Universality, tolerance, chaos and order

Dedicated to Endre Szemerédi, for his 70th birthday

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January 25, 2010

Abstract

What is the minimum possible number of edges in a graph that contains a copy of every graph on n vertices with maximum degree a most k ? This question, as well as several related variants, received a considerable amount of attention during the last decade. In this short survey we describe the known results focusing on the main ideas in the proofs, discuss the remaining open problems, and mention a recent application in the investigation of the complexity of subgraph containment problems.

1 Introduction

For a family $\mathcal{H}$ of graphs, a graph G is *$\mathcal{H}$-universal* if it contains a copy of any $H \in \mathcal{H}$. The construction of sparse universal graphs for various families arises in the study of VLSI circuit design. See, for example, [13] and [21] for applications motivating the study of universal graphs with a small number of edges for various families of graphs. There is an extensive literature on universal graphs. In particular, universal graphs for forests have been studied in [12], [19], [20], [25], and universal graphs for planar graphs and other related families have been investigated in [3], [11], [12], [15], [16], [35].

Universal graphs for general bounded-degree graphs have also been considered extensively. For positive integers $k > 2$ and n, let $\mathcal{H}(k,n)$ denote the family of all graphs on n vertices with maximum degree at most k. Various deterministic and randomized constructions of sparse $\mathcal{H}(k,n)$-universal graphs have been found by several researchers, including constructions that satisfy certain fault-tolerance properties, as well as constructions of sparse Ramsey graphs for the problem, namely, sparse graphs for which every two-edge coloring contains a monochromatic $\mathcal{H}(k,n)$-universal graph.

In this survey we discuss the main constructions, including some of those that are not optimal, focusing on the methods used, that apply several interesting ideas. These combine probabilistic techniques with results about graph coloring, tools from matching theory and properties of high girth expanders, as well as sparse versions of the regularity lemma of Szemerédi.

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Note that a simple counting argument mentioned in [6] shows that any $\mathcal{H}(k,n)$-universal graph must contain at least $\Omega(n^{2-2/k})$ edges, and a construction showing this is tight is given in [5]. Nevertheless we believe that many of the previous, sub-optimal constructions are interesting in their own right. The study of the Ramsey type question mentioned above is more complicated, and the known estimates for this question are not tight.

2 The strong chromatic number and universal graphs

Even the fact that there are $\mathcal{H}(k,n)$-universal graphs with at most $O(n^{2-\epsilon_k})$ edges, for some $\epsilon_k > 0$ is not obvious. The first construction given in [6] establishes this fact. It is based on the notion of the strong chromatic number of a graph and provides an extremely simple construction of $\mathcal{H}(k,n)$-universal graphs with at most $O(n^{2-c/k \log k})$ edges. The construction is in fact so simple that for any n which is a power of $3k-1$, say, $n = (3k-1)^s$, it is a graph $G = G(k,s)$ that can be described in one (short) sentence, as follows. The vertices are all vectors of length s over the alphabet $\{1, 2, \ldots, 3k-1\}$, and two are adjacent if and only if they differ in all coordinates.

Let H be a graph with $|V(H)| = n$. If t divides n we say that H is *strongly t-colorable* if for any partition of $V(H)$ into pairwise disjoint sets V_i, each of cardinality t precisely, there is a proper t-vertex coloring of H in which each color class intersects each V_i in exactly one vertex. If t does not divide n, we say that H is strongly t-colorable if the graph obtained from H by adding to it $t\lceil n/t \rceil - n$ isolated vertices is strongly t-colorable. The *strong chromatic number* of H is the minimum t such that H is strongly t-colorable.

The notion of strong chromatic number is studied in [2], where it shown that the strong chromatic number of any graph with maximum degree k is at most bk, for some (large) absolute constant b. The constant has been improved substantially by Haxell [27], who showed that the estimate bk above can be replaced by $3k-1$.

