# Conflict-Free colorings of Shallow Discs

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**Abstract**

We prove that any collection of $n$ discs in which each one intersects at most $k$ others, can be colored with at most $O(\log^3 k)$ colors so that for each point $p$ in the union of all discs there is at least one disc in the collection containing $p$ whose color differs from that of all other members of the collection that contain $p$. This is motivated by a problem on frequency assignments in cellular networks, and improves the best previously known upper bound of $O(\log n)$ when $k$ is much smaller than $n$.

## 1 Introduction

A coloring of a family $\mathcal{S}$ of $n$ regions of some fixed type (such as discs, pseudo-discs, axis-parallel rectangles, etc.), is called *conflict-free* (CF for short) if for each point $p \in \cup_{b \in \mathcal{S}} b$ there is at least one region $b \in \mathcal{S}$ that contains $p$ in its interior, whose color is unique among all regions in $\mathcal{S}$ that contain $p$ in their interior (in this case we say that $p$ is being 'served' by that color).

The study of such colorings, which originated in [3] and [9], was motivated by the problem of frequency assignment in cellular networks. Specifically, cellular networks are heterogeneous networks with two different types of nodes: base-stations (that act as servers) and clients. The base-stations are interconnected by an external fixed backbone network. Clients are connected only to base stations; connections between clients and base-stations are implemented by radio links. Fixed frequencies are assigned to base-stations to enable links to clients. Clients, on the other hand, continuously scan frequencies in search of a base-station with good reception. The fundamental problem of frequency assignment in cellular networks is to assign frequencies to base-stations so that every client is served by some base-station, in the sense that it lies within the range of the station and no other station within its reception range has the same frequency. The goal is to minimize the number of assigned frequencies since the frequency spectrum is limited and costly.

Suppose we are given a set of $n$ antennas (base-stations). The location of each antenna and its radius of transmission is fixed and is given (as a disc in the plane). Even et al. [3] have shown that one can find an assignment of frequencies to the antennas with a total of at most $O(\log n)$ frequencies such that each antenna is assigned one of the frequencies and the resulting assignment

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is free of conflicts, in the sense that, for each point $p$ in the plane, there exists at least one antenna $a$, whose disc of transmission contains $p$ so that $a$'s frequency is different from those of all other antennas that can reach $p$. Furthermore, it was shown that this bound is worst-case optimal.

Thus, Even et al. have shown that any family of $n$ discs in the plane has a CF-coloring with $O(\log n)$ colors and that this bound is tight in the worst case. Furthermore, such a coloring can be found in polynomial time. Other variants and extensions of these coloring problems have drawn the attention of researchers and have been the focus of several recent papers (see, e.g., [3, 4, 5, 6, 8, 9, 10]). In this paper we study conflict-free colorings of $n$ discs such that each disc intersects at most $k$ other discs (we assume that $k \ll n$). The lower bound of $\Omega(\log n)$ for CF-colorings of $n$ discs, given in [3] is such that every disc intersects all other discs. In view of the motivation of frequency assignment, it is natural to assume that in a realistic setting, the number of discs that are intersected by some given disc is bounded by some parameter $k$ which is much smaller than the total number of the given discs. In this case the only known lower bound for the number of colors needed in any CF-coloring is $\Omega(\log k)$. In this paper, we provide an upper bound of $O(\log^3 k)$ on the number of colors needed for CF-coloring such discs. This improves the known upper bound of $O(\log n)$, whenever $k$ is much smaller than $n$.

## 2 Coloring shallow discs

Let $D$ be a set of discs. For a point $p \in \bigcup_{d \in D} d$, put $d(p) = \{d \in D | p \in d\}$. For a given integer $k$, put $H_{\leq k}(D) = (D, E_{\leq k}(D))$, where $E_{\leq k}(D) = \{d(p) \mid |d(p)| \leq k \text{ and } p \in \bigcup_{d \in D} d\}$. We refer to $d(p)$ as the *depth* (or *level*) of $p$. Let $\chi : D \longrightarrow \{1, ..., i\}$ be a coloring of $D$. For a point $p \in \bigcup_{d \in D} d$, we say that $d(p)$ is *conflict-free* if there exists at least one disc $s \in d(p)$ with a unique color (i.e., no other disc in $d(p)$ gets the same color as $s$). $\chi$ is called a *conflict-free coloring* if every point $p \in \bigcup_{d \in D} d$, $d(p)$ is conflict-free.

