RAINBOW STACKINGS OF RANDOM EDGE-COLORINGS

NOGA ALON, COLIN DEFANT, AND NOAH KRAVITZ

Abstract. A *rainbow stacking* of r-edge-colorings $\chi_1, \ldots, \chi_m$ of the complete graph on n vertices is a way of superimposing $\chi_1, \ldots, \chi_m$ so that no edges of the same color are superimposed on each other. We determine a sharp threshold for r (as a function of m and n) governing the existence and nonexistence of rainbow stackings of random r-edge-colorings $\chi_1, \ldots, \chi_m$.

1. Introduction

Let $\mathfrak{S}_n$ denote the symmetric group of permutations of the set $[n] \coloneqq \{1, \ldots, n\}$. Let K_n denote the complete graph with vertex set $[n]$ and edge set $\binom{[n]}{2}$. Consider a set $\mathcal{C}_r$ of r colors, and let $\chi_1, \ldots, \chi_m \colon \binom{[n]}{2} \to \mathcal{C}_r$ be edge-colorings of K_n. A *rainbow stacking* of $\chi_1, \ldots, \chi_m$ is a tuple $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_m) \in \mathfrak{S}_n^m$ such that for each edge $e \in \binom{[n]}{2}$, the colors

$$\chi_1(\sigma_1^{-1}(e)), \ldots, \chi_m(\sigma_m^{-1}(e))$$

are all distinct (where $\sigma_k^{-1}(\{i, j\}) \coloneqq \{\sigma_k^{-1}(i), \sigma_k^{-1}(j)\}$). Less formally, a rainbow stacking is a way of stacking copies of K_n with the colorings $\chi_1, \ldots, \chi_m$ on top of each other so that no edge is stacked above another edge of the same color (see Figure 1).

We are interested in the existence of rainbow stackings, especially when $\chi_1, \ldots, \chi_m \colon \binom{[n]}{2} \to \mathcal{C}_r$ are independent uniformly random colorings. When m is fixed and n is growing, we wish to determine which values of r (in terms of n) guarantee the existence or nonexistence of rainbow stackings. In what follows, the phrase *with high probability* always means *with probability tending*

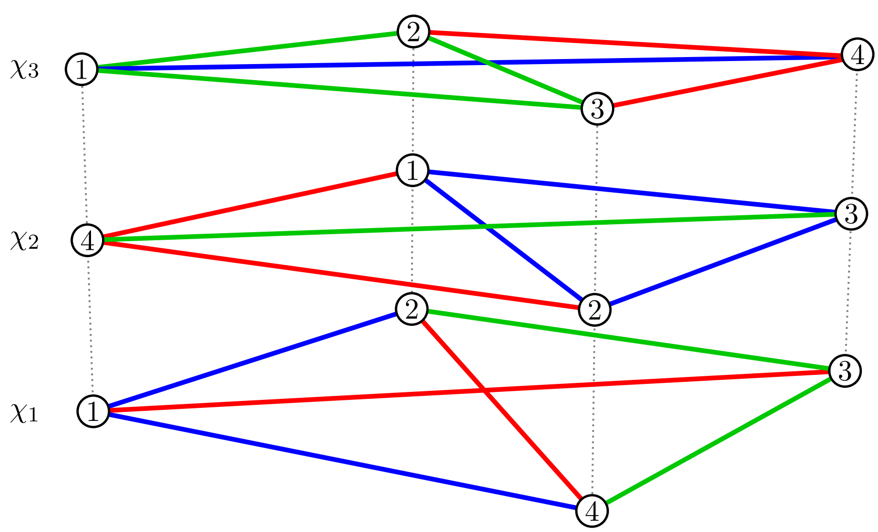

Figure 1. A rainbow stacking $\boldsymbol{\pi} \in \mathfrak{S}_4^3$ of 3 edge-colorings of K_4, where there are $r = 3$ total colors. The permutations in the tuple $\boldsymbol{\pi}$ are (in one-line notation) $\pi_1 = 1234$, $\pi_2 = 2431$, and $\pi_3 = 1243$.

to 1 *as* $n \to \infty$. Our main theorem determines a sharp threshold that governs whether rainbow stackings exist with high probability or do not exist with high probability.

Using the first-moment method, one can quickly find a upper bound on r that guarantees the nonexistence of a rainbow stacking with high probability, as follows. Let $\chi_1, \dots, \chi_m \colon \binom{[n]}{2} \to \mathcal{C}_r$ be independent uniformly random edge-colorings. For each $\boldsymbol{\sigma} \in \mathfrak{S}_n^m$, let $Z_{\boldsymbol{\sigma}}$ be the indicator function for the event that $\boldsymbol{\sigma}$ is a rainbow stacking of $\chi_1, \dots, \chi_m$, and let $Z \coloneqq \sum_{\boldsymbol{\sigma}} Z_{\boldsymbol{\sigma}}$ be the total number of rainbow stackings of $\chi_1, \dots, \chi_m$. Note that Z is always a multiple of $n!$; indeed, if $(\sigma_1, \dots, \sigma_m)$ is a rainbow stacking of $\chi_1, \dots, \chi_m$, then so is each tuple of the form $(\tau\sigma_1, \dots, \tau\sigma_m)$ for $\tau \in \mathfrak{S}_n$. For each $\boldsymbol{\sigma} \in \mathfrak{S}_n^m$, the expectation of $Z_{\boldsymbol{\sigma}}$ is exactly

$$E_{n,m,r} \coloneqq \prod_{i=1}^{m-1} \left(1 - \frac{i}{r}\right)^{\binom{n}{2}}.$$

Consequently,

$$\mathbb{E}[Z] = n!^m E_{n,m,r} \le n!^m \left(\prod_{i=1}^{m-1} e^{-i/r}\right)^{\binom{n}{2}} = n! \exp\left((m-1)\log(n!) - \binom{m}{2}\binom{n}{2} \cdot \frac{1}{r}\right). \tag{1}$$

If there is a function $\omega \colon \mathbb{N} \to \mathbb{R}$ such that $\lim_{n\to\infty} \omega(n) = \infty$ and

$$r \le \frac{m\binom{n}{2}}{2\log(n!)} - \frac{\omega(n)}{(\log n)^2},$$

then

$$m\binom{n}{2} \cdot \frac{1}{r} - 2\log(n!) \to \infty,$$

so $\mathbb{E}[Z] = o_m(n!)$ as $n \to \infty$. In this case, since the value of Z is always a nonnegative integer multiple of $n!$, Markov's Inequality implies that

$$\mathbb{P}[Z > 0] = \mathbb{P}[Z \ge n!] \le \mathbb{E}[Z]/n! = o_m(1),$$

so with high probability, there are no rainbow stackings of $\chi_1, \dots, \chi_m$. This establishes the first half of the following theorem; the proof of the second half is the main work of this paper.