Given a graph H on $n = (3k-1)^s$ vertices and maximum degree at most k, we have to show it is a subgraph of $G(k,s)$. Partition the vertices of H arbitrarily into sets of size $3k-1$, and, using the fact that the strong chromatic number of H is at most $3k-1$, find a proper $3k-1$-coloring c_1 of it in which each set is multicolored. This provides a partition of the vertices of G into $3k-1$ independent sets of equal size. Partition each of them into new sets of size $3k-1$ each, and find a proper $3k-1$ coloring c_2 in which each of these new sets is multicolored. We now have an ordered pair of colors $(c_1(v), c_2(v))$ for each vertex v, all $(3k-1)^2$ color classes are of equal size, and the colors of any pair of adjacent vertices differ in both coordinates. Continuing in this manner s steps, and then mapping the vertex v of H to the vertex $(c_1(v), c_2(v), \ldots, c_s(v))$ of $G(k,s)$, provides the required embedding of H as a spanning subgraph of $G(k,s)$. The construction for general n is similar, see [6] for more details.

A related construction is given in [3]. Instead of using the notion of the strong chromatic number of a graph, it is based on the fact that if H is an arbitrary graph on n vertices with maximum degree at most k, and $V_1, V_2, \ldots, V_m$ is an arbitrary partition of the set of its vertices into pairwise disjoint sets, each of size at least $\frac{c_k}{\epsilon^2} \log n$, then there are two disjoint independent sets of H, each containing at least a fraction of $(\frac{1}{k+1} - \epsilon)$ of each V_i. This is proved by ordering the vertices of H randomly along a line, defining one independent set to be the set of all vertices that appear before all their neighbors, and the other to be the set of all vertices that appear after all their neighbors. A simple probabilistic argument given in [3] (which conveniently applies the Hajnal Szemerédi Theorem [28]) shows that the desired result holds with positive probability. This can now be used in a recursive way that resembles the one in the construction based on strong coloring to construct relatively sparse universal graphs for $\mathcal{H}(k,n)$.

3 Random universal fault tolerant graphs

It is not surprising that random graphs with appropriate number of vertices and edge-density are $\mathcal{H}(k,n)$-universal with high probability. This is proved in [6]. Let $G(m,p)$ denote, as usual, the random graph on m labelled vertices in which each pair of distinct vertices forms an edge, randomly and independently, with probability p. We say that $G(m,p)$ satisfies a property *asymptotically almost surely*, or *a.a.s.* for short, if the probability it satisfies it tends to 1 as m tends to infinity.

Theorem 3.1 ([6]) *For every $\epsilon > 0$ there exists a positive constant $c = c(\varepsilon)$ such that, for every $k > 2$, the random graph $G(\lceil(1+\varepsilon)n\rceil, p)$ with $p = cn^{-1/k}(\log n)^{1/k}$ is* a.a.s. *$\mathcal{H}(k,n)$-universal. Consequently, for $n > n_0(k)$ there is an $\mathcal{H}(k,n)$-universal graph G with $\lceil(1+\varepsilon)n\rceil$ vertices and at most $(1+\varepsilon)^2 cn^{2-1/k}(\log n)^{1/k}$ edges.*

It turns out that if we restrict our attention to bipartite graphs with maximum degree k, then random graphs satisfy, a.a.s., a stronger property. Let $\mathcal{H}(k,n,n)$ denote the set of all bipartite graphs with n vertices in each color class and maximum degree at most k. For a real number α, where $0 < \alpha < 1$, we say that a graph G is *α-fault-tolerant* with respect to a family of graphs $\mathcal{H}$, if every subgraph of G with at least a $1-\alpha$ fraction of the edges of G is $\mathcal{H}$-universal. Note that restricting to bipartite graphs is unavoidable here, as for any graph G, there is a bipartite subgraph G' of G with at least half the edges of G.

