ANNALES UNIV. SCI. BUDAPEST., SECT. MATH. **66** (2023), 91–98

ON SUMS AND PRODUCTS ALONG THE EDGES, II

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(Received December 4, 2023)

Abstract. This note is a continuation of an earlier paper by the authors [1]. We describe improved constructions addressing a question of Erdős and Szemerédi on sums and products of real numbers along the edges of a graph. We also add a few observations about related versions of the problem.

1. Introduction

In this note, we describe an improved construction addressing a question of Erdős and Szemerédi about sums and products along the edges of a graph. We also mention some related problems. The main improvement is obtained by modifying the construction in [1], which works for real numbers, instead of the integers considered there.

In their original paper Erdős and Szemerédi [5] considered sum and product along the edges of graphs. Let G_n be a graph on n vertices, $v_1, v_2, \dots, v_n$, with n^{1+c} edges for some real $c > 0$. Let $\mathcal{A}$ be an n-element set of real numbers, $\mathcal{A} = \{a_1, a_2, \dots, a_n\}$. The *sumset of* $\mathcal{A}$ *along* G_n, denoted by $\mathcal{A} +_{G_n} \mathcal{A}$, is the set $\{a_i + a_j \mid (i, j) \in E(G_n)\}$. The product set along G_n is defined similarly,

$$\mathcal{A} \cdot_{G_n} \mathcal{A} = \{a_i \cdot a_j \mid (i, j) \in E(G_n)\}.$$

The Strong Erdős–Szemerédi Conjecture, which was refuted in [1], is the following.

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[1] Research supported in part by an OTKA NK 133819 grant
[2] Research supported in part by an NSERC and an OTKA NK 133819 grant

Conjecture 1. [5] *For every $c >$ and $\varepsilon > 0$, there is a threshold, n_0, such that if $n \geq n_0$ then for any n-element subset of reals $\mathcal{A} \subset \mathbb{R}$ and any graph G_n with n vertices and at least n^{1+c} edges*

$$|\mathcal{A} +_{G_n} \mathcal{A}| + |\mathcal{A} \cdot_{G_n} \mathcal{A}| \geq |\mathcal{A}|^{1+c-\varepsilon}.$$

Now, the question is to find dense graphs with small sumset and product set along the edges. Here we extend the construction in [1]. The improvement follows by considering real numbers instead of integers only.

2. Constructions

2.1. Sum-product along edges with real numbers

Here, we extend our earlier construction so that we get better bounds in a range of edge densities. In our previous paper, for arbitrary large m_0, we constructed a set of integers, $\mathcal{A}$, and a graph on $|\mathcal{A}| = m \geq m_0$ vertices, G_m, with $\Omega(m^{5/3}/\log^{1/3} m)$ edges such that

$$|\mathcal{A} +_{G_m} \mathcal{A}| + |\mathcal{A} \cdot_{G_m} \mathcal{A}| = O\left((|\mathcal{A}| \log |\mathcal{A}|)^{4/3}\right).$$

Thus we had a graph on m vertices and roughly m^{2-c} edges with roughly m^{2-2c} sums and products along the edges for $c = 1/3$. In the following construction, we show a similar bound in a range covering all $0 \leq c \leq 2/5$. In what follows, it is convenient to ignore the logarithmic terms. We thus use now the common notation $f = \tilde{O}(g)$ for two functions $f(n)$ and $g(n)$ to denote that there are absolute positive constants c_1, c_2 so that $f(n) \leq c_1 g(n)(\log g(n))^{c_2}$ for all admissible values of n. The notation $f = \tilde{\Omega}(g)$ means that $g = \tilde{O}(f)$ and $f = \tilde{\Theta}(g)$ denotes that $f = \tilde{\Omega}(g)$ and $g = \tilde{O}(f)$.