**Theorem 2.1.** *Let $D$ be a set of $n$ planar discs and let $k > 1$ be an integer. Assume that no disc in $D$ intersects more than $k$ other discs in $D$. Then $D$ can be conflict-free colored with only $O(\log^3 k)$ colors.*

Before proceeding to the proof of Theorem 2.1, we need a few technical lemmas:

**Lemma 2.2.** *Let $D$ be a finite set of $n$ planar discs. Then $|E_{\leq k}(D)| = O(kn)$.*

*Proof.* Let $\mathcal{A}(D)$ denote the arrangement of the bounding circles of the discs in $D$. We may assume that $D$ is in general position, in the sense that no three distinct circles pass through a common point. This can be enforced by a slight perturbation of the discs, which does not decrease $|E_{\leq k}(D)|$. Let $S_{\leq k}(D)$ be the set of vertices of the arrangement $\mathcal{A}(D)$ that lie in the interior of at most $k$ discs of $D$. By the analysis of Clarkson and Shor [2], we have $|S_{\leq k}(D)| = O\big(k^2 \mathcal{U}(\frac{n}{k})\big)$, where $\mathcal{U}(n)$ is the maximum complexity of the union of any $n$ discs (i.e., the maximum possible number of vertices of an arrangement of $n$ discs that are at level 0). Let $F_{\leq k}$ denote the set of all cells in $\mathcal{A}(D)$ with level at most $k$ (i.e., the set of connected components of the complement of the union of the bounding circles that are contained in at most $k$ discs). Obviously $|E_{\leq k}(D)| \leq |F_{\leq k}(D)|$. We charge a cell $f \in F_{\leq k}(D)$ to one of the vertices on the boundary $\partial f$, if $\partial f$ has vertices. Thus, the only cells unaccounted for by this charging scheme are the cells that have no vertices on their boundary. However, it is easy to check that the number of such cells is only $O(n)$, as we can charge such a cell to the disc $d$ whose bounding circle forms its outer boundary. It is easily seen that in this charging scheme, a vertex is charged at most four times, since it can belong to the boundary of at most four cells. Note also that every charged vertex is contained in at most $k$ discs of $D$ and therefore belongs to $S_{\leq k}(D)$. Thus $|E_{\leq k}(D)| \leq |F_{\leq k}(D)| \leq 4 \cdot S_{\leq k}(D) + n = O(k^2\mathcal{U}(\frac{n}{k}) + n)$. By the result of [7], $\mathcal{U}(n) \leq 6n - 12$. Thus $|E_{\leq k}(D)| \leq O(nk)$. This completes the proof of the lemma. □

**Lemma 2.3.** *Let $D$ be a set of $n$ planar discs, and let $\ell > 1$ be an integer. Then $D$ can be colored with $O(\ell^2)$ colors such that all hyperedges in $E_{\leq \ell}(D)$ are conflict-free.*

*Proof.* Lemma 2.2 implies that there exists a constant $c$ such that for any finite collection of discs $D$ there exists a disc $d \in D$ which belongs to at most $c\ell^2$ hyperedges in $E_{\leq \ell}(D)$. Indeed, the sum $\sum_{s \in D} deg(s) \leq \ell\,|E_{\leq \ell}(D)| = O(\ell^2 n)$, where $deg(s)$ (*degree* of $s$) is the number of hyperedges of $E_{\leq \ell}(D)$ that contain $s$. Thus, the average degree is at most $c\ell^2$ for some constant $c$, and therefore there must exist a disc with degree at most $c\ell^2$. We show that $c\ell^2 + 1$ colors suffice for our coloring. We proceed by induction on the number $n$ of discs in $D$. Let $d \in D$ be a disc that belongs to at most $c\ell^2$ hyperedges of $E_{\leq \ell}(D)$. By the induction hypothesis, the set $D \setminus \{d\}$ can be colored with at most $c\ell^2 + 1$ colors such that all hyperedges in $E_{\leq \ell}(D \setminus \{d\})$ are conflict-free. For each of these sets $S \in E_{\leq \ell}(D \setminus \{d\})$ there is at least one color $c(S)$ that is unique in $S$. If we assign $d$ a color distinct from $c(S)$, then the set $S \cup \{d\}$ is also conflict-free. Since $d$ belongs to at most $c\ell^2$ sets in $E_{\leq \ell}(D)$, we can color $d$ with a color distinct from all the unique colors found in $E_{\leq \ell}(D \setminus \{d\})$ such that the coloring of $E_{\leq \ell}(D)$ is conflict-free. This completes the proof of the lemma. □

**Lemma 2.4.** *Let $D$ be a set of discs such that every disc intersects at most $k$ other ones. Then there is a constant $A$ such that $D$ can be colored with two colors (red and blue) and such that for every face $f \in \mathcal{A}(D)$ with depth at least $A \ln k$, there are at least $\frac{|d(f)|}{3}$ red discs containing $f$ and at least $\frac{|d(f)|}{3}$ blue discs containing $f$, where $d(f)$ is the set of all discs containing the face $f$.*

*Proof.* Consider a random coloring of the discs in $D$, where each disc $d \in D$ is colored independently red or blue with equal probability. For a face $f$ of the arrangement $\mathcal{A}(D)$ with $|d(f)| \geq A \ln k$, let $A_f$ denote the "bad" event that either less than $\frac{|d(f)|}{3}$ of the discs in $d(f)$ or more than $\frac{2|d(f)|}{3}$ of them are colored blue. By the Chernoff inequality (see, e.g., [1]) we have:

$$Pr[A_f] \leq 2e^{-\frac{|d(f)|}{72}} \leq 2e^{-\frac{A \ln k}{72}}$$

We claim that for every face $f$, the event $A_f$ is mutually independent of all but at most $O(k^3)$ other events. Indeed $A_f$ is independent of all events $A_s$ for which $d(s) \cap d(f) = \emptyset$. By assumption, $|d(f)| \leq k+1$. Observe also that a disc that contains $f$, can contain at most $k^2$ other faces, simply because the arrangement of $k$ discs consists of at most $k^2$ faces. Hence, the claim follows.

