Triangle-free graphs of diameter 2

Noga Alon, Princeton, nalon@math.princeton.edu

Abstract

We show that for any $0 < \varepsilon < 1/6$ and $n > n_0(\varepsilon)$, one can add to any triangle-free graph on n vertices with maximum degree at most $n^{1/2-\varepsilon}$ less than $3n^{2-\varepsilon}$ edges, transforming it to a triangle-free graph with diameter 2. This settles, in a strong form, an open problem of Erdős and Gyárfás.

1 Introduction

The following problem was raised by Erdős and Gyárfás ([6], see also [2], Problem number 134).

Problem 1.1. *Let $\varepsilon, \delta > 0$ be two fixed positive reals, and suppose n is large as a function of ε, δ. Let G be a triangle-free graph on n vertices with maximum degree smaller than $n^{1/2-\varepsilon}$. Can G be made into a triangle-free graph with diameter 2 by adding at most δn^2 edges ?*

They proved that the conclusion holds if the maximum degree is at most $\log n / \log \log n$. It is mentioned in [6] that Simonovits showed that this does not necessarily hold if the maximum degree is $C\sqrt{n}$ for some large fixed C. In fact, maximum degree $(1+o(1))\sqrt{n/2}$ suffices as shown by the incidence graph of the lines and points of a projective plane of order p. This is a bipartite $(p+1)$-regular (triangle-free, of course) graph on $n = 2(p^2+p+1)$ vertices, where p is a prime power. Any two vertices of the same vertex class in this graph have a common neighbor. Therefore one cannot add any edge connecting two vertices of the same vertex class without creating a triangle. It follows that in order to reduce the diameter to 2 one must add all missing edges between pairs of nonadjacent vertices that do not lie in the same vertex class. The number of these missing edges is $(1/4 - o(1))n^2 = \Omega(n^2)$.

We describe a short proof of the following, which settles the Erdős-Gyárfás problem in a strong form. Here and in what follows we do not make any effort to optimize the absolute constants.

Theorem 1.2. *Let $G = (V, E)$ be a triangle-free graph with n vertices and maximum degree $d \le c(n)\sqrt{n}$, where*

$$2\frac{(\log n)^{1/3}}{n^{1/6}} \le c = c(n) \le \frac{1}{10}$$

and n is sufficiently large. Then one can add to G at most $2.5cn^2$ edges and get a triangle-free graph of diameter 2.

Note that by taking $c(n) = n^{-\varepsilon}$ the above theorem implies that if in the Erdős-Gyárfás problem the maximum degree is at most $n^{1/2-\varepsilon}$ then it suffices to add at most $O(n^{2-\varepsilon})$ edges.

Another problem of Erdős and Pach dealing with triangle-free graphs appears right before the problem above in [6], see also [2], Problem 133.

Problem 1.3. *Let $f(n)$ denote the smallest integer for which there is a triangle-free graph G on n vertices, diameter 2 and maximum degree $f(n)$. What is the order of growth of $f(n)$?*

Erdős and Pach conjectured that $f(n)/\sqrt{n}$ tends to infinity as n tends to infinity. Erdős also mentions in [6] that Simonovits observed that the Kneser graph $K(3m-1, m)$ whose vertices are all m-subsets of a $(3m-1)$-element set, in which two vertices are connected iff the corresponding subsets are disjoint, shows that for infinitely many values of n, $f(n) \leq n^{1-c}$ for some fixed $c > 0$. Indeed, this construction shows that

$$f(n) \leq n^{(1+o(1))\frac{2}{3H(1/3)}} = n^{0.7182..},$$

where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function. Erdős adds that it is not impossible that this graph gives the smallest possible value of $f(n)$. We observe that this is not the case.

Fact 1.4. *In the notation above, $f(n) \leq O(\sqrt{n \log n})$*

Problem 1.3 remains open. Clearly $f(n) \geq (1-o(1))\sqrt{n}$ as any graph with maximum degree f and diameter 2 can have at most $1 + f + f(f-1) = f^2 + 1$ vertices. We suspect that $f(n)$ is $O(\sqrt{n})$ and maybe even $f(n) = (1+o(1))\sqrt{n}$. It is known that for $r \in \{1, 2, 3, 7\}$ and possibly also for $r = 57$ there is an r-regular triangle-free graph of diameter 2 with $r^2 + 1$ vertices. Although the Hoffman-Singleton Theorem [10] asserts that there are no such graphs for additional values of r, it is possible that there are triangle-free graphs with maximum degree r and $(1-o(1))r^2$ vertices for infinitely many values of r.

2 Proofs

Proof of Theorem 1.2. Let $G = (V, E)$, d and $c = c(n)$ be as in the statement of the theorem. Throughout the proof we assume, whenever this is needed, that $n > n_0$ where n_0 is a sufficiently large constant. We first apply (a variant of) the triangle-free process for $m = c^2 n^{3/2}$ steps as follows. Starting with $G = G_0$, in each step i for $1 \leq i \leq m$ let G_i be obtained from G_{i-1} by adding a random edge chosen uniformly among all pairs of nonadjacent vertices of G_{i-1} that are both of degree smaller than $2c\sqrt{n}$ and that do not have a common neighbor. Note that by construction the maximum degree of G_m (and hence of all the graphs during the process) is at most $2c\sqrt{n}$. In addition, by construction, G_m (and all graphs during the process) are triangle-free.