Theorem 2. *For arbitrary large m_0, and parameter α, where $0 \leq \alpha \leq 1/5$, there is a set of reals, $\mathcal{A}$, and a graph on $|\mathcal{A}| = m \geq m_0$ vertices, G_m, with*

$$\tilde{\Omega}\left(m^{2-2\alpha}\right)$$

edges such that

$$|\mathcal{A} +_{G_m} \mathcal{A}| + |\mathcal{A} \cdot_{G_m} \mathcal{A}| = \tilde{O}\left(|\mathcal{A}|^{2-4\alpha}\right).$$

PROOF. It is easier to describe the construction using prime numbers only. We get a slightly larger exponent in the hidden logarithmic factor, but we ignore these factors here. The set of primes is denoted by $\mathbb{P}$ here. We define the set $\mathcal{A}$ and then the graph using the parameter α.

$$\mathcal{A} := \left\{ \frac{u\sqrt{w}}{\sqrt{v}} \mid u, v, w \in \mathbb{P} \text{ distinct and } v, w \le n^{\alpha}, u \le n^{1-2\alpha} \right\}.$$

Clearly, distinct choices of 3-tuples u, v, w lead to distinct reals. Thus, with this choice of parameters, the size of $\mathcal{A}$ is $\tilde{\Theta}(n)$. We are going to define a graph G_m with vertex set $\mathcal{A}$, where $|\mathcal{A}| = m = \tilde{\Theta}(n)$. Two elements, $a, b \in \mathcal{A}$ are connected by an edge if in the definition of $\mathcal{A}$ above $a = \frac{u\sqrt{w}}{\sqrt{v}}$ and $b = \frac{z\sqrt{v}}{\sqrt{w}}$. Since the degree of every vertex here is $\tilde{\Theta}(n^{1-2\alpha})$ the number of edges is

$$\tilde{\Omega}\left(m^{2-2\alpha}\right).$$

The products of pairs of elements of $\mathcal{A}$ along an edge of G_m are integers of size at most

$$n^{2-4\alpha} = \tilde{O}\left(m^{2-4\alpha}\right).$$

The sums along the edges are of the form

$$\frac{u\sqrt{w}}{\sqrt{v}} + \frac{z\sqrt{v}}{\sqrt{w}} = \frac{wu + vz}{\sqrt{vw}}.$$

The number of possibilities for the denominator is at most $n^{2\alpha}$ and the numerator is a positive integer of size at most $2n^{1-\alpha}$, hence the number of sums is, at most

$$O(n^{1+\alpha}) = \tilde{O}\left(m^{2-(1-\alpha)}\right).$$

The sum is asymptotically smaller than the product set, as long as $1-\alpha > 4\alpha$, i.e. $\alpha < 1/5$. ∎

Based on this construction, one can easily get examples of sparser graphs, simply taking smaller copies of G_m and leaving other vertices isolated.

THEOREM 3. *For every parameters $0 \le \nu \le 3/5$ and n_0 there are $n > n_0$, an n-element set of reals, $\mathcal{A} \subset \mathbb{R}$, and a graph H_n with $\tilde{\Omega}(n^{1+\nu})$ edges such that*

$$|\mathcal{A} +_{H_n} \mathcal{A}| + |\mathcal{A} \cdot_{H_n} \mathcal{A}| = \tilde{O}\left(|\mathcal{A}|^{3(1+\nu)/4}\right).$$

PROOF. The construction of Theorem 2 with $\alpha = 1/5$ supplies a set of m reals and a graph with $\tilde{\Omega}(m^{8/5})$ edges so that the number of sums and products along the edges is at most $\tilde{O}(m^{6/5})$. Take this construction with $m = n^{5(1+\nu)/8} (\leq n)$ and add to it $n - m$ isolated vertices assigning to them arbitrary distinct reals that differ from the ones used already. ∎

A similar statement holds for integers too.

