Two notions of unit distance graphs

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Abstract

A *faithful (unit) distance graph* in $\mathbb{R}^d$ is a graph whose set of vertices is a finite subset of the d-dimensional Euclidean space, where two vertices are adjacent if and only if the Euclidean distance between them is exactly 1. A *(unit) distance graph* in $\mathbb{R}^d$ is any subgraph of such a graph.

In the first part of the paper we focus on the differences between these two classes of graphs. In particular, we show that for any fixed d the number of faithful distance graphs in $\mathbb{R}^d$ on n labelled vertices is $2^{(1+o(1))dn\log_2 n}$, and give a short proof of the known fact that the number of distance graphs in $\mathbb{R}^d$ on n labelled vertices is $2^{(1-1/\lfloor d/2\rfloor+o(1))n^2/2}$. We also study the behavior of several Ramsey-type quantities involving these graphs.

In the second part of the paper we discuss the problem of determining the minimum possible number of edges of a graph which is not isomorphic to a faithful distance graph in $\mathbb{R}^d$.

1 Introduction

1.1 Background

We study the differences between the following two well-known notions of (unit) distance graphs:

Definition 1. A graph $G = (V, E)$ is a *(unit) distance graph in* $\mathbb{R}^d$*,* if $V \subset \mathbb{R}^d$ and $E \subseteq \{(x, y) : x, y \in V, |x - y| = 1\}$, where $|x - y|$ denotes the Euclidean distance between x and y.

Definition 2. A graph $G = (V, E)$ is a *faithful (unit) distance graph in* $\mathbb{R}^d$*,* if $V \subset \mathbb{R}^d$ and $E = \{(x, y) : x, y \in V, |x - y| = 1\}$.

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We say that a graph G is realized as a (faithful) distance graph in $\mathbb{R}^d$, if it is isomorphic to some (faithful) distance graph in $\mathbb{R}^d$. Denote by $\mathcal{D}(d)$ ($\mathcal{D}_n(d)$) the set of all labelled distance graphs in $\mathbb{R}^d$ (of order n). Similarly, denote by $\mathcal{FD}(d)$ ($\mathcal{FD}_n(d)$) the set of all labelled faithful distance graphs in $\mathbb{R}^d$ (of order n).

Distance graphs appear in the investigation of two well-studied problems. The first is the problem of determining the chromatic number $\chi(\mathbb{R}^d)$ of the d-dimensional space:

$$\chi(\mathbb{R}^d) = \min\{m \in \mathbb{N} : \mathbb{R}^d = H_1 \cup \ldots \cup H_m : \forall i, \forall x, y \in H_i, \quad |x - y| \neq 1\}.$$

The second is the investigation of the maximum possible number $f_2(n)$ of pairs of points at unit distance apart in a set of n points in the plane $\mathbb{R}^2$. Distance graphs arise naturally in the context of both problems. Indeed,

$$\chi(\mathbb{R}^d) = \max_{G \in \mathcal{D}(d)} \chi(G) = \max_{G \in \mathcal{FD}(d)} \chi(G),$$
$$f_2(n) = \max_{G \in \mathcal{D}_n(2)} |E(G)| = \max_{G \in \mathcal{FD}_n(2)} |E(G)|.$$

Thus, in the study of these two extremal problems it does not matter whether we consider distance graphs or faithful distance graphs. However, there is a substantial difference between the sets $\mathcal{D}(d)$ and $\mathcal{FD}(d)$. This difference is discussed in the theorems that appear in what follows.

1.2 The main results

The first theorem provides some classes of graphs that are (or are not) distance or faithful distance graphs in $\mathbb{R}^d$. A surprising aspect of Theorem 1.1 is that for any d there are bipartite graphs that are not faithful distance graphs in $\mathbb{R}^d$.

Theorem 1.1. *1. Any d-colorable graph can be realized as a distance graph in $\mathbb{R}^{2d}$.*

2. Let $d \in \mathbb{N}$ and $d \geqslant 4$. Consider the graph $K' = K_{d,d} - H$, where H is a matching of size $d - 3$. Then the graph K' is not realizable as a faithful distance graph in $\mathbb{R}^d$.

3. Any bipartite graph with maximum degree at most d in one of its parts so that no three vertices of degree d in this part have exactly the same set of neighbors is realizable as a faithful distance graph in $\mathbb{R}^d$.

The next theorem shows that in any dimension d there are far more distance graphs than faithful distance graphs.

Theorem 1.2. *1. For any $n, d \in \mathbb{N}, n \geqslant 2d$, we have $|\mathcal{FD}_n(d)| \leqslant \binom{n(n-1)}{nd}$. Therefore, for any fixed d, $|\mathcal{FD}_n(d)| = 2^{(1+o(1))dn \log_2 n}$.*

2. (A. Kupavskii, A. Raigorodskii, M. Titova, [6]). For any fixed $d \in \mathbb{N}$ we have $|\mathcal{D}_n(d)| = 2^{\left(1 - \frac{1}{[d/2]} + o(1)\right)\frac{n^2}{2}}$.

