ON SUMS AND PRODUCTS ALONG THE EDGES, II

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Abstract. This note is a continuation of an earlier paper of the authors [1]. We describe improved constructions addressing a question of Erdős and Szemerédi on sums and products of real numbers along the edges of a graph. We also add a few observations about related versions of the problem.

1. Introduction

In this note, we describe an improved construction addressing a question of Erdős and Szemerédi about sums and products along the edges of a graph. We also mention some related problems. The main improvement is obtained by a simple modification of the construction in [1] which works for real numbers, instead of the integers considered there.

In their original paper Erdős and Szemerédi [5] considered sum and product along the edges of graphs. Let $G_n$ be a graph on $n$ vertices, $v_1, v_2, \ldots, v_n$, with $n^{1+c}$ edges for some real $c > 0$. Let $A$ be an $n$-element set of real numbers, $A = \{a_1, a_2, \ldots, a_n\}$. The sumset of $A$ along $G_n$, denoted by $A + G_n A$, is the set $\{a_i + a_j | (i, j) \in E(G_n)\}$. The product set along $G_n$ is defined similarly, $A \cdot G_n A = \{a_i \cdot a_j | (i, j) \in E(G_n)\}$.

The Strong Erdős-Szemerédi Conjecture, which was refuted in [1], is the following.

Conjecture 1. [5] For every $c > $ and $\varepsilon > 0$, there is a threshold, $n_0$, such that if $n \geq n_0$ then for any $n$-element subset of reals $A \subset \mathbb{R}$ and any graph $G_n$ with $n$ vertices and at least $n^{1+c}$ edges

$$|A + G_n A| + |A \cdot G_n A| \geq |A|^{1+c-\varepsilon}.$$ 

Now the question is to find dense graphs with small sumset and product set along the edges. Here we extend the construction in [1]. The improvement follows by considering real numbers, instead of integers only.

2. Constructions

2.1. Sum-product along edges with real numbers. Here we extend our earlier construction so that we get better bounds in a range of edge densities. In our previous paper for arbitrary large $m_0$, we constructed a set of integers, $A$, and a graph on $|A| = m \geq m_0$ vertices, $G_m$, with $\Omega(m^{5/3}/\log^{1/3} m)$ edges such that

$$|A + G_m A| + |A \cdot G_m A| = O\left((|A| \log |A|)^{4/3}\right).$$ 

Thus we had a graph on $m$ vertices and roughly $m^{2-c}$ edges with roughly $m^{2-2c}$ sums and products along the edges for $c = 1/3$. In the following construction, we show a similar bound in a range covering all $0 \leq c \leq 2/5$. In what follows, it is convenient to ignore the logarithmic terms. We thus use now the common notation $f = \tilde{O}(g)$ for two functions $f(n)$ and $g(n)$ to denote that there are absolute positive constants $c_1, c_2$ so that $f(n) \leq c_1 g(n)(\log g(n))^{c_2}$ for all admissible values of $n$. The notation $f = \tilde{\Omega}(g)$ means that $g = \tilde{O}(f)$ and $f = \tilde{\Theta}(g)$ denotes that $f = \tilde{\Omega}(g)$ and $g = \tilde{O}(f)$.

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The sums along the edges are of the form $$\sum_{\text{pair}}$$

The products of pairs of elements of $$\mathbb{P}$$ exponent in the hidden logarithmic factor, but we are anyway ignoring these factors here. The set of integer of size at most 2 copies of $$\mathbb{G}$$

Theorem 2. For arbitrary large $$m_0$$, and parameter $$\alpha$$, where $$0 \leq \alpha \leq 1/5$$, there is a set of reals, $$\mathcal{A}$$, and a graph on $$|\mathcal{A}| = m \geq m_0$$ vertices, $$G_m$$, with $$\tilde{\Omega} \left( m^{2-2\alpha} \right)$$ edges such that $$\mathcal{A} + G_m \mathcal{A} + |\mathcal{A} \cdot G_m \mathcal{A} = \tilde{O} \left( |\mathcal{A}|^{2-4\alpha} \right).$$

Proof: It is easier to describe the construction using prime numbers only. We get a slightly larger exponent in the hidden logarithmic factor, but we are anyway ignoring these factors here. The set of primes is denoted by $$\mathbb{P}$$ here. We define the set $$\mathcal{A}$$ first and then the graph using the parameter $$\alpha$$.

$$\mathcal{A} := \left\{ \frac{uv\sqrt{w}}{\sqrt{v}} \mid u, v, w \in \mathbb{P} \text{ distinct and } v, w \leq n^{\alpha}, u \leq n^{1-2\alpha} \right\}.$$  