THEOREM 4. *For every parameters $0 \leq \nu \leq 2/3$ and n_0 there are $n > n_0$, an n-element set of integers $\mathcal{A}$, and a graph H_n with $\tilde{\Omega}(n^{1+\nu})$ edges such that*

$$|\mathcal{A} +_{H_n} \mathcal{A}| + |\mathcal{A} \cdot_{H_n} \mathcal{A}| = \tilde{O}\left(|\mathcal{A}|^{4(1+\nu)/5}\right).$$

This follows as in the real case by starting with the construction of [1] that gives a set of m integers and a graph with $\tilde{\Omega}(m^{5/3})$ edges so that the number of sums and products along the edges is at most $\tilde{O}(m^{4/3})$. This construction with $m = n^{3(1+\nu)/5} \leq n$ together with $n - m$ isolated vertices with arbitrary $n - m$ new integers implies the statement above.

2.2. Matchings

For sparser graphs, a particular variant of the sum-product problem for integers is the following:

PROBLEM 5. *Given two n-element sets of integers, $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ let us define a sumset and a product set as*

$$S = \{a_i + b_i \mid 1 \leq i \leq n\} \quad \text{and} \quad P = \{a_i \cdot b_i \mid 1 \leq i \leq n\}.$$

Erdős and Szemerédi asked if

$$|P| + |S| = \Omega(n^{1-\varepsilon}) \tag{1}$$

for some constant $c > 0$.

The best-known lower bound is due to Chang [3], who proved that

$$|P| + |S| \geq n^{1/2} \log^{1/48} n.$$

It was shown recently in [9] that under the assumption of a special case of the Bombieri-Lang conjecture [2], $|P| + |S| = \Omega(n^{3/5})$, even for multisets.

THEOREM 6 ([9]). *Let* $M = \{(a_i, b_i) \mid 1 \le i \le n\}$ *be a set of distinct pairs of integers. If* P *and* S *are defined as above, then under the hypothesis of the Bombieri-Lang conjecture* $|P| + |S| = \Omega(n^{3/5})$.

Our previous constructions give an upper bound. The statement of Theorem 6 cannot be improved beyond an extra $1/5$ in the exponent, i.e. one can not expect a $|P| + |S| = \Omega(n^{1-\varepsilon})$ type bound for multisets.

THEOREM 7. *For arbitrarily large* n*, there is a matching* M *of size* n*, with* n *distinct pairs of integers* (a_i, b_i)*, so that* $|P| + |S| = \tilde{O}(n^{4/5})$.

PROOF. If multisets are allowed, and the only requirement is that the pairs assigned to distinct edges of the matching are distinct, then any construction of a graph with n edges yields a construction of a matching of size n. It thus follows from [1, Theorem 3] (or from Theorem 4 here) that for the multiset version there is, for arbitrarily large n, an example of a matching M of size n as above, with n distinct pairs of integers (a_i, b_i), so that $|P| + |S| = \tilde{O}(n^{4/5})$. ∎

3. Lower bounds

In [1], we followed Elekes' method using point-line incidence bounds to give a lower bound on the sum-product problem along the edges of a graph. For sparser graphs, Oliver Roche-Newton improved our bound, extending the range where a non-trivial bound can be established. He proved the following

THEOREM 8 (Theorem 6.1 in [6]). *For arbitrary set of reals,* $\mathcal{A}$*, and a graph on* $|\mathcal{A}| = m$ *vertices,* G_m*, with*

$$\tilde{\Omega}\left(m^{2-2\alpha}\right)$$

edges the following bound holds:

$$|\mathcal{A} +_{G_m} \mathcal{A}| + |\mathcal{A} \cdot_{G_m} \mathcal{A}| = \tilde{\Omega}\left(|\mathcal{A}|^{\frac{9-12\alpha}{8}}\right).$$

The result follows from applying an Elekes-Szabó type bound on the intersection size of polynomials and Cartesian products. Roche-Newton used the bound from [7]. However, a better result follows from the recent improvement in [10].