Theorem 3.2 ([6]) *For every $k > 2$ and $0 < \alpha < 1$ there exist constants $c > 0$ and $C > 0$ such that* a.a.s. *the random graph $G(Cn,p)$ is α-fault-tolerant with respect to $\mathcal{H}(k,n,n)$, where $p = c(\log n/n)^{1/2k}$. Consequently, for $n > n_0(k)$ there is a graph G with $O(n)$ vertices and at most $O(n^{2-1/2k}(\log n)^{1/2k})$ edges, which is α-fault-tolerant with respect to $\mathcal{H}(k,n,n)$.*

It has been shown in [8] (see also [37] for a related result) that, given any *fixed*, particular $H \in \mathcal{H}(k,n)$, the graph H is *a.a.s.* a subgraph of $G(n,p)$, for $p = cn^{-\frac{1}{k}}\log^{1/k} n$, where c is a sufficiently large constant independent of n. By a simple averaging argument, this implies that $G(n,p)$ *a.a.s.* contains **almost every** $H \in \mathcal{H}(k,n)$ as a subgraph. This, however, does not suffice to show that a random graph $G = G((1+\varepsilon)n,p)$ *a.a.s* contains **every** $H \in \mathcal{H}(k,n)$ as a subgraph for a fixed $\varepsilon > 0$, as stated in Theorem 3.1. To prove this statement, one first shows that the random graph G satisfies a.a.s. certain properties concerning the number and distribution of sets of common neighbors of arbitrary sets of vertices of size at most k. It is then possible to apply Hall's theorem and show that any graph that satisfies these properties is $\mathcal{H}(k,n)$-universal. See [6] for more details.

The proof of Theorem 3.2 is more complicated. It is based on a combination of a sparse version of the regularity lemma with a hypergraph packing result proved in [36] and several additional ideas. A related problem regarding the construction of sparse fault tolerant graphs is discussed in [1].

4 Universal graphs and products of expanders

A different approach for constructing sparse $\mathcal{H}(k,n)$-universal graphs is described in [4], [5], following an initial construction given in [7]. The first result gives such universal graphs with exactly n vertices.

Theorem 4.1 ([4]) *For every $k > 2$ there exists an (explicitly constructible) $\mathcal{H}(k,n)$-universal graph T with n vertices and at most $c(k)n^{2-2/k}\log^{4/k} n$ edges, for some constant $c(k)$.*

The graphs in the second result have more vertices, but have an optimal number of edges, up to a constant factor.

Theorem 4.2 ([5]) *For every $k > 2$ there exist positive constants $c_1 = c_1(k)$ and $c_2 = c_2(k)$ so that for every n there is an (explicitly constructible) $\mathcal{H}(k,n)$-universal graph G with at most $c_1 n$ vertices and at most $c_2 n^{2-2/k}$ edges.*

The construction in the two results above are similar, but the proofs of universality are different. In particular, unlike the proof in [5], the proof that the construction of [4] is $\mathcal{H}(k,n)-$universal has the intriguing property that it is probabilistic (although the construction is explicit). We proceed with a description of the construction in [5].

Let $k > 2$ be an integer and put $m = 20n^{1/k}$. Let F be a constant degree high girth expander on m vertices. Specifically, we assume that F is an (m,d,λ)-graph, where d is an appropriate absolute constant. This means that F is d-regular and all its eigenvalues but the largest have absolute value at most λ. It is convenient to assume that F is Ramanujan, that is, $\lambda \leq 2\sqrt{d-1}$. We also assume that the girth of F is at least $\frac{2}{3}\log m/\log(d-1)$. Explicit constructions of such high girth expanders, for every $d = p+1$, where p is a prime congruent to 1 modulo 4, have been given in [31], [32]. Let $G = G_{k,n}$ be the graph whose vertex set is $V(G) = (V(F))^k$, where two vertices $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$ are adjacent iff there exist at least two indices i such that x_i and y_i are within distance 4 in F. Note that G has $m^k = O(n)$ vertices and $O(nm^{k-2}) = O(n^{2-2/k})$ edges.

Theorem 4.2 follows by showing that the graph $G_{k,n}$ is $\mathcal{H}(k,n)$-universal. This is done by establishing a graph decomposition result, and by combining it with some properties of high girth expanders. A sketch of the argument follows.