Theorem 1.1. *Fix an integer $m \ge 2$ and a function $\omega \colon \mathbb{N} \to \mathbb{R}$ such that $\lim_{n\to\infty} \omega(n) = \infty$. For each $n \ge 1$, let $\chi_1, \dots, \chi_m \colon \binom{[n]}{2} \to \mathcal{C}_r$ be independent uniformly random r-edge-colorings. If $r = r(n) \ge 1$ satisfies*

$$r \le \frac{m\binom{n}{2}}{2\log(n!)} - \frac{\omega(n)}{(\log n)^2}, \tag{2}$$

then with high probability, there does not exist a rainbow stacking of $\chi_1, \dots, \chi_m$. If

$$r \ge \frac{m\binom{n}{2}}{2\log(n!)} + \frac{2m-1}{3} + \frac{m}{2\log n} + \frac{\omega(n)}{(\log n)^2}, \tag{3}$$

then with high probability, there exists a rainbow stacking of $\chi_1, \dots, \chi_m$.

2. Existence of Rainbow Stackings

We will use the second-moment method to prove the second statement of Theorem 1.1. We already computed the first moment of Z in Section 1. The second moment of Z is

$$\mathbb{E}[Z^2] = \mathbb{E}\left[\left(\sum_{\boldsymbol{\sigma} \in \mathfrak{S}_n^m} Z_{\boldsymbol{\sigma}}\right)^2\right] = \sum_{\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathfrak{S}_n^m} \mathbb{E}[Z_{\boldsymbol{\sigma}} Z_{\boldsymbol{\tau}}].$$

For each $k \in [m]$, define the new coloring $\chi'_k : \binom{[n]}{2} \to \mathcal{C}_r$ by $\chi'_k(e) \coloneqq \chi_k(\sigma_k^{-1}(e))$. Now, $Z_{\boldsymbol{\sigma}} Z_{\boldsymbol{\tau}}$ is the indicator function of the event that for each $e \in \binom{[n]}{2}$, the colors

$$\chi'_1(e), \dots, \chi'_m(e)$$

are all distinct and the colors

$$\chi'_1(\sigma_1 \tau_1^{-1}(e)), \dots, \chi'_m(\sigma_m \tau_m^{-1}(e))$$

are all distinct. Hence, $Z_{\boldsymbol{\sigma}} Z_{\boldsymbol{\tau}}$ has the same distribution as $Z_{\mathbf{id}} Z_{(\sigma_1\tau_1^{-1},\dots,\sigma_m\tau_m^{-1})}$, where we write $\mathbf{id} = (\mathrm{id}, \dots, \mathrm{id})$ for the tuple in $\mathfrak{S}_n^m$ whose components are all equal to the identity permutation $\mathrm{id} \in \mathfrak{S}_n$. Consequently,

$$\mathbb{E}[Z^2] = n!^m \sum_{\boldsymbol{\pi} \in \mathfrak{S}_n^m} \mathbb{E}[Z_{\mathbf{id}} Z_{\boldsymbol{\pi}}].$$

We derive an explicit formula for each $[Z_{\mathbf{id}} Z_{\boldsymbol{\pi}}]$ as follows.

For each $e \in \binom{[n]}{2}$, let $\beta_1(e), \dots, \beta_m(e)$ be m copies of e. Consider the m-partite graph $G_{\boldsymbol{\pi}}$ with vertex set $V(G_{\boldsymbol{\pi}}) = \{\beta_k(e) : e \in \binom{[n]}{2}, k \in [m]\}$ in which $\beta_k(e)$ is adjacent to $\beta_{k'}(e')$ if and only if $k \neq k'$ and either $e = e'$ or $\pi_k(e) = \pi_{k'}(e')$. The edge-colorings $\chi_1, \dots, \chi_m$ naturally induce a vertex-coloring of $G_{\boldsymbol{\pi}}$, where $\beta_k(e)$ is assigned the color $\chi_k(e)$. Observe that $\mathbf{id}$ and $\boldsymbol{\pi}$ are both rainbow stackings of $\chi_1, \dots, \chi_m$ if and only if the induced vertex-coloring of $G_{\boldsymbol{\pi}}$ is proper. For example, if $\boldsymbol{\pi}$ and χ_1, χ_2, χ_3 are as depicted in Figure 1, then $G_{\boldsymbol{\pi}}$ and its induced coloring are shown in Figure 2. Although $\boldsymbol{\pi}$ is a rainbow stacking of χ_1, χ_2, χ_3, the identity tuple $\mathbf{id}$ is not; this is why there are pink edges in Figure 2 whose endpoints have the same color.

The quantity $\mathbb{E}[Z_{\mathbf{id}} Z_{\boldsymbol{\pi}}]$ is equal to $r^{-m\binom{n}{2}} N_{\boldsymbol{\pi}}$, where $N_{\boldsymbol{\pi}}$ is the number of proper r-vertex-colorings of $G_{\boldsymbol{\pi}}$. Hence, we will study how $N_{\boldsymbol{\pi}}$ depends on $\boldsymbol{\pi}$.

The graph $G_{\boldsymbol{\pi}}$ has $m\binom{n}{2}$ vertices and

$$2\binom{m}{2}\binom{n}{2} - \left| \left\{ (k, k', e) \in [m] \times [m] \times \binom{[n]}{2} : k < k' \text{ and } \pi_k(e) = \pi_{k'}(e) \right\} \right|$$

edges. For each $k < k'$, we have

$$\left| \left\{ e \in \binom{[n]}{2} : \pi_k(e) = \pi_{k'}(e) \right\} \right| = \binom{f_{k,k'}(\boldsymbol{\pi})}{2} + t_{k,k'}(\boldsymbol{\pi}),$$

where $f_{k,k'}(\boldsymbol{\pi})$ and $t_{k,k'}(\boldsymbol{\pi})$ denote the number of fixed points and the number of 2-cycles (respectively) of $\pi_k^{-1}\pi_{k'}$ (viewed as a permutation of $[n]$). Define the weight

$$\mathrm{wt}(\boldsymbol{\pi}) \coloneqq \sum_{1 \le k < k' \le m} \mathrm{wt}_{k,k'}(\boldsymbol{\pi}),$$

where

$$\mathrm{wt}_{k,k'}(\boldsymbol{\pi}) \coloneqq \binom{f_{k,k'}(\boldsymbol{\pi})}{2} + t_{k,k'}(\boldsymbol{\pi}).$$

Then the number of edges of $G_{\boldsymbol{\pi}}$ is

$$2\binom{m}{2}\binom{n}{2} - \mathrm{wt}(\boldsymbol{\pi}).$$

The following lemma provides an upper bound on $N_{\boldsymbol{\pi}}$ in terms of the weight $\mathrm{wt}(\boldsymbol{\pi})$.