THEOREM 9 (Theorem 1.4 in [10]). *Let $f \in \mathbb{C}[x, y, z]$ be an irreducible polynomial. Then, at least one of the following is true.*

(A) For all finite sets $A, B, C \subset \mathbb{R}$ with $|A| \leq |B| \leq |C|$, we have

$$|(A \times B \times C) \cap Z(f)| = \tilde{O}(|A||B||C|)^{4/7} + |B||C|^{1/2},$$

where the implicit constant depends on the degree of f.

(B) After possibly permuting the coordinates x, y, z, we have $f(x, y, z) = g(x, y)$, for some bivariate polynomial g.

(C) f encodes additive group structure.[1]

Now, we state a new lower bound on the size of the sumset and product set along the edges of a graph.

THEOREM 10. *For arbitrary set of reals, $\mathcal{A}$, and a graph on $|\mathcal{A}| = m$ vertices, G_m, with*

$$\tilde{\Omega}\left(m^{2-2\alpha}\right)$$

edges the following bound holds:

$$|\mathcal{A} +_{G_m} \mathcal{A}| + |\mathcal{A} \cdot_{G_m} \mathcal{A}| = \tilde{\Omega}\left(|\mathcal{A}|^{\frac{5-7\alpha}{4}}\right).$$

PROOF. For the proof, we can follow the arguments in [6] and use the new Elekes-Szabó type bound from Theorem 9. We consider the zero set of the polynomial

$$f(x, y, z) = x(y - x) - z,$$

and its intersection with the Cartesian product $\mathcal{A} \times \{\mathcal{A} +_{G_m} \mathcal{A}\} \times \{\mathcal{A} \cdot_{G_m} \mathcal{A}\}$. Every edge in G_m which connects vertices a and b determines an intersection point, by $x = a$, $y = a + b$ and $z = ab$. This is the polynomial variant of Elekes' original sum-product bound in [4] where he considered lines $\alpha(X - \beta) - Y = 0$ with $\alpha, \beta \in \mathcal{A}$ and $X \in \mathcal{A} + \mathcal{A}, Y \in \mathcal{A}\mathcal{A}$. As it was shown in [6], for this polynomial, Part A applies from Theorem 9. From that, we have the bound

$$m^{2-2\alpha} = \tilde{O}\left(\left(|\mathcal{A}||\mathcal{A} +_{G_m} \mathcal{A}||\mathcal{A} \cdot_{G_m} \mathcal{A}|\right)^{4/7} + |\mathcal{A} +_{G_m} \mathcal{A}||\mathcal{A} \cdot_{G_m} \mathcal{A}|^{1/2}\right)$$

which implies

$$|\mathcal{A} +_{G_m} \mathcal{A}| + |\mathcal{A} \cdot_{G_m} \mathcal{A}| = \tilde{\Omega}\left(|\mathcal{A}|^{\frac{5-7\alpha}{4}}\right). \qquad \blacksquare$$

[1]When $f(x, y, z)$ is of the special form $h(x, y) - z$, then f encodes additive structure if and only if h has the form $h(x, y) = p(q(x) + r(y))$ or $h(x, y) = p(q(x)r(y))$ for univariate polynomials p, q, r.

4. Remarks

There is still a gap between the lower bound and our construction. It is inevitable as long as the original sum-product conjecture is open. Our construction goes to the conjectured optimum as the graph gets denser. As the density increases, the lower bound approaches Elekes' bound, $\Omega(n^{5/4})$.

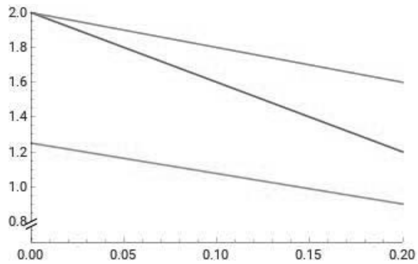

FIGURE 1. The exponents in the upper and lower bounds when the number of edges is $m^{2-2\alpha}$ (top line) and $0 < \alpha < 1/5$

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