Claim : With high probability G_m does not contain an independent set of size $5cn$.

Proof: Fix an independent set U of $5cn$ vertices of $G = G_0$. We estimate the probability that it stays independent in G_m. Since the maximum degree in each G_i is at most $2c\sqrt{n}$, the number of pairs of vertices in U that have a common neighbor is at most

$$n\binom{2c\sqrt{n}}{2} < 2c^2n^2.$$

In addition, the total number of vertices whose degrees have been increased already to $2c\sqrt{n}$ is at most $2cn$ (since the total number of edges added is at most $c^2n^{3/2}$ so the graph consisting of all added edges can have at most $2cn$ vertices of degree at least $c\sqrt{n}$). If follows that in every G_i during the process there are at least

$$\binom{|U|-2cn}{2} - 2c^2n^2 > 2c^2n^2$$

pairs of vertices of U that are of degree smaller than $2c\sqrt{n}$ and that do not have a common neighbor. Each such pair can be chosen as the selected random edge in each step, and the probability none of these edges have been chosen during the process is at most

$$(1-4c^2)^m \le e^{-4c^4n^{3/2}}.$$

There are at most

$$\binom{n}{5cn} \le 2^{H(5c)n} < 2^{10c\log(1/c)n}$$

possible sets U, where H is the binary entropy function. Our choice of c ensures that

$$2^{10c\log(1/c)n}e^{-4c^4n^{3/2}} = o(1).$$

The assertion of the claim thus follows by the union bound.

Returning to the proof of the theorem, fix a graph G_m satisfying the conclusion of the claim and add to it, repeatedly, edges to make it a maximal (with respect to containment) triangle-free graph. In other words, as long as there is a pair of nonadjacent vertices with no common neighbor, add the edge connecting them. This creates a triangle-free graph G' of diameter 2, and its independence number is at most that of G_m, which is smaller than $5cn$, by the claim. This implies that the maximum degree in G' is smaller than $5cn$, as the set of all neighbors of any vertex is an independent set. Therefore G' contains at most $2.5cn^2$ edges, completing the proof of Theorem 1.2. □

Proof of Fact 1.4. By the known results about the Ramsey number $R(3,t)$ proved in [3], [7] (improving the constant in the earlier estimate of Kim [11]) there is a triangle-free graph G on n vertices with independence number at most $(\sqrt{2}+o(1))\sqrt{n\log n}$. Starting with this graph, as long as it has a nonadjacent pair of vertices with no common neighbor, add an edge connecting this pair. The resulting graph at the end of this process is a triangle-free graph on n vertices with independence number at most $(\sqrt{2}+o(1))\sqrt{n\log n}$. This implies that its maximum degree is at most $(\sqrt{2}+o(1))\sqrt{n\log n}$, since the neighborhood of any vertex is an independent set. The assertion of the fact follows. □

Remark: After posting this note I learned from Ishay Haviv that Problem 1.3 has been solved up to a constant factor already in the 80s by Hanson and Seyffarth [10] who proved that $f(n) \leq 2\sqrt{n}$. Additional constructions appear in [4], [9]. The problem of deciding whether or not $f(n) = (1+o(1))\sqrt{n}$ remains open. The constructions in all three papers above are Cayley graphs of Abelian groups, and it is easy to see that using such a construction cannot provide graphs with maximum degree smaller than $(\sqrt{2}+o(1))\sqrt{n}$. A better upper estimate of $(2/\sqrt{3}+o(1))\sqrt{n}$ is described in [5]. See also [1] for some related constructions.

Acknowledgment I thank Ishay Haviv for telling me about the results in [8], [4], [9].

References

[1] Noga Alon, Paul Erdös, Ron Holzman and Michel Krivelevich, On k-saturated graphs with restrictions on the degrees, J. Graph Theory 23 (1996), 1-20.

[2] Thomas Bloom, Erdős' Open Problems, https://www.erdosproblems.com/all

[3] Tom Bohman and Peter Keevash, Dynamic concentration of the triangle-free process, Random Structures Algorithms 58 (2021), no. 2, 221–293.

[4] W. E. Clark and J. Pedersen, Sum-free sets in vector spaces over GF(2), Journal of Combinatorial Theory, Series A 61 (1992), 222–229.

[5] Zoltán Füredi and Akos Seress, Maximal triangle-free graphs with restrictions on the degrees. J. Graph Theory 18 (1994), no. 1, 11–24.

[6] Paul Erdős, Some old and new problems in various branches of combinatorics, Discrete Math. 165/166 (1997), 227–231.

[7] Gonzalo Fiz Pontiveros, Simon Griffiths and Robert Morris, The triangle-free process and the Ramsey number $R(3,k)$, Mem. Amer. Math. Soc. 263 (2020), no. 1274, v+125 pp.

[8] D. Hanson and K. Seyffarth, k-saturated graphs of prescribed maximum degree, Congressus Numerantium 44 (1984), 127–138.

[9] Ishay Haviv and Dan Levy, Symmetric complete sum-free sets in cyclic groups, Israel J. Math. 227 (2018), 931–956.

[10] Alan J. Hoffman an Robert R. Singleton, Moore graphs with diameter 2 and 3, IBM Journal of Research and Development, 5 (1960), 497–504.

[11] Jeong Han Kim, The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$, Random Structures Algorithms 7 (1995), no. 3, 173–207.