Let $A$ be a constant such that:

$$e \cdot 2e^{-\frac{A \ln k}{72}} \cdot 2k^3 < 1$$

By the Lovasz' Local Lemma, (see, e.g., [1]) we have:

$$Pr[\bigwedge_{|d(f)| \geq A \ln k} \bar{A}_f] > 0$$

In particular, this means that there exists a coloring for which every face $f$ with $|d(f)| \geq A \ln k$ has at least $\frac{|d(f)|}{3}$ red discs containing $f$ and at least $\frac{|d(f)|}{3}$ blue discs containing it, as asserted. This completes the proof of the lemma. □

**Proof of Theorem 2.1:** Consider a coloring of $D$ by two colors as in Lemma 2.4. Let $B_1$ denote the set of discs in $D$ colored blue. We will color the discs of $B_1$ with $O(\ln^2 k)$ colors such that $E_{\leq 2A \ln k}(B_1)$ is conflict-free, and recursively color the discs in $D \setminus B_1$ with colors disjoint from those used to color $B_1$. This is done again by splitting the discs into a set of red discs and a set $B_2$ of blue discs with the properties guaranteed by Lemma 2.4. We repeat this process until every face of the arrangement $\mathcal{A}(D')$ (of the set $D'$ of all remaining discs) has depth at most $A \ln k$. At that time, we color $D'$ with $O(\ln^2 k)$ colors as described in Lemma 2.3. We claim that this coloring scheme uses only $O(\ln^3 k)$ colors and is indeed conflict-free. To see that this coloring scheme is a valid conflict-free coloring, consider a point $p \in \bigcup_{d \in D} d$. Let $i$ be the largest index for which $d(p) \cap B_i \neq \emptyset$. If $i$ does not exist (namely, $d(p) \cap B_i = \emptyset \ \forall i$) then by Lemma 2.4 $|d(p)| \leq A \ln k$. However, this means that $d(p) \in E_{\leq A \ln k}$ and thus $d(p)$ is conflict-free by the coloring of the last step. If $|d(p) \cap B_i| \leq 2A \ln k$ then by the way we colored $B_i$, $d(p)$ is conflict-free. Assume then, that $|d(p) \cap B_i| > 2A \ln k$. Let $x$ denote the number of discs containing $p$ at step $i$. By the property of the coloring of step $i$, we have that $x \geq 3A \ln k$. This means that after removing $B_i$, the face containing $p$ is also contained in at least $A \ln k$ other discs. Hence, $p$ must also belong to a disc of $B_{i+1}$, a contradiction to the maximality of $i$. To argue about the number of colors used by the above procedure, note that in each prune step, we reduce the depth of every face with depth $i \geq A \ln k$ by a factor of at least $\frac{2}{3}$. We started with a set of discs such that the maximal depth is $k+1$. After the first step, the maximal depth is $\frac{2}{3}k$ and for each step we used $O(\ln^2 k)$ colors so, in total we have that the maximum number of colors $f(k, r)$, needed for CF-coloring a family of discs with maximum depth $r$ such that each disc intersects at most $k$ others satisfies the recursion:

$$f(k, r) \leq O(\ln^2 k) + f(k, \frac{2}{3}r).$$

This gives $f(k, r) = O(\ln^2 k \log r)$. Since, in our case $r \leq k + 1$, we obtain the asserted upper bound. This completes the proof of the theorem. □

# 3 Discussion

- Theorem 2.1 works almost verbatim for any family of regions (not necessarily convex) with linear union complexity. Thus, for example, our result applies to families of pseudo-discs (i.e., a family of simple Jordan regions, each pair of whose boundaries intersect at most twice), since pseudo-discs have linear union complexity (see, e.g., [7]).

- The proof of Theorem 2.1 is non-constructive since it uses the Lovasz' Local Lemma. However, we can use the known algorithmic versions of the Local Lemma (c.f., e.g., [1], Chapter 5) to obtain a constructive proof of Theorem 2.1[1].

- As mentioned in the Introduction, the only lower bound we have for the problem studied here is $\Omega(\log k)$ which is obvious from taking the lower bound construction of [3] with $k$ discs. It would be interesting to close the gap between this lower bound and the upper bound $O(\log^3 k)$ obtained in this paper.

- Another interesting open problem is to obtain a CF-coloring of discs with maximum depth $k + 1$ (i.e., no point is covered by more than $k + 1$ discs) with only polylogarithmic (in $k$) many colors. Obviously, the assumption of this paper that a disc can intersect at most $k$

[1] In the constructive version we get a weaker constant in the $O(\ln^3 k)$ bound.

others is much stronger and implies maximum depth $k+1$. However, the converse is not true. Assuming only bounded depth does not imply the former. In bounded depth, we still might have discs intersecting many (possibly all) other discs.

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