Lemma 2.1. *Let $m \ge 2$ be an integer. If $\boldsymbol{\pi} \in \mathfrak{S}_n^m$ and $r = r(n)$ satisfies $n^2/r^3 = o(1)$, then*

$$N_{\boldsymbol{\pi}} \le (1 + o_m(1)) r^{m\binom{n}{2}} E_{n,m,r}^2 e^{\mathrm{wt}(\boldsymbol{\pi})/(r-(2m-1)/3)}.$$

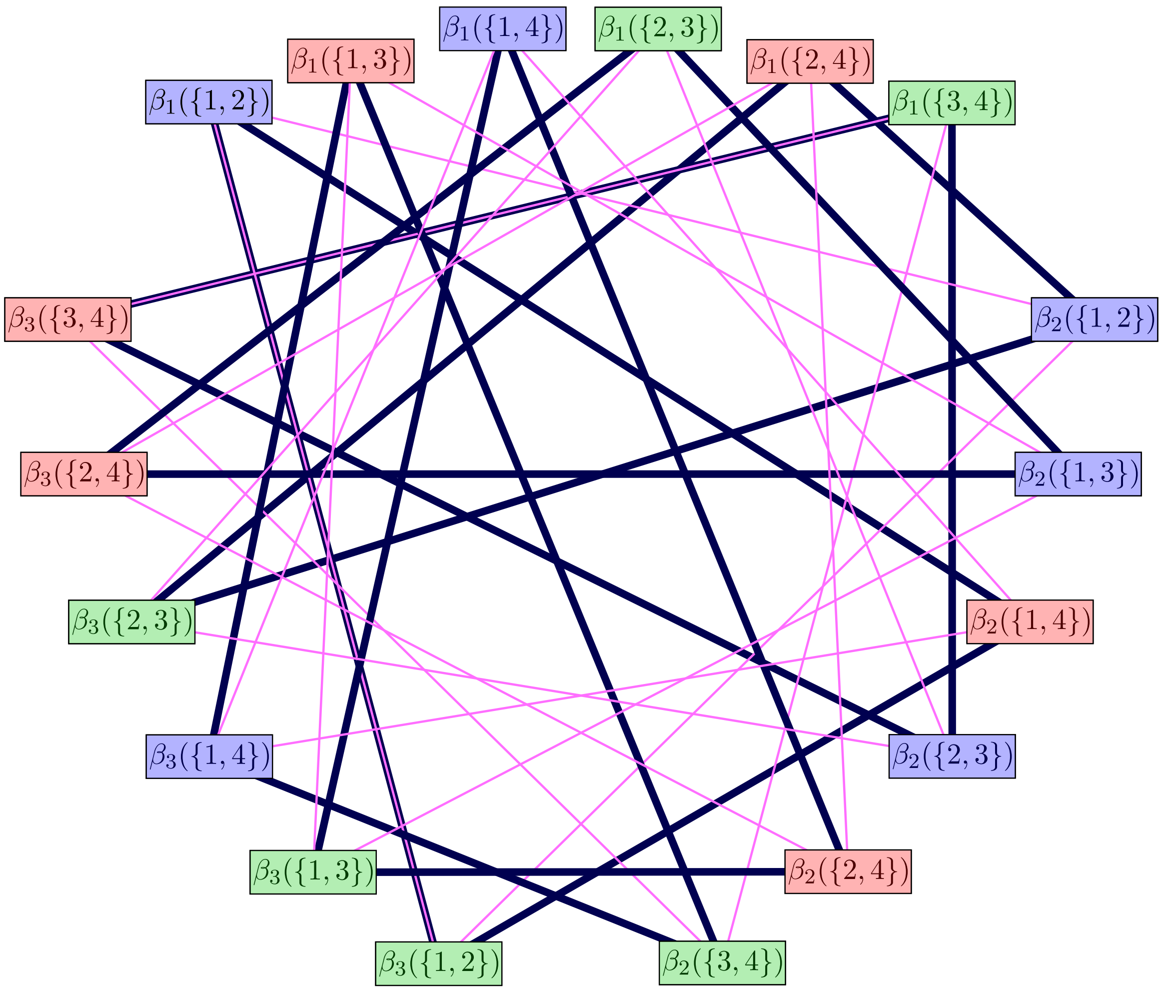

FIGURE 2. The graph $G_{\boldsymbol{\pi}}$, where $\boldsymbol{\pi} = (1234, 2431, 1243)$ is as depicted in Figure 1. Edges of $G_{\boldsymbol{\pi}}$ of the form $\{\beta_k(e), \beta_{k'}(e)\}$ are drawn in thin pink, while those of the form $\{\beta_k(e), \beta_{k'}(e')\}$ with $\pi_k(e) = \pi_{k'}(e')$ are drawn in **thick navy**. Edges of the form $\{\beta_k(e), \beta_{k'}(e)\}$ with $\pi_k(e) = \pi_{k'}(e)$ are drawn with **both colors**. The vertex-coloring of $G_{\boldsymbol{\pi}}$ is induced by the edge-colorings χ_1, χ_2, χ_3 in Figure 1.

Proof. We use the entropy method. Let $\chi : V(G_{\boldsymbol{\pi}}) \to \mathcal{C}_r$ be a uniformly random proper r-coloring. Then the entropy of χ is

$$H(\chi) = \log(N_{\boldsymbol{\pi}}),$$

where $H(\cdot)$ denotes the base-e entropy function. Let $\sigma \in \mathfrak{S}_m$ be a permutation. We will reveal the values of χ on the vertices $\beta_{\sigma(1)}(e)$, then the vertices $\beta_{\sigma(2)}(e)$, and so on until the vertices $\beta_{\sigma(m)}(e)$. For each stage, let $\chi^{\sigma}_{<k}$ denote the partial coloring on the vertices $\beta_{\sigma(k')}(e)$ for $k' < k$ and $e \in \binom{[n]}{2}$. Then the chain rule and the subadditivity of entropy give that

$$H(\chi) \le \sum_{k=1}^{m} \sum_{e \in \binom{[n]}{2}} H(\chi(\beta_{\sigma(k)}(e)) \mid \chi^{\sigma}_{<k}).$$

We will estimate the summands appearing on the right-hand side of this inequality individually.

