On the Capacity of Digraphs

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Abstract

For a digraph $G = (V, E)$ let $w(G^n)$ denote the maximum possible cardinality of a subset S of V^n in which for every ordered pair $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ of members of S there is some $1 \leq i \leq n$ such that $(u_i, v_i) \in E$. The *capacity* $C(G)$ of G is $C(G) = lim_{n \mapsto \infty}[\ (w(G^n))^{1/n}]$. It is shown that for any digraph G with maximum outdegree d, $C(G) \leq d + 1$. It is also shown that for every n there is a tournament T on $2n$ vertices whose capacity is at least $\sqrt{n}$, whereas the maximum number of vertices in a transitive subtournament in it is only $O(\log n)$. This settles a question of Körner and Simonyi.

1 Introduction

For a digraph $G = (V, E)$ and for a positive integer n, let $w(G^n)$ denote the maximum possible cardinality of a subset S of V^n in which for every ordered pair $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ of members of S there is some i, $1 \leq i \leq n$ such that (u_i, v_i) is a directed edge of G. It is easy to see that the function $g(n) = w(G^n)$ is super-multiplicative, and hence the limit

$$\lim_{n \mapsto \infty}[\ (w(G^n))^{1/n}]$$

exists and is equal to the supremum of the quantity in the square brackets. This limit, denoted by $C(G)$, is called the *capacity* of the digraph.

The study of the capacity of directed graphs was introduced by Körner and Simonyi and by Gargano, Körner and Vaccaro in [10], [6], where the authors study the quantity $\sum(G) = \log C(G)$, which they call the *Sperner capacity* of G, and show that it generalizes the Shannon capacity of an undirected graph ([11]). In several subsequent papers [7], [9] they apply some properties of this invariant in the asymptotic solution of various problems in extremal set theory.

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A *tournament* T is a digraph in which for every pair u, v of distinct vertices exactly one of the ordered pairs (u, v), (v, u) is a directed edge. The tournament T is *transitive* if there is a linear order on its vertices and (u, v) is a directed edge iff u is smaller than v in this order.

It is easy to see that the capacity $C(T_n)$ of the transitive tournament on n vertices is n. Therefore, the capacity of any tournament that contains a transitive subtournament on n vertices is at least n. Using algebraic techniques, Calderbank, Frankl, Graham, Li and Shepp [3] proved that the capacity of the cyclically directed triangle is 2, namely, the number of vertices in the largest transitive subtournament in it. Blokhuis [2] gave a simpler proof of this result. This inspired the following conjecture of Körner and Simonyi.

Conjecture 1.1 ([9]) *For every tournament T, the capacity $C(T)$ is the maximum number of vertices in a transitive subtournament of T.*

In this note we first observe that the algebraic method in [3] and [2] (which is a modification of a method of Haemers [8]) implies the following result.

Theorem 1.2 *The capacity of any directed graph D with maximum outdegree d^+ and maximum indegree d^- satisfies $C(D) \leq d^+ + 1$ and $C(D) \leq d^- + 1$.*

In particular, any tournament T with maximum outdegree d that contains a transitive subtournament on $d+1$ vertices satisfies $C(T) = d+1$. An example of such a tournament on $2d+1$ vertices is the cyclic tournament C_{2d+1} whose vertices are the numbers $\{0, 1, 2, \ldots, 2d\}$ in which (i, j) is a directed edge iff $(j-i)$ *mod* $(2d)$ is between 1 and d. Therefore, these tournaments satisfy the assertion of the above conjecture. Moreover, it is not difficult to check that the above theorem implies that the conjecture holds for all tournaments with at most 4 vertices. This is because if the maximum indegree and the maximum outdegree of a tournament on 4 vertices are both 3, then it is necessarily transitive and its capacity is 4, and if the maximum outdegree or the maximum indegree is 2, then it contains a transitive tournament of size 3 and hence its capacity is 3.

Our second result, however, is that Conjecture 1.1 is false in the following strong sense.

Theorem 1.3 *For every integer n there is a tournament T on $2n$ vertices of capacity $C(T) \geq \sqrt{n}$ in which the maximum number of vertices in any transitive subtournament is at most $4\log_2 n + 2$.*

The proof of the above theorem is by a probabilistic construction, which resembles one of the results in [1].