4.1 A graph-decomposition result

A *homomorphism* from a graph Z to a graph T is a mapping of the vertices of Z to those of T such that adjacent vertices in Z are mapped to adjacent ones in T. Note that there is an injective homomorphism from Z to T iff Z is a subgraph of T.

The *k-th power* T^k of a graph $T = (V(T), E(T))$ is the graph whose vertices are the vertices of T, and two are adjacent iff the distance between them in T is at most k. Let $P = P_n$ denote the path on n vertices, that is, the graph whose set of vertices is $[n] = \{1, 2, \ldots, n\}$, where i, j are connected iff $|i - j| = 1$.

An *augmentation* of a graph $T = (V, E)$ is any graph obtained from T by choosing an arbitrary (possibly empty) subset $U \subset V$, adding a new set U' of $|U|$ vertices, and adding a matching between U and U'. Thus, an augmentation of T is obtained from it by connecting new vertices of degree 1 to some of its vertices.

Call a graph *thin* if its maximum degree is at most 3 and each connected component of it is either an augmentation of a path or of a cycle, or a graph with at most two vertices of degree 3. It is easy to check that every thin graph H on n vertices is a (spanning) subgraph of the forth power of the path P_n, that is, there is a bijective homomorphism from each such H to P_n^4.

Theorem 4.3 ([4]) *Let $k \geq 2$ be an integer, and let H be an arbitrary graph of maximum degree at most k. Then there are k spanning subgraphs $H_1, H_2, \ldots, H_k$ of H such that each H_i is thin, and every edge of H lies in precisely two graphs H_i.* ■

The assertion of the theorem for even values of k is an immediate consequence of Petersen's Theorem (c.f., e.g., [39]). The proof for odd values of k requires some work based on techniques from Matching Theory.

4.2 A sketch of the universality of $G_{k,n}$

To prove Theorem 4.2 we have to show that every graph $H \in \mathcal{H}(k, n)$ is a subgraph of $G = G_{k,n}$.

Given such an $H = (V, E)$, let $H_1, H_2, \ldots, H_k$ be as in Theorem 4.3, and note that as all of them are spanning subgraphs of H, the set of vertices of each of them is V. As each H_i is thin, there are injective homomorphisms $g_i : V \mapsto [n]$ from H_i to P_n^4. The main part of the proof is to show that there are homomorphisms $f_i : [n] \mapsto V(F)$ from the path P_n to the expander F, such that the mapping $f : V(H) \mapsto V(G)$ given by $f(v) = (f_1(g_1(v)), f_2(g_2(v), \ldots, f_k(g_k(v))$ forms an injective homomorphism from H to G, thus implying that H is a subgraph of G. To do so, we define each f_i as a homomorphism from the path P_n to F, given by a non-backtracking walk. Since the girth of F exceeds 4, this ensures that each composition $f_i(g_i(\cdot))$ is a homomorphism from H_i to the forth power F^4 of F. By the definition of G, this implies that f is indeed a homomorphism from H to G. Indeed, for any pair u, v of adjacent vertices of H there are two indices i such that u, v are adjacent in H_i, as each edge of H is covered by two of the graphs H_i. For each such index i, $g_i(u)$ and $g_i(v)$ are distinct and within distance 4 in P, implying that $f_i(g_i(u))$ and $f_i(g_i(v))$ are distinct and within distance 4 in F, that is, they are adjacent in F^4. Hence $f(u)$ and $f(v)$ are adjacent in G, and f is a homomorphism, as needed.

The crucial part of the proof is to show that the homomorphisms f_i can be defined so that f is injective. This is done by a careful analysis, based on the spectral properties of the expander F. The full details can be found in [5].