For each color $c \in \mathcal{C}_r$, each vertex $\beta_\ell(e)$, and each partial proper coloring χ' of the other vertices of $G_{\boldsymbol{\pi}}$, we have

$$\mathbb{P}[\chi(\beta_\ell(e)) = c \mid \chi \text{ and } \chi' \text{ agree wherever } \chi' \text{ is defined}] \leq 1/(r-2m+2). \tag{4}$$

Indeed, because $\beta_\ell(e)$ has at most $2m-2$ neighbors, there are at most $2m-2$ forbidden values of $\chi(\beta_\ell(e))$; since the remaining colors are equally likely, each one occurs with probability at most $1/(r-2m+2)$.

Now, consider a single permutation σ and a single vertex $\beta_{\sigma(k)}(e)$. Let $y = y_k^\sigma(e)$ be such that $2(k-1)-y$ is the number of distinct colors already assigned by χ to the neighbors of $\beta_{\sigma(k)}(e)$ that are of the form $\beta_{\sigma(k')}(e')$ with $k' < k$. Then there are at most $r-2(k-1)+y$ possibilities for $\chi(\beta_{\sigma(k)}(e))$, and the entropy of $\chi(\beta_{\sigma(k)}(e))$ conditioned on this partial coloring is at most

$$\log(r-2(k-1)+y) = \log(r-2(k-1)) + \log\left(1 + \frac{y}{r-2(k-1)}\right) \leq \log(r-2(k-1)) + \frac{y}{r-2(k-1)}.$$

Summing over all of the possibilities for the partial coloring $\chi^\sigma_{<k}$, we find that

$$H(\chi(\beta_{\sigma(k)}(e)) \mid \chi^\sigma_{<k}) \leq \log(r-2(k-1)) + \mathbb{E}[y]\frac{1}{r-2(k-1)}. \tag{5}$$

The next task is estimating $\mathbb{E}[y]$.

For each triple (ℓ, ℓ', e) with $\ell < \ell'$ and $e \in \binom{[n]}{2}$, let

$$x(\ell, \ell', e) = \begin{cases} 1 & \text{if } \pi_\ell(e) = \pi_{\ell'}(e); \\ 0 & \text{otherwise.} \end{cases}$$

We record for future reference the identity

$$\sum_{\ell < \ell'} \sum_{e \in \binom{[n]}{2}} x(\ell, \ell', e) = \mathrm{wt}(\boldsymbol{\pi}). \tag{6}$$

The neighbors of $\beta_{\sigma(k)}(e)$ already colored by $\chi^\sigma_{<k}$ are the vertices $\beta_{\sigma(k')}(e)$ and $\beta_{\sigma(k')}(\pi^{-1}_{\sigma(k')}\pi_{\sigma(k)}(e))$ for $k' < k$. Counting collisions, we find that the number of such vertices is

$$2(k-1) - \sum_{k' < k} x(\sigma(k), \sigma(k'), e).$$

The vertices $\beta_{\sigma(k')}(e)$ for $k' < k$ form a clique in $G_{\boldsymbol{\pi}}$; likewise, the vertices $\beta_{\sigma(k')}(\pi^{-1}_{\sigma(k')}\pi_{\sigma(k)}(e))$ for $k' < k$ form a clique in $G_{\boldsymbol{\pi}}$. So the pairs of such vertices receiving the same color form a matching, and the number of such pairs is at least $y - \sum_{k'<k} x(\sigma(k), \sigma(k'), e)$. Each pair of vertices $\beta_{\sigma(k'')}(e), \beta_{\sigma(k')}(\pi^{-1}_{\sigma(k')}\pi_{\sigma(k)}(e))$, for $k', k'' < k$, receives the same color with probability at most $1/(r-2m+2)$ by (4), so

$$\begin{aligned}
\mathbb{E}[y] &\leq \frac{(k-1-\sum_{k'<k} x(\sigma(k), \sigma(k'), e))^2}{r-2m+2} + \sum_{k'<k} x(\sigma(k), \sigma(k'), e) \\
&\leq \frac{(k-1)(k-1-\sum_{k'<k} x(\sigma(k), \sigma(k'), e))}{r-2m+2} + \sum_{k'<k} x(\sigma(k), \sigma(k'), e) \\
&= \frac{(k-1)^2}{r-2m+2} + \left(1 - \frac{k-1}{r-2m+2}\right) \sum_{k'<k} x(\sigma(k), \sigma(k'), e) \\
&= \frac{(k-1)^2}{r-2m+2} + \left(\frac{r-2(k-1)}{r-k+1} + O_m(1/r^2)\right) \sum_{k'<k} x(\sigma(k), \sigma(k'), e).
\end{aligned}$$

Substituting this into (5), summing over k and e, and averaging over $\sigma \in \mathfrak{S}_m$ gives that

$$H(\chi) \leq \mathop{\mathbb{E}}_{\sigma\in\mathfrak{S}_m} \sum_{k=1}^{m} \sum_{e\in\binom{[n]}{2}} \left(\Omega_k + \Psi^{\sigma}_{k,e} + O_m(1/r^3)\right), \tag{7}$$

where

$$\Omega_k = \log(r-2(k-1)) + \frac{(k-1)^2}{(r-2m+2)(r-2(k-1))} \quad \text{and} \quad \Psi^{\sigma}_{k,e} = \frac{\sum_{k'<k} x(\sigma(k),\sigma(k'),e)}{r-k+1}.$$

We compute that

$$\begin{aligned}\Omega_k &= \log(r-2(k-1)) + \frac{(k-1)^2/r}{r-2(k-1)} + O_m(1/r^3)\\ &= \log(r-2(k-1)) + \log\left(1+\frac{(k-1)^2/r}{r-2(k-1)}\right) + O_m(1/r^3)\\ &= \log(r-2(k-1)+(k-1)^2/r) + O_m(1/r^3)\\ &= \log r + 2\log(1-(k-1)/r) + O_m(1/r^3);\end{aligned}$$

the crucial point is the identity $r-2(k-1)+(k-1)^2/r = r(1-(k-1)/r)^2$. Consequently,

$$\begin{aligned}\mathop{\mathbb{E}}_{\sigma\in\mathfrak{S}_m} \sum_{k=1}^{m} \sum_{e\in\binom{[n]}{2}} \Omega_k &= m\binom{n}{2}\log r + 2\binom{n}{2}\sum_{k=1}^{m}\log(1-(k-1)/r) + O_m(n^2/r^3)\\ &= m\binom{n}{2}\log r + 2\log(E_{n,m,r}) + o_m(1).\end{aligned}$$