The rest of this note is organized as follows. In Section 2 we present the proof of Theorem 1.2 and in Section 3 we present the proof of Theorem 1.3. The final Section 4 contains some open problems.

2 Bounding the capacity via the degrees

In this section we prove Theorem 1.2. We use the method of [2]; a related proof can be given following the technique in [3]. Let $D = (V, E)$ be a directed graph on a set $V = \{1, 2, \ldots, k\}$ of k vertices, with maximum outdegree d^+ and maximum indegree d^-. We prove that $C(D) \le d^- + 1$. The bound in terms of the maximum outdegree follows, either by repeating the same argument with the obvious modifications or by observing that the capacity of any digraph equals to that of the digraph obtained from it by reversing the direction of all edges. Let n be a positive integer, and let S be a subset of cardinality $w(D^n)$ of V^n, in which for every ordered pair $(u_1, \ldots, u_n), (v_1, \ldots, v_n)$ of members of S there is some i such that $(u_i, v_i) \in E$. Associate each member $v = (v_1, v_2, \ldots, v_n)$ of S with a polynomial $P_v = P_v(x_1, x_2, \ldots, x_n)$ (over the rationals, say) defined by

$$P_v(x_1, \ldots, x_n) = \prod_{i=1}^{n} \prod_{j \in N^-(v_i)} (x_i - j),$$

where here $N^-(v_i)$ denotes the set of all in-neighbors of v in D, that is, the set of all vertices u of D such that $(u, v) \in E$.

Note that for every $v = (v_1, \ldots, v_n) \in S$, $P_v(v_1, \ldots, v_n) \ne 0$, since $v_i \notin N^-(v_i)$ for all i. On the other hand, if $u = (u_1, \ldots, u_n) \in S$, $u \ne v$, then $P_v(u_1, \ldots, u_n) = 0$, since there is some i for which $u_i \in N^-(v_i)$. It follows that the set of polynomials $\{P_v : v \in S\}$ is linearly independent (since if $\sum_{v \in S} c_v P_v(x_1, \ldots, x_n) = 0$ then, by substituting $(x_1, \ldots, x_n) = (v_1, \ldots, v_n)$ we conclude that $c_v = 0$). Since each P_v is a polynomial of degree at most d^- in each variable, the number of these polynomials does not exceed the dimension of the space of polynomials in n variables of degree at most d^- in each variable, which is $(d^- + 1)^n$. Therefore $|S| = w(D^n) \le (d^- + 1)^n$, implying that $C(D) \le d^- + 1$ and completing the proof. $\Box$

3 Pseudo-random tournaments with large capacities

In this section we prove Theorem 1.3. We need the following known lemma, due to Erdős and Moser ([4], see also [5]), whose short probabilistic proof is presented here, for the sake of completeness.

Lemma 3.1 *For every n there exists a tournament on n vertices containing no transitive subtournament on more than $2 \log_2 n + 1$ vertices.*

Proof. Let T be a random tournament on a set V of n labeled vertices obtained by choosing, for each pair of distinct vertices u, v in V, randomly and independently, either (u, v) or (v, u) to be a directed edge of T, where both choices are equally likely. For each fixed ordered set K of k vertices of T, the probability that K forms a transitive subtournament in T with respect to the linear order of its elements is precisely $2^{-\binom{k}{2}}$. Therefore, the probability that T contains a transitive tournament on k vertices is at most

$$n(n-1)\cdots(n-k+1)2^{-\binom{k}{2}} < [n2^{-(k-1)/2}]^k < 1,$$

provided $k > 2\log_2 n + 1$. Therefore, with positive probability, T contains no transitive tournament on more than $2\log_2 n + 1$ vertices, completing the proof. □