5 A Ramsey type problem

Theodore Motzkin is credited with the observation that complete disorder, or total chaos, is impossible. This sentence captures the essence of Ramsey Theory. Indeed, Ramsey theory implies that in great generality, every sufficiently large system must contain a substantial ordered sub-system. The quantitative version of this statement for graphs of bounded degree has been considered some 25 years ago by Chvátal, Rödl, Szemerédi and Trotter in [22]. Their main result asserts that the Ramsey number of any graph H on n vertices and maximum degree k is at most $O(n)$. That is, for any fixed k there exists a constant c so that for any graph H on n vertices with maximum degree k, any two coloring of the edges of the complete graph on ck vertices contains a monochromatic copy of H. In a recent paper of Kohayakawa, Rödl, Schacht and Szemerédi [30] it is shown that the complete graph can be replaced by a sparser graph, with only $O(n^{2-1/k}\log^{1/k} n)$ edges. In fact, a random graph with cn vertices and $Cn^{2-1/k}\log^{1/k} n$ edges satisfies this property with high probability, where c and C are appropriate constants. Moreover, this random graph satisfies, a.a.s., the above Ramsey-type property for all such graphs H simultaneously. Indeed, any two coloring of its edges contains a monochromatic $\mathcal{H}(k,n)$-universal graph. Note that as described in the previous sections, the minimum possible number of edges of any $\mathcal{H}(k,n)$-universal graph is $\Theta(n^{2-2/k})$. The random graph considered here has a somewhat larger number of edges, but satisfies a much stronger condition.

The proof described in [30] is a delicate application of the regularity method, adapted to an appropriate sparse setting. The regularity method, which was initially based on the regularity lemma of Szemerédi proved in [38], turned out to be one of the most powerful tools in Extremal Graph Theory, with applications in other areas including Combinatorial Number Theory and theoretical Computer Science. The initial applications in Graph Theory considered only dense graphs, but it later turned out that sparse versions can be useful as well. The main ingredient in the proof of [30] is an embedding lemma, that enables one to embed bounded degree graphs of linear order in graphs with sufficiently strong pseudo-random properties. A useful phenomenon here is the fact that regularity is typically inherited at a scale that is much finer than the scale at which it is assumed. The detailed proof can be found in [30].

6 Balanced homomorphisms and subgraph containment problems

The Color Coding technique, introduced in [10], supplies a method for deciding if a given input graph G on n vertices contains a copy of a prescribed graph H with t vertices and treewidth w, in time $2^{O(t)}n^{O(w)}$. This means that the H-subgraph problem for graphs H with bounded treewidth is fixed-parameter tractable when the parameter is the size of the graph H. See [23] for the definition of fixed-parameter tractability, and [34] for the definition of treewidth. It is more convenient to consider a somewhat better understood problem, which we call here the colored H-subgraph problem. The input to this problem is a graph G whose vertices are colored by the numbers $\{1, 2, \ldots, h\}$ that represent the h vertices of H, and the objective is to decide whether or not there is a copy of H in G, in which the vertex playing the role of $i \in V(H)$ is colored i.

The work of Marx [33], improving earlier ideas of Grohe [26] shows that in fact, for every graph H, the treewidth of H essentially captures the complexity of this problem. More precisely, this means that if the Exponential Time Hypothesis of [29] holds, that is, 3-SAT on m variables cannot be solved in time $2^{o(m)}$, then there is no algorithm that solves the colored H-subgraph problem on an n vertex graph in time $n^{o(w/\log w)}$, where H is a fixed graph and $w = w(H)$ is its treewidth. Note that, as usual, the little-o notation here means that formally one has to consider an infinite family of graphs H, and the term $o(w/\log w)$ is a quantity whose ratio to $w/\log w$ tends to zero as w tends to infinity. We will, however, apply here and in what follows a slight abuse of notation, and use the o terminology even when discussing a fixed graph H, having the formal interpretation in mind. Note also that it has been proved already in [17], [18] that under the Exponential Time Hypothesis there is no algorithm that solves the K_w-subgraph containment problem for a clique of size w on an input graph on n vertices in time $n^{o(w)}$, and the novelty in the results of [26] and [33] is to show that the treewidth is the crucial parameter capturing the complexity of the problem for any graph H, and not only for cliques.