Next, using (6), the formula for the sum of the first $m-1$ squares, and the hypothesis that $n^2/r^3 = o(1)$, we find that

$$\begin{aligned}\mathop{\mathbb{E}}_{\sigma\in\mathfrak{S}_m} \sum_{k=1}^{m} \sum_{e\in\binom{[n]}{2}} \Psi^{\sigma}_{k,e} &= \sum_{\ell<\ell'}\sum_{e} x(\ell,\ell',e) \mathop{\mathbb{E}}_{\sigma\in\mathfrak{S}_m} \frac{1}{r-\max\{\sigma^{-1}(\ell),\sigma^{-1}(\ell')\}+1}\\ &= \mathrm{wt}(\boldsymbol{\pi}) \mathop{\mathbb{E}}_{\sigma\in\mathfrak{S}_m} \frac{1}{r-\max\{\sigma^{-1}(1),\sigma^{-1}(2)\}+1}\\ &= \mathrm{wt}(\boldsymbol{\pi}) \sum_{j=2}^{m} \frac{(j-1)/\binom{m}{2}}{r-j+1}\\ &= \mathrm{wt}(\boldsymbol{\pi}) \left(\frac{1}{r} + \sum_{j=2}^{m}\frac{(j-1)^2/\binom{m}{2}}{r^2} + O_m(1/r^3)\right)\\ &= \mathrm{wt}(\boldsymbol{\pi}) \left(\frac{1}{r} + \frac{2m-1}{3r^2}\right) + O_m(n^2/r^3)\\ &= \frac{\mathrm{wt}(\boldsymbol{\pi})}{r-(2m-1)/3} + o_m(1).\end{aligned}$$

Substituting these bounds back into (7), we conclude that

$$N_{\boldsymbol{\pi}} \leq (1+o_m(1)) r^{m\binom{n}{2}} E^2_{n,m,r} e^{\mathrm{wt}(\boldsymbol{\pi})/(r-(2m-1)/3)},$$

as desired. □

For convenience, let

$$\widehat{r} = r - (2m-1)/3.$$

If r satisfies (3), then it follows immediately from Lemma 2.1 that

$$\mathbb{E}[Z_{\mathbf{id}} Z_{\boldsymbol{\pi}}] \le (1+o_m(1)) E_{n,m,r}^2 e^{\mathrm{wt}(\boldsymbol{\pi})/\widehat{r}} \tag{8}$$

for each $\boldsymbol{\pi} \in \mathfrak{S}_n^m$, so

$$\mathbb{E}[Z^2] \le (1+o_m(1)) E_{n,m,r}^2 n!^m \sum_{\boldsymbol{\pi} \in \mathfrak{S}_n^m} e^{\mathrm{wt}(\boldsymbol{\pi})/\widehat{r}}. \tag{9}$$

Our goal is to obtain an upper bound on the sum on the right-hand side of (9). The following proposition captures the central estimate of the proof.

Proposition 2.2. *If r satisfies (3), then*

$$\sum_{\boldsymbol{\pi} \in \mathfrak{S}_n^m} e^{\mathrm{wt}(\boldsymbol{\pi})/\widehat{r}} = n!^m (1+o(1)).$$

Proposition 2.2 tells us that if r satisfies (3), then

$$\mathbb{E}[Z^2] = (1+o_m(1))(E_{n,m,r} n!^m)^2 = (1+o_m(1))\mathbb{E}[Z]^2$$

(using (1) for the second equality), so $\mathrm{Var}(Z) = o_m(\mathbb{E}[Z])$. Then Chebyshev's Inequality gives that

$$\mathbb{P}[Z=0] = o_m(1),$$

which proves the second part of Theorem 1.1. Thus, the remainder of this section will be devoted to proving Proposition 2.2. Assume in what follows that r satisfies (3) or, equivalently, that

$$\widehat{r} \ge \frac{m\binom{n}{2}}{2\log(n!)} + \frac{m}{2\log n} + \frac{\omega(n)}{(\log n)^2}. \tag{10}$$

Each of the $n!^m$ summands on the left-hand side of the equation in Proposition 2.2 is at least 1, so we must show that very few summands can be significantly larger than 1. To accomplish this, we will split the sum according to the values of $f_{k,k'}(\boldsymbol{\pi})$ and $t_{k,k'}(\boldsymbol{\pi})$.

Let $L(\boldsymbol{\pi})$ be the sequence obtained by listing the pairs $(k,k') \in [m] \times [m]$ with $k < k'$ in decreasing order of $\mathrm{wt}_{k,k'}(\boldsymbol{\pi})$ (breaking ties arbitrarily). Now, let us construct a subsequence $\vec{p}(\boldsymbol{\pi}) = (p_1(\boldsymbol{\pi}), \dots, p_{m-1}(\boldsymbol{\pi}))$ of $L(\boldsymbol{\pi})$ recursively as follows. Let $p_1(\boldsymbol{\pi})$ be the first pair in $L(\boldsymbol{\pi})$. For $2 \le i \le m-1$, let $p_i(\boldsymbol{\pi})$ be the first pair in $L(\boldsymbol{\pi})$ such that $p_i(\boldsymbol{\pi}) \notin \{p_1(\boldsymbol{\pi}), \dots, p_{i-1}(\boldsymbol{\pi})\}$ and the (undirected) graph on the vertex set $[m]$ with the edge set $\{p_1(\boldsymbol{\pi}), \dots, p_i(\boldsymbol{\pi})\}$ is acyclic (where we are identifying the ordered pair (k,k') with the unordered pair $\{k,k'\}$). In other words, $\vec{p}(\boldsymbol{\pi})$ is the lexicographically first subsequence of $L(\boldsymbol{\pi})$ whose entries form the edges of a spanning tree of the complete graph on $[m]$. Notice that $\vec{p}(\boldsymbol{\pi})$ is uniquely determined by $L(\boldsymbol{\pi})$.