It is clear that distinct choices of 3-tuples $$u, v, w$$ lead to distinct reals. Thus with this choice of parameters, the size of $$\mathcal{A}$$ is $$\Theta(n)$$. We are going to define a graph $$G_m$$ with vertex set $$\mathcal{A}$$, where $$|\mathcal{A}| = m = \Theta(n)$$. Two elements, $$a, b \in \mathcal{A}$$ are connected by an edge if in the definition of $$\mathcal{A}$$ above $$a = \frac{uv\sqrt{w}}{\sqrt{v}}$$ and $$b = \frac{wz\sqrt{v}}{\sqrt{w}}$$. Since the degree of every vertex here is $$\Theta(n^{1-2\alpha})$$ the number of edges is $$\tilde{\Omega} \left( m^{2-2\alpha} \right).$$

The products of pairs of elements of $$\mathcal{A}$$ along an edge of $$G_m$$ are integers of size at most $$n^{2-4\alpha} = \tilde{O} \left( m^{2-4\alpha} \right).$$

The sums along the edges are of the form

$$\frac{u\sqrt{w}}{\sqrt{v}} + \frac{z\sqrt{v}}{\sqrt{w}} = \frac{wu + vz}{\sqrt{vw}}.$$  

The number of possibilities for the denominator is at most $$n^{2\alpha}$$ and the numerator is a positive integer of size at most $$2n^{1-\alpha}$$, hence the number of sums is, at most

$$O(n^{1+\alpha}) = \tilde{O} \left( m^{2-(1-\alpha)} \right).$$

The sum is asymptotically smaller than the product set, as long as $$1 - \alpha > 4\alpha$$, i.e. $$\alpha < 1/5$$.  

Based on this construction, one can easily get examples of sparser graphs, simply taking smaller copies of $$G_m$$ and leaving other vertices isolated.

Theorem 3. For every parameters $$0 \leq \nu \leq 3/5$$ and $$n_0$$ there are $$n > n_0$$, an $$n$$-element set of reals, $$\mathcal{A} \subset \mathbb{R}$$, and a graph $$H_n$$ with $$\tilde{\Omega}(n^{1+\nu})$$ edges such that

$$|\mathcal{A} + H_n \mathcal{A} | + |\mathcal{A} \cdot H_n \mathcal{A} | = \tilde{O} \left( |\mathcal{A}|^{3(1+\nu)/4} \right).$$

Proof: The construction of Theorem 2 with $$\alpha = 1/5$$ supplies a set of $$m$$ reals and a graph with $$\tilde{\Omega}(m^{8/5})$$ edges so that the number of sums and products along the edges is at most $$\tilde{O}(m^{6/5})$$. Take this construction with $$m = n^{5(1+\nu)/8} (\leq n)$$ and add to it $$n - m$$ isolated vertices assigning to them arbitrary distinct reals that differ from the ones used already.  

A similar statement holds for integers too.

Theorem 4. For every parameters $$0 \leq \nu \leq 2/3$$ and $$n_0$$ there are $$n > n_0$$, an $$n$$-element set of integers $$\mathcal{A}$$, and a graph $$H_n$$ with $$\tilde{\Omega}(n^{1+\nu})$$ edges such that

$$|\mathcal{A} + H_n \mathcal{A} | + |\mathcal{A} \cdot H_n \mathcal{A} | = \tilde{O} \left( |\mathcal{A}|^{4(1+\nu)/5} \right).$$

This follows as in the real case by starting with the construction of [1] that gives a set of $$m$$ integers and a graph with $$\tilde{\Omega}(m^{5/3})$$ edges so that the number of sums and products along the edges is at most $$\tilde{O}(m^{1/3})$$. This construction with $$m = n^{3(1+\nu)/8} \leq n$$ together with $$n - m$$ isolated vertices with arbitrary $$n - m$$ new integers implies the statement above.
2.2. Matchings. A particular variant of the sum-product problem for integers is the following:

**Problem 5.** Given two \( n \)-element sets of integers, \( A = \{a_1, \ldots, a_n\} \) and \( B = \{b_1, \ldots, b_n\} \) let us define a sumset and a product set as

\[
S = \{a_i + b_i | 1 \leq i \leq n\} \quad \text{and} \quad P = \{a_i \cdot b_i | 1 \leq i \leq n\}.
\]

Erdős and Szemerédi conjectured that

\[
|P| + |S| = \Omega(n^{1/2 + c})
\]

for some constant \( c > 0 \).

The best-known lower bound is due to Chang [3], who proved that

\[
|P| + |S| \geq n^{1/2} \log^{1/48} n.
\]

It was shown recently in [9] that under the assumption of a special case of the Bombieri-Lang conjecture [2], one can take \( c = 1/10 \) in equation (1), i.e. \( |P| + |S| = \Omega(n^{3/5}) \), even for multisets.

**Theorem 6.** [9] Let \( M = \{(a_i, b_i) | 1 \leq i \leq n\} \) be a set of distinct pairs of integers. If \( P \) and \( S \) are defined as above, then under the hypothesis of the Bombieri-Lang conjecture \( |P| + |S| = \Omega(n^{1/2 + c}) \) with \( c = 1/10 \).