A (rough) sketch of the proof in [33] is the following. Given a 3-SAT formula with m variables and a linear number of clauses (which is known to be as difficult as the general case, see [29]), represent it by a graph F with $O(m)$ edges. A function mapping each vertex of F to a connected subset of H is called an *embedding of depth d* (of F into H) if the endpoints of each edge of F are mapped to sets that are within distance 1 or 0 in H, and the inverse image of every vertex of H is of size at most d.

The crucial step in the proof is to use the fact that the treewidth of H is w in order to show that F (and in fact any graph with $O(m)$ edges) has an embedding of depth at most $O(m \log w/w)$ into H.

Next, construct a colored graph G by replacing each vertex i of H by an independent set of size $2^{O(m \log w/w)}$, representing all possible assignments to the variables of the formula mapped to this vertex by the above embedding. All vertices of this set are assigned the color i. The edges of G can now be defined in such a way that each satisfying assignment will correspond to a colored copy of H in G, and vice versa.

If we can now solve the colored H-subgraph problem for G in time $n^{o(w/\log w)}$, where $n = |V(H)|2^{O(m \log w/w)}$ is the number of vertices of G, we will be able to solve the satisfiability instance in time $2^{o(m)}$, contradicting the Exponential Time Hypothesis.

The main combinatorial part of the argument above is the proof that if the treewidth of H is w, then any graph with m edges can be embedded in it in a balanced way as described above. A natural problem, raised in [33], is whether the $\log w$ term in this embedding result can be omitted; this will make the result tight, up to a constant factor. It turns out that some of the techniques discussed in the present paper can be used to settle this embedding question, show that the logarithmic term is needed, and prove several interesting facts about balanced embeddings of the above type, which supply, in particular, a large class of graphs H for which the colored H-subgraph problem on an n vertex input graph cannot be solved in time $n^{o(|V(H)|)}$ assuming the Exponential Time Hypothesis. These results will appear in [9], here we merely include a brief outline.

The first result proved in [9] is the following.

Proposition 6.1 ([9]) *For every fixed integer $k > 2$, real $1/4 > \epsilon > 0$, integer $w > w_0(\epsilon, k)$ and for* *every even $m > m_0(w)$ the following holds. Let $F = (V, E)$ be a random k-regular graph on m vertices. Then a.a.s., for every coloring of the vertices of F by w colors, so that each color appears at most $\frac{m}{w^{1-\epsilon}}$ times, and for any choice of a set S at most $w^{2-\frac{2}{k}-3\epsilon}$ pairs of colors, there are at least ϵm edges of F whose endpoints are not colored by one of the pairs in S.*

This is proved by estimating the number of k-regular graphs F on m vertices for which there is a set S as above and a coloring with less than ϵm edges whose endpoints are not colored by a pair in S. The estimate obtained shows that this number is much smaller than the total number of k-regular graphs on m vertices. Note that the exponent $2 - 2/k$ which appears here (up to the additive error 3ϵ), is the same exponent that appears in the minimum possible number of edges of an $\mathcal{H}(k, n)$-universal graph. Indeed it turns out that the corresponding problems are closely related.

The above proposition implies that the $\log w$ term in the embedding result of Marx [33] mentioned above is indeed needed, as stated in the next corollary. This settles a problem raised in [33].

Corollary 6.2 ([9]) *Let H be a 3-regular graph with w vertices. Then, for all even $m > m_0(w)$, there exists a 3-regular graph F on m vertices so that any embedding of F into H is of depth at least $\Omega(\frac{m \log w}{w})$.*

Note that since the above applies to a 3-regular expander H, whose treewidth is $\Theta(w)$, this shows that the $\log w$-term is needed in the embedding result of [33].