Writing $p_\ell(\boldsymbol{\pi}) = (k_\ell, k'_\ell)$, we can use standard rearrangement inequalities to find that

$$\mathrm{wt}(\boldsymbol{\pi}) \le \sum_{\ell=1}^{m-1} \ell \left[\binom{f_\ell(\boldsymbol{\pi})}{2} + t_\ell(\boldsymbol{\pi}) \right],$$

where $f_\ell(\boldsymbol{\pi}) := f_{k_\ell,k'_\ell}(\boldsymbol{\pi})$ and $t_\ell(\boldsymbol{\pi}) := t_{k_\ell,k'_\ell}(\boldsymbol{\pi})$. Note that the number of permutations in $\mathfrak{S}_n$ with f fixed points and t 2-cycles is at most

$$\binom{n}{f}\binom{n-f}{2t}(2t-1)!!(n-f-2t)! = \frac{n!}{f!2^t t!}.$$

Given a sequence L and tuples $(f_1, \dots, f_{m-1})$ and $(t_1, \dots, t_{m-1})$, the number of tuples $\boldsymbol{\pi} \in \mathfrak{S}_n^m$ satisfying $L(\boldsymbol{\pi}) = L$ and $f_\ell(\boldsymbol{\pi}) = f_\ell$ and $t_\ell(\boldsymbol{\pi}) = t_\ell$ for all $1 \le \ell \le m-1$ is at most

$$n!^m \prod_{\ell=1}^{m-1} \frac{1}{f_\ell! 2^{t_\ell} t_\ell!};$$

here, we are crucially using the fact that the entries of $\vec{p}(\boldsymbol{\pi})$ form the edge set of an acyclic graph.

Let Υ denote the set of tuples $(f_1, \dots, f_{m-1}, t_1, \dots, t_{m-1})$ of integers satisfying the following conditions:

- $0 \le f_1, \dots, f_{m-1} \le n$;
- $0 \le t_1, \dots, t_{m-1} \le n/2$;
- $\binom{f_1}{2} + t_1 \ge \cdots \ge \binom{f_{m-1}}{2} + t_{m-1}$.

Let

$$\Upsilon_\le \coloneqq \{(f_1, \dots, f_{m-1}, t_1, \dots, t_{m-1}) \in \Upsilon : \tbinom{f_1}{2} + t_1 \le \tbinom{\log n}{2}\}$$

and

$$\Upsilon_\ge \coloneqq \{(f_1, \dots, f_{m-1}, t_1, \dots, t_{m-1}) \in \Upsilon : \tbinom{f_1}{2} + t_1 \ge \tbinom{\log n}{2}\}.$$

(We remark that in these definitions, the particular choice of $\binom{\log n}{2}$ for the cutoff is not important.) For each $\boldsymbol{\pi} \in \mathfrak{S}_n^m$, the tuple $(f_1(\boldsymbol{\pi}), \dots, f_{m-1}(\boldsymbol{\pi}), t_1(\boldsymbol{\pi}), \dots, t_{m-1}(\boldsymbol{\pi}))$ belongs to $\Upsilon_\le$ or $\Upsilon_\ge$. The tuples $\boldsymbol{\pi}$ with $(f_1(\boldsymbol{\pi}), \dots, f_{m-1}(\boldsymbol{\pi}), t_1(\boldsymbol{\pi}), \dots, t_{m-1}(\boldsymbol{\pi})) \in \Upsilon_\le$ will end up contributing the main term of $n!^m(1 + o(1))$ in Proposition 2.2, while the other tuples will end up contributing only to the $o(n!^m)$ error term.

Let us begin with $\Upsilon_\le$. If $\boldsymbol{\pi} \in \mathfrak{S}_n^m$ is such that $(f_1(\boldsymbol{\pi}), \dots, f_{m-1}(\boldsymbol{\pi}), t_1(\boldsymbol{\pi}), \dots, t_{m-1}(\boldsymbol{\pi})) \in \Upsilon_\le$, then

$$\mathrm{wt}(\boldsymbol{\pi}) \le \binom{m}{2}\binom{\log n}{2}.$$

Since there are at most $n!^m$ such tuples $\boldsymbol{\pi}$, the sum of $e^{\mathrm{wt}(\boldsymbol{\pi})/\widehat{r}}$ over these tuples is at most

$$n!^m \exp\left(\binom{m}{2}\binom{\log n}{2}\frac{1}{\widehat{r}}\right) \le n!^m \exp\left(O_m((\log n)^3/n)\right) = n!^m(1 + o(1)),$$

where we have used (3).

We now turn to $\Upsilon_\ge$. Since there are at most $\binom{m}{2}!$ possibilities for the sequence $L(\boldsymbol{\pi})$, the sum of $e^{\mathrm{wt}(\boldsymbol{\pi})/\widehat{r}}$ over all tuples $\boldsymbol{\pi} \in \mathfrak{S}_n^m$ corresponding to a given tuple $T = (f_1, \dots, f_{m-1}, t_1, \dots, t_{m-1}) \in \Upsilon_\ge$ is at most $\binom{m}{2}! n!^m X(T)$, where

$$X(T) \coloneqq \prod_{\ell=1}^{m-1} \left(\frac{1}{f_\ell! 2^{t_\ell} t_\ell!} \exp\left(\left[\binom{f_\ell}{2} + t_\ell\right]\frac{\ell}{\widehat{r}}\right)\right). \tag{11}$$

We wish to find a uniform upper bound on $X(T)$ as T ranges over the elements of $\Upsilon_\ge$.

We first require a technical lemma. Define $g, h \colon \mathbb{R}_{\ge 1} \to \mathbb{R}_{\ge 0}$ by $g(x) = \frac{x(x-1)}{2}$ and $h(x) = \frac{1+\sqrt{1+8x}}{2}$ so that $g(h(x)) = x$. Let Γ denote the gamma function.

Lemma 2.3. *If t and K are integers such that $0 \le t \le K$, then*

$$\Gamma(h(K - t) + 1)2^t t! \ge \Gamma(h(K) + 1).$$

Proof. It suffices to prove that $\Gamma(h(K - (t-1)) + 1)2^{t-1}(t-1)! \le \Gamma(h(K-t) + 1)2^t t!$ whenever $1 \le t \le K$. The identity $g(h(x)) = x$ implies that $h'(x) = \frac{1}{g'(h(x))} = \frac{2}{2h(x)-1}$. Because $h''(x) < 0$ for all $x > 1$, we have

$$h(x+1) \le h(x) + h'(x) = h(x) + \frac{2}{2h(x) - 1}. \tag{12}$$

It is routine to verify that

$$\frac{\Gamma(z)}{\Gamma(z+\frac{2}{2z-1})} \geq \frac{1}{2} \tag{13}$$

for every real number $z \geq 2$. Assume $1 \leq t \leq K$. Since $h(K-t)+1 \geq h(0)+1 = 2$, we can set $x = K-t$ in (12) and set $z = h(K-t)+1$ in (13) to find that

$$\frac{\Gamma(h(K-t)+1)}{\Gamma(h(K-t+1)+1)} \geq \frac{\Gamma(h(K-t)+1)}{\Gamma(h(K-t)+1+\frac{2}{2h(K-t)-1})} \geq \frac{1}{2}.$$