Proof of Theorem 1.3. Given n, there exists, by the last lemma, a tournament R on the set of n vertices $\{c_1, c_2, \ldots, c_n\}$ containing no transitive tournament on more than $2\log_2 n+1$ vertices. Let T be a tournament on the set of $2n$ vertices $\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$ in which for all $1 \leq i, j \leq n$, (a_i, a_j) is a directed edge of T iff (c_i, c_j) is a directed edge of R, (b_i, b_j) is a directed edge of T iff (c_j, c_i) is a directed edge of R, and the edges connecting the vertices a_i with the vertices b_j are oriented arbitrarily. Note that T contains a copy of R on the vertices a_i, and a reversed copy of R on the vertices b_j. Thus, any transitive tournament in T cannot contain more than $2\log_2 n + 1$ vertices a_i and cannot contain more than $2\log_2 n+1$ vertices b_j, implying that the maximum cardinality of a transitive tournament in T is at most $4\log_2 n+2$. On the other hand, the set $S = \{(a_i, b_i) : 1 \leq i \leq n\}$ is a set of cardinality n is V^2, and for each two distinct i and j, either (a_i, a_j) is an edge of T or (b_i, b_j) is an edge of T, by construction. Hence $w(T^2) \geq n$, and thus

$$C(T) = \underset{n}{Sup}[w(T^n)^{1/n}] \geq w(T^2)^{1/2} \geq \sqrt{n}.$$

This completes the proof. □

Remark: The constants in the last theorem can be improved, and we made no attempt to optimize them. If, in the definition of T in the proof above, one orients every edge between an a_i and a b_j towards b_j, then the set $S \cup \{(b_i, a_i) : 1 \leq i \leq n\}$ shows that $w(T^2) \geq 2n$ implying that for this T, $C(T) \geq \sqrt{2n}$. On the other hand, a random orientation of the edges between the vertices a_i and the vertices b_j will reduce the size of the maximum transitive subtournament, leaving $C(T)$ at least $\sqrt{n}$.

4 Concluding remarks and open problems

It may be interesting to determine the smallest possible size of a tournament for which the assertion of Conjecture 1.1 fails. As mentioned in the introduction, the conjecture is true for tournaments with at most 4 vertices, and together with Tibor Szabó and Gábor Tardos we verified it for all tournaments with at most 5 vertices as well. For a prime $p \equiv 3(mod\ 4)$, the *quadratic tournament* Q_p is the tournament whose vertices are the numbers $0, 1, 2, \ldots p-1$, in which (i, j) is a directed edge iff $j-i$ is a quadratic residue modulo p. This tournament is isomorphic to the tournament obtained from it by reversing the direction of all edges (since the mapping $x \mapsto -x$ is such an isomorphism.) Therefore, the set $S = \{(x, -x) : 0 \leq x \leq p-1\}$ implies that $w(Q_p^2) \geq p$, showing that $C(Q_p) \geq \sqrt{p}$. It seems likely that the largest number of vertices in a transitive subtournament of Q_p is much smaller than $\sqrt{p}$ and may even be polylogarithmic in p. If true, however, the proof of this should be very difficult, as it would imply much better bounds for the well studied number theoretic problem of estimating the size of the smallest quadratic nonresidue modulo a prime p than those known. It is, however, feasible to check via computer the size of the largest transitive subtournament in Q_p for modest values of p, and thus obtain smaller counterexamples to Conjecture 1.1 than the ones supplied by the probabilistic approach (even with optimized constants). Indeed, together with Szabó we found, using a computer, that for $p = 67$ the quadratic tournament Q_{67} contains no transitive subtournament on 9 vertices and as its capacity is at least $\sqrt{67} > 8$ it supplies an explicit, relatively small counterexample to the conjecture.

Another interesting problem related to the content of this note is the estimation of the typical capacity of a random tournament on n vertices. The following conjecture seems plausible.

Conjecture 4.1 *There is an absolute constant c so that the probability that the capacity of a random tournament on n vertices exceeds $c \log_2 n$ tends to 0 as n tends to infinity.*

It is not too difficult to show that with high probability the capacity of such a tournament is $o(n)$. Let $t(T)$ denote the maximum number of vertices in a transitive subtournament of T. It would be interesting to estimate the maximum possible value of the ratio $C(T)/t(T)$, as T ranges over all tournaments of n vertices. Our results here show that this maximum is at least $\Omega(\sqrt{n}/\log n)$, but we suspect it might be bigger.

Another interesting problem, suggested by Körner, is to characterize all tournaments T in which for every subtournament T', $C(T') = t(T')$.

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