Here is a sketch of the proof of the corollary. Take $k = 3$, $\epsilon = \frac{1}{100}$ and a sufficiently large w in Proposition 6.1. Assuming the assertion of the Corollary does not hold, let F be a random cubic graph on m vertices satisfying the assertion of the proposition. Fix an embedding of the required type of F in H in which the maximum size of the inverse image of a vertex of H is of size smaller than $\frac{\epsilon^2 m \log w}{3w}$. Then there are less than $\frac{\epsilon m}{3}$ vertices of F that are mapped onto sets of size at least $\epsilon \log w$, and the total number of edges they touch is less than ϵm. Let V' denote the set of all vertices of F mapped to sets of size at least $\epsilon \log w$, and let E' denote the set of all edges they touch.

For each vertex v of F choose an arbitrary vertex of H in the connected subgraph to which it is mapped, and let this vertex be the color of v. This defines a coloring of the vertices of F by w colors (corresponding to the vertices of H), and no color appears more than $\frac{\epsilon^2 m \log w}{3w} < \frac{m}{w^{1-\epsilon}}$ times. Let S be the set of all pairs of colors x, y (=pairs of vertices x, y of H) so that the distance in H between x and y does not exceed $2\epsilon \log w = 0.02 \log w$. Since H is 3-regular, $|S| \leq O(w \cdot 2^{2\epsilon \log w}) \leq O(w^{1.02}) < w^{4/3-3/100}$. It follows that there must be at least ϵm edges of F whose endpoints are not colored by a pair of colors in S. As $|E'| < \epsilon m$ there is such an edge uv that does not belong to E', that is, it does not touch a vertex of V'. But this means that both u and v are mapped onto sets of size at most $\epsilon \log m$, and hence the properties of our embedding imply that the distance between their colors in H is at most $2\epsilon \log m$, contradicting the fact that this pair of colors does not belong to S. This completes the proof of the corollary.

The value $2 - 2/k$ (up to the 3ϵ additive error) in the exponent in Proposition 6.1 is tight in a strong sense. Indeed, if the set S in the proposition is allowed to contain $\Theta(w^{2-2/k})$ pairs, then every k-regular graph has a coloring of the required type in which the endpoints of every edge are colored by a pair in S. Moreover, there is always such a coloring with nearly equal color classes (and even exactly equal, if the number of vertices of F is divisible by w), and such a coloring can be obtained by a homomorphism into an appropriate graph with w vertices.

To state the precise result we need a few definitions. Call a homomorphism f from a graph F to a graph H *nearly balanced* if for every two vertices u, v of H, the ratio between $|f^{-1}(u)|$ and $|f^{-1}(v)|$ is at most 1.1 and at least $\frac{1}{1.1}$. The homomorphism is called *perfectly balanced* if all quantities $|f^{-1}(u)|$ are exactly equal. Note that this means that the graph F is a spanning subgraph of the $|V(F)|/|V(H)|$-blowup of H, that is, the graph obtained from H by replacing each of its vertices by an independent set of size $|V(F)|/||V(H)|$, and each of its edges by a complete bipartite graph between the corresponding sets.

Theorem 6.3 ([9]) *Let T be an arbitrary regular connected graph. Let H be the graph whose vertex set is $V(T)^k$ in which two vertices are connected iff in at least two coordinates they are within distance 4 in T. Let w denote the number of vertices of H. Then, for every k-regular graph F with $m > m_0(w)$ vertices, there is a nearly balanced homomorphism of F into H.*

The proof is similar to that given in [4], and is based on the decomposition result described in Section 4 and the fact that the random walk on T converges to a uniform distribution. Starting with a bounded degree T, and combining the construction above with a bounded degree expander on all vertices of H, as done in [7], we can obtain many explicit constructions of graphs H on w vertices with $O(w^{2-2/k})$ edges, so that every k-regular graph whose number of vertices $n \gg w$ is divisible by w admits a perfectly balanced homomorphism into H. Thus, the appropriate blow-ups of the graphs H are $\mathcal{H}(k,n)$-universal (their number of edges is much bigger than the minimum possible, but they have a very special structure).

By the results of [33] and their proofs, the construction in Theorem 6.3 (for $k = 3$) also provides many examples of graphs H with w vertices and maximum degree $O(w^{1/3})$ so that, assuming the Exponential Time Hypothesis of [29], the colored H-subgraph problem on an n vertex graph cannot be solved in time $n^{o(w)}$.