Therefore,

$$\frac{\Gamma(h(K-t)+1)2^t t!}{\Gamma(h(K-(t-1))+1)2^{t-1}(t-1)!} = 2t\frac{\Gamma(h(K-t)+1)}{\Gamma(h(K-t+1)+1)} \geq t \geq 1,$$

as desired. □

An immediate consequence of Lemma 2.3 is that for each integer $K \geq 0$, the maximum value of

$$\frac{1}{f!2^t t!}\exp\left(\left[\binom{f}{2}+t\right]\frac{\ell}{\widehat{r}}\right),$$

over all $f, t \in \mathbb{Z}_{\geq 0}$ satisfying $\binom{f}{2}+t = K$, is at most

$$\frac{1}{\Gamma(h(K)+1)}\exp\left(\binom{h(K)}{2}\frac{\ell}{\widehat{r}}\right).$$

Applying this estimate to each multiplicand in the definition of $X(T)$, we find that

$$\begin{aligned}\max_{T\in\Upsilon_{\geq}} X(T) &\leq \max_{\substack{n\geq f_1\geq\cdots\geq f_{m-1}\geq 0,\\ f_1\geq \log n}} \prod_{\ell=1}^{m-1}\frac{1}{\Gamma(f_\ell+1)}\exp\left(\binom{f_\ell}{2}\cdot\frac{\ell}{\widehat{r}}\right)\\ &= \max_{f_1\in[\log n,n]}\exp(\varphi_{1/\widehat{r}}(f_1))\max_{f_2\in[0,f_1]}\exp(\varphi_{2/\widehat{r}}(f_2))\cdots\max_{f_{m-1}\in[0,f_{m-2}]}\exp(\varphi_{(m-1)/\widehat{r}}(f_{m-1})),\end{aligned} \tag{14}$$

where the f_ℓ's run over intervals of real numbers and we have set

$$\varphi_q(f) \coloneqq -\log\Gamma(f+1)+q\binom{f}{2}.$$

We will now study the behavior of the functions φ_q.

Since the logarithm of the gamma function is convex (by the Bohr–Mollerup Theorem; see, e.g., [3]) and the function $f \mapsto \binom{f}{2}$ is concave, the function φ_q is also concave for all $q \geq 0$. In particular, the maximum value of φ_q on an interval is always assumed at one of the endpoints of the interval. So the maximum over f_{m-1} is achieved when either $f_{m-1} = 0$ or $f_{m-1} = f_{m-2}$. In the former case, we simply remove this term (since $\varphi_q(0) = 0$ for all q). In the latter case, we "incorporate" the f_{m-1} term into the preceding f_{m-2} term by noting that

$$\varphi_{(m-2)/\widehat{r}}(f_{m-2})+\varphi_{(m-1)/\widehat{r}}(f_{m-2}) = 2\varphi_{(m-3/2)/\widehat{r}}(f_{m-2}).$$

We then obtain the same dichotomy for the maximum over f_{m-2}, and we continue in this manner until we reach f_1, where the maximum occurs when either $f_1 = n$ or $f_1 = \log n$. Thus, there is some $1 \leq s \leq m-1$ such that

$$\max_{T\in\Upsilon_{\geq}} X(T) \leq \max\left\{\exp\left(\sum_{\ell=1}^{s}\varphi_{\ell/\widehat{r}}(\log n)\right), \exp\left(\sum_{\ell=1}^{s}\varphi_{\ell/\widehat{r}}(n)\right)\right\}.$$

Then the sum of $e^{\mathrm{wt}(\boldsymbol{\pi})/\widehat{r}}$ over all $\boldsymbol{\pi} \in \mathfrak{S}_n^m$ with $(f_1(\boldsymbol{\pi}), \dots, f_{m-1}(\boldsymbol{\pi}), t_1(\boldsymbol{\pi}), \dots, t_{m-1}(\boldsymbol{\pi})) \in \Upsilon_{\geq}$ is thus at most

$$\binom{m}{2}!(n^2/2)^{m-1} n!^m \max\left\{\exp\left(\sum_{\ell=1}^{s} \varphi_{\ell/\widehat{r}}(\log n)\right), \exp\left(\sum_{\ell=1}^{s} \varphi_{\ell/\widehat{r}}(n)\right)\right\},$$

so to prove Proposition 2.2, it suffices to show that

$$(m-1)\log(n^2/2) + \max\left\{\sum_{\ell=1}^{s} \varphi_{\ell/\widehat{r}}(\log n), \sum_{\ell=1}^{s} \varphi_{\ell/\widehat{r}}(n)\right\} \to -\infty$$

as $n \to \infty$. We first check that

$$\begin{aligned}(m-1)\log(n^2/2) + \sum_{\ell=1}^{s} \varphi_{\ell/\widehat{r}}(\log n) &= (m-1)\log(n^2/2) - s\log(\Gamma(\log n + 1)) + \binom{s+1}{2}\binom{\log n}{2} \cdot \frac{1}{\widehat{r}} \\ &= -s\log n \log\log n + O_m(\log n)\end{aligned}$$

tends to $-\infty$ (with room to spare in the asymptotic condition on $\widehat{r}$ in (10)). We next consider

$$(m-1)\log(n^2/2) + \sum_{\ell=1}^{s} \varphi_{\ell/\widehat{r}}(n) = (m-1)\log(n^2/2) - s\log(n!) + \binom{s+1}{2}\binom{n}{2} \cdot \frac{1}{\widehat{r}}. \tag{15}$$

If $s < m-1$, then the right-hand side of (15) is $-(1+o(1))s(1-(s+1)/m)n\log n$, which certainly tends to $-\infty$. If $s = m-1$, then the right-hand side of (15) becomes

$$(m-1)\left[\log(n^2/2) - \log(n!) + \frac{m}{2}\binom{n}{2} \cdot \frac{1}{\widehat{r}}\right],$$

which tends to $-\infty$ by (10). This finishes the proof of Proposition 2.2 and hence also of Theorem 1.1.

