\ell}D \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

are all distinct and there are at least $1 + 2 + \ldots + 2^{n-1} = 2^n - 1$ of them. Note that if $\vec{v}(a_2, \dots, a_\ell) = \begin{bmatrix} u \\ t \end{bmatrix}$, then $t < u \le 4^n \le N$, since the maximum entry of $(A^2D)^n$ is at most 4^n. Note further that the ratio $\frac{t}{u}$ has the required continued fraction expansion by (2). In addition, we know that $\gcd(t, u) = 1$ since multiplying a vector with coprime coordinates by either A or D preserves this property (since $\gcd(x + y, y) = \gcd(x, y) = \gcd(x, x + y)$).

Now by the same argument behind Lemma 2.1 we conclude that either at least $2^{n/2} - 1$ of the vectors have distinct first coordinates or at least $2^{n/2} - 1$ of them have distinct sums. In the former case we are done immediately, and in the latter we may multiply our vectors by A to obtain a vector with first coordinate being the sum, which are in particular then all distinct and still smaller than N. In either case we get at least $2^{n/2} - 1 \ge N^{1/4}/2 - 1$ distinct $u < N$ for which there is a coprime t with the desired properties. □

We note that one can easily improve the exponent in Theorem 1.4 by for example by restricting attention in the above proof to only vectors which have precisely $n/2$ exponents equal to $1/2$. It can also be improved a bit further by relaxing either the restriction $a_i \in \{1, 2\}$ to allow them to be smaller than a fixed constant C or by removing the requirement that the second entry is one. However, the limit of our approach here is always $N^{1/2}$ due to the use of the argument behind Lemma 2.1.

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