7 Concluding remarks and open problems

- As mentioned in Section 5, it is shown in [30] that there is a graph G with $O(n^{2-1/k}\log^{1/k} n)$ edges so that every two-edge coloring of it contains a monochromatic copy of an $\mathcal{H}(k,n)$-universal graph. The only lower bound known for the minimum possible number of edges of such a graph is $\Omega(n^{2-2/k})$, namely, the minimum possible number of edges of an $\mathcal{H}(k,n)$-universal graph. The problem of closing the gap between the upper and lower bound, raised in [30], seems difficult. Another interesting problem is that of finding an explicit construction of a graph G as above.

- The $\mathcal{H}(k,n)$-universal graph constructed in [5] has an optimal number of edges up to a constant factor, but its number of vertices is (much) bigger than n. By combining it with an appropriate expander, as done in [7], one can reduce the number of vertices to $(1+\epsilon)n$, for any fixed $\epsilon > 0$, increasing the number of edges only by a constant factor (depending on ϵ). It remains open to decide if there are $\mathcal{H}(k,n)$-universal graphs with n vertices and $O_k(n^{2-2/k})$ edges. Note that the construction in [4] provides $\mathcal{H}(k,n)$-universal graphs with n vertices, but their number of edges exceeds that of the graphs constructed in [5] by a logarithmic factor.

- The results of Grohe [26] and Marx [33], described in Section 6 apply to general binary Constraint Satisfaction Problems (CSPs, for short), showing that if the naturally defined graph corresponding to a general binary CSP has treewidth w, then, assuming the Exponential Time Hypothesis of [29], there is no algorithm that solves the problem in time $d^{o(w/\log w)}$, where d is the size of the domain of each variable of the CSP problem. This is tight, up to the $\log w$ term in the exponent, and the results of [9] discussed in Section 6 imply that the method in [33] does not suffice to close this $\log w$ gap.

- In [9] it is shown that for every fixed $\delta > 0$ there are families of graphs H on w vertices with maximum degree at most w^δ, so that the colored H-subgraph problem on an input graph on n vertices cannot be solved in time $n^{o(w)}$, assuming the Exponential Time Hypothesis. It will be interesting to decide if there are sparser examples H with the same property. In particular, if H is a cubic expander, or a random cubic graph on w vertices, it is not clear if the colored H-subgraph problem on an n-vertex input graph can be solved in time $n^{o(w)}$.

- In Proposition 6.1 it is shown that almost every k-regular graph F on m vertices does not admit a vertex coloring by w colors so that the number of pairs of colors appearing in the endpoints of edges of F is smaller than $w^{2-2/k-3\epsilon}$. It will be interesting to find an explicit graph F with this property, for some fixed small value of ϵ, say $\epsilon = 1/100$.

- Corollary 6.2 implies that the $\log w$-term in the embedding result of [33] cannot be omitted. It is still plausible to suspect that the $\log w$-term can be omitted in the result about the complexity of the colored H-subgraph problem, but the proof of this statement, if true, will require a different argument.

- The problem of determining or estimating the minimum possible number of *vertices* of *induced*-universal graphs for bounded degree graphs has also been considered by various authors. Butler [14] showed that for every even k there is a graph G on $O(n^{k/2})$ vertices that contains every $H \in \mathcal{H}(k,n)$ as an induced subgraph. This is tight up to a constant factor. For odd values of k the situation is more complicated. The construction of Butler gives an induced $\mathcal{H}(k,n)$-universal graph with $O(n^{\lceil k/2 \rceil})$ vertices, and this has been improved in [24] to $O(n^{\lceil k/2 \rceil - 1/k} \log^{2+2/k} n)$ by applying the construction in [4]. The methods in [5] can in fact be used to get a tight bound of $O(n^{k/2})$ for odd values of k as well. We omit the details.

Acknowledgment I would like to thank Dániel Marx for helpful comments.

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