3. Further Remarks

3.1. **Comments on the proof.** When $m = 2$, our proof of Theorem 1.1 simplifies considerably but is still nontrivial. In this case, each graph $G_{\boldsymbol{\pi}}$ is a disjoint union of edges (corresponding to edges fixed by $\pi_1\pi_2^{-1}$) and even-length cycles. One can then bound $N_{\boldsymbol{\pi}}$ using the known formulas for the chromatic polynomials of cycles in lieu of Lemma 2.1. Moreover, the proof of Proposition 2.2 simplifies because the list $L(\boldsymbol{\pi})$ consists of the single element $(1,2)$ and it is not necessary to extract the subsequence $\vec{p}(\boldsymbol{\pi})$.

An examination of our proofs shows that Theorem 1.1 continues to hold in the regime where m grows reasonably slowly with n. To optimize this dependence (which we have not attempted), one should tweak some of the parameters appearing in our proof (for instance, the cutoff $\binom{\log n}{2}$ in the definitions of $\Upsilon_{\leq}, \Upsilon_{\geq}$); we leave the details to the curious reader.

3.2. **Sharp thresholds.** Theorem 1.1 shows that the existence problem for rainbow stackings exhibits a sharp threshold, in the sense that the transition from having no rainbow stackings with high probability to having rainbow stackings with high probability occurs within an interval of length roughly $(2m-1)/3$. It is natural to ask if the transition is even sharper; in particular, we pose the following problem.

Problem 3.1. *Determine whether or not there exists a function $r_0\colon \mathbb{N} \to \mathbb{R}$ such that the following holds. For each $n \geq 1$, let $\chi_1, \dots, \chi_m\colon \binom{[n]}{2} \to \mathcal{C}_r$ be independent uniformly random r-edge-colorings. If $r = r(n)$ satisfies $r(n) < r_0(n)$, then with high probability, there does not exist a rainbow stacking of $\chi_1, \dots, \chi_m$. If $r = r(n)$ satisfies $r(n) > r_0(n)$, then with high probability, there exists a rainbow stacking of $\chi_1, \dots, \chi_m$.*

In the past, the second-moment method has often been effective for obtaining analogous sharp results. A well-known example is the proof of the 2-point concentration of the independence number of the Erdős–Rényi random graph $G(n, 1/2)$ (see, e.g., [2,4,5]). Since, however, the expected value of this quantity is only $O(\log n)$, such a sharp concentration is less dramatic than the sharp transition for rainbow stackings, where the critical value of r is on the order of $n/\log n$.

3.3. **Rainbow stackings of deterministic edge-colorings.** It seems interesting to find sufficient (deterministic) conditions for the existence of rainbow stackings, even when $m = 2$. Proper edge-colorings provide a natural starting point. We note that not every pair of proper edge-colorings has a rainbow stacking.

Proposition 3.2. *If $n = 2^k - 2$ for some integer $k \geq 2$, then there is a pair of proper edge-colorings of K_n with no rainbow stackings.*

Proof. We provide an explicit construction of such a pair of colorings, based on a construction described in [1] (in the context of transversals in Latin squares). Let $\mathbb{F}_2^k$ denote the elementary abelian 2-group of rank k. Let $u_1, v_1, u_2, v_2 \in \mathbb{F}_2^k$ be such that $u_1 \neq v_1$, $u_2 \neq v_2$, and $u_1+v_1 = u_2+v_2$. For each $i \in \{1, 2\}$, let us identify the set $\mathbb{F}_2^k \setminus \{u_i, v_i\}$ with $[n]$ arbitrarily and define the coloring $\chi_i \colon \binom{\mathbb{F}_2^k \setminus \{u_i, v_i\}}{2} \to \mathbb{F}_2^k$ by $\chi_i(\{x, y\}) \coloneqq x + y$. It is clear that χ_1, χ_2 are proper edge-colorings.

We will show that the colorings χ_1, χ_2 do not admit a rainbow stacking. Consider a bijection $\sigma \colon \mathbb{F}_2^k \setminus \{u_1, v_1\} \to \mathbb{F}_2^k \setminus \{u_2, v_2\}$. We claim that there are distinct elements $x, y \in \mathbb{F}_2^k \setminus \{u_1, v_1\}$ such that $x + \sigma(x) = y + \sigma(y)$. Indeed, if this were not the case, then the quantities $z + \sigma(z)$ for $z \in \mathbb{F}_2^k \setminus \{u_1, v_1\}$ would all be distinct. Then, since $\sum_{z \in \mathbb{F}_2^k} z = 0$, the quantity $\sum_{z \in \mathbb{F}_2^k \setminus \{u_1, v_1\}} (z + \sigma(z))$ would be the sum of two distinct elements of $\mathbb{F}_2^k$, so it would be nonzero. At the same time, our choice of u_1, v_1, u_2, v_2 ensures that

$$\sum_{z \in \mathbb{F}_2^k \setminus \{u_1, v_1\}} (z + \sigma(z)) = -(u_1 + v_1) - (u_2 + v_2) = 0.$$

This contradiction establishes the claim.

Take x, y as in the claim. The fact that $\chi_1(\{x, y\}) = x + y = \sigma(x) + \sigma(y) = \chi_2(\sigma(\{x, y\}))$ shows that σ is not a rainbow stacking. □

We remark that the Cayley sum-graph construction in the proof of Proposition 3.2 does not work when n is sufficiently large and $n \neq 2^k - 2$. Indeed, in this case, Müyesser and Pokrovskiy showed [7, Theorem 1.4] that for any n-element subsets A and B of $\mathbb{F}_2^k$, there exists a bijection $\sigma \colon A \to B$ such that the sums of the form $a + \sigma(a)$ for $a \in A$ are all distinct.

Motivated by these observations and by a conjecture of Ryser about the existence of transversals in Latin squares of odd order (see [6, 8]), we ask the following question.

Question 3.3. *Is it true that when n is odd, every pair of proper edge-colorings of K_n admits a rainbow stacking?*

We remark that the answer to Question 3.3 is "yes" when $n = 3$ (by inspection) and when $n = 5$ (by computer search). It seems that a general affirmative resolution of this question would be difficult; it may be easier to start with proper edge-colorings in which no color appears a large number of times.

3.4. **Hypergraphs.** It could be interesting to extend our work to random edge-colorings of complete d-uniform hypergraphs for $d > 2$.

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Department of Mathematics, Princeton University, Princeton, NJ 08544, USA and Schools of Mathematics and Computer Science, Tel Aviv University, Tel Aviv 69978, Israel
Email address: `nalon@math.princeton.edu`

Department of Mathematics, Harvard University, Cambridge, MA 02138, USA
Email address: `colindefant@gmail.com`

Department of Mathematics, Princeton University, Princeton, NJ 08540, USA
Email address: `nkravitz@princeton.edu`