If multisets are allowed, and the only requirement is that the pairs assigned to distinct edges of the matching are distinct, then any construction of a graph with \( n \) edges yields a construction of a matching of size \( n \). It thus follows from [1, Theorem 3] (or from Theorem 4 here) that for the multiset version there is, for arbitrarily large \( n \), an example of a matching \( M \) of size \( n \) as above, with \( n \) distinct pairs of integers \((a_i, b_i)\), so that \( |P| + |S| = \tilde{O}(n^{1/5}) \). This shows that the statement of Theorem 6 cannot be improved beyond an extra \( 1/5 \) in the exponent.

3. Lower bounds

In [1], we followed Elekes’ method using point-line incidence bounds to give a lower bound on the sum-product problem along the edges of a graph. For sparser graphs, Oliver Roche-Newton improved our bound, extending the range where a non-trivial bound can be established. He proved the following

**Theorem 7 (Theorem 6.1 in [6]).** For arbitrary set of reals, \( A \), and a graph on \(|A| = m \) vertices, \( G_m \), with

\[
\tilde{\Omega}(m^{2-2\alpha})
\]

edges the following bound holds:

\[
|A + G_m \cdot A| + |A \cdot G_m \cdot A| = \tilde{\Omega}
\]

\[
(|A|^{9-12\alpha}).
\]

The result follows from applying an Elekes-Szabó type bound on the intersection size of polynomials and Cartesian products. Roche-Newton used the bound from [7], however, a better result follows from the recent improvement in [10].

**Theorem 8.** [Theorem 1.4 in [10]] Let \( f \in \mathbb{C}[x, y, z] \) be an irreducible polynomial. Then at least one of the following is true.

(A) For all finite sets \( A, B, C \subseteq \mathbb{R} \) with \(|A| \leq |B| \leq |C|\), we have

\[
|(A \times B \times C) \cap Z(f)| = \tilde{O}(|A||B||C|^{4/7} + |B||C|^{1/2}),
\]

where the implicit constant depends on the degree of \( f \).

(B) After possibly permuting the coordinates \( x, y, z \), we have \( f(x, y, z) = g(x, y) \), for some bivariate polynomial \( g \).

(C) \( f \) encodes additive group structure.\(^1\)

\(^1\)When \( f(x, y, z) \) is of the special form \( h(x, y) = z \), then \( f \) encodes additive structure if and only if \( h \) has the form \( h(x, y) = p(q(x) + r(y)) \) or \( h(x, y) = p(q(x)r(y)) \) for univariate polynomials \( p, q, r \).
Now we state a new lower bound on the size of the sumset and product set along the edges of a graph.

**Theorem 9.** For arbitrary set of reals, \( A \), and a graph on \(|A| = m \) vertices, \( G_m \), with
\[
\tilde{\Omega} \left( m^{2-2\alpha} \right)
\]
edges the following bound holds:
\[
|A + G_m A| + |A \cdot G_m A| = \tilde{\Omega} \left( |A|^{\frac{5-7\alpha}{4}} \right).
\]

**Proof:** For the proof we can follow the arguments in [6] and use the new Elekes-Szabó type bound from Theorem 8. We consider the zero set of the polynomial
\[
f(x, y, z) = x(y - x) - z,
\]
and its intersection with the Cartesian product \( A \times \{A + G_m A\} \times \{A \cdot G_m A\} \). Every edge in \( G_m \) which connects vertices \( a \) and \( b \) determines an intersection point, by \( x = a, y = a + b \) and \( z = ab \).

This is the polynomial variant of Elekes’ original sum-product bound in [4] where he considered lines \( \alpha(X - \beta) - Y = 0 \) with \( \alpha, \beta \in A \) and \( X \in A + A, Y \in AA \). As it was shown in [6], for this polynomial Part A applies from Theorem 8. From that, we have the bound
\[
m^{2-2\alpha} = \tilde{O} \left( (|A||A + G_m A||A \cdot G_m A|)^{4/7} + |A + G_m A||A \cdot G_m A|^{1/2} \right)
\]
which implies
\[
|A + G_m A| + |A \cdot G_m A| = \tilde{\Omega} \left( |A|^{\frac{5-7\alpha}{4}} \right).
\]

\]

4. Remarks

There is still a gap between the lower bound and our construction. It is inevitable as long as the original sum-product conjecture is open. Our construction goes to the conjectured optimum as the graph is getting denser. The lower bound approaches Elekes’ bound [4].

![Figure 1](image.png)

**Figure 1.** The exponents in the upper and lower bounds when the number of edges is \( m^{2-2\alpha} \) (top line) and \( 0 < \alpha < 1/5 \)

**References**


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