By simple calculations one can obtain the following corollary from the upper bound in part 1 of Theorem 1.2:

Corollary 1.3. *1. If $d = d(n) = o(n)$, then we have $|\mathcal{FD}_n(d)| = 2^{o(n^2)}$.*

2. If $d = d(n) \leqslant cn$, where $0 < c < 1/2$ and $H(c) < 1/2$ (here $H(z) = -z\log_2 z - (1-z)\log_2(1-z)$ is the binary entropy function), then there exists a constant $c' = c'(c) < 1/2$ such that $|\mathcal{FD}_n(d)| \leqslant 2^{c'n^2(1+o(1))}$.

As proved by B. Bollobás [2], with high probability the random graph $G(n, 1/2)$ has chromatic number $(1+o(1))\frac{n}{2\log_2 n}$. By part 1 of Theorem 1.1, any k-colorable graph is realizable as a distance graph in $\mathbb{R}^{2k}$. It means that if $d = d(n) \geqslant c\frac{n}{\log_2 n}$, where $c > 1$, then $|\mathcal{D}_n(d)| = (1+o(1))2^{\frac{n(n-1)}{2}}$. In other words, for such d almost every graph on n vertices can be realized as a distance graph in $\mathbb{R}^d$. This is very different from the behaviour of faithful distance graphs, as shown in Corollary 1.3.

We next consider the following extremal problem.

Problem 1. *Determine the minimum possible number $g(d)$ of edges of a graph G which is not realizable as a faithful distance graph in $\mathbb{R}^d$.*

An intriguing question here is whether or not for any $d \geqslant 4$, $g(d) = \binom{d+2}{2}$. In other words, does K_{d+2} have the minimal number of edges among the graphs that are not realizable as faithful distance graphs in $\mathbb{R}^d$, $d \geqslant 4$. Interestingly, this is not the case in $\mathbb{R}^3$, since the graph $K_{3,3}$ is not realizable as a faithful distance graph in $\mathbb{R}^3$ and it has fewer edges than K_5.

We restrict our attention here to bipartite graphs, studying the following problem.

Problem 2. *Determine the minimum possible number $g_2(d)$ of edges of a bipartite graph K which is not realizable as a faithful distance graph in $\mathbb{R}^d$.*

Note that the minimum number of vertices such a K can have equals $2d$, as follows from parts 2 and 3 of Theorem 1.1.

Theorem 1.4. *For any $d \geqslant 4$ we have $\binom{d+2}{2} \leqslant g_2(d) \leqslant \binom{d+3}{2} - 6$.*

Remark.

After completing this manuscript we learned that some of the questions discussed here have already been studied by Erdős and Simonovits in [4] and by Maehara in [7]. It seems that Maehara was unaware of the paper [4]. We proceed with a brief comparison between the the results of these two papers and our results here. Part 3 of Theorem 1.1 slightly improves the bipartite case of Theorem 2 from [7], which states the following: if a graph G has maximum degree d and $\chi(G) = k$, then G is faithful distance in $\mathbb{R}^D$, where $D = \binom{k}{2}(d+1)$. In part 2 of Theorem 1.1 we present a graph which is not realizable as a faithful distance graph in $\mathbb{R}^d$. Constructions of such graphs can be found in both papers [4] and [7]. The construction of Erdős and Simonovits (given in Proposition 1 of [4]) is similar to the construction we use, however, it is slightly worse in terms of the number of vertices and edges (the smallest known construction, which is used in the proof of the upper bound in Theorem 1.4, is a bipartite graph with parts $A = \{a_1, \dots, a_d\}, B = \{b_1, \dots, b_d\}$ and the set of edges $E = \{(a_i, b_j) : i > j\} \cup \{(a_i, b_j) : i \leqslant 3\}$). The graph used by Maehara is much bigger than both graphs used by us and by Erdős and Simonovits. Note that our graph is in some sense best possible, as follows from the assertion of part 3 of Theorem 1.1).

The main results of both papers [7] and [8] establish bounds on the dimension in which a graph can be realized as a faithful distance graph in terms of the maximum degree and the chromatic number (see Theorem 2.8 in the present paper). Similar bounds were already proved by Erdős and Simonovits (see Theorem 6 in [4]), and their bound differs from the bound of Rödl and Maehara only by 1 (Erdős and Simonovits prove that any graph with maximum degree k can be realized as a faithful distance graph in $\mathbb{R}^{2k+1}$, while Rödl and Maehara prove that such graphs can be realized in $\mathbb{R}^{2k}$, the proofs rely on similar constructions).

A question analogous to Problem 2, for distance graphs instead of faithful distance graphs, was asked in [4] (problem 5 in [4]).

1.3 More on the difference between distance and faithful distance graphs

We study two Ramsey-type quantities related to distance and faithful distance graphs.

Definition 3. *The (faithful) distance Ramsey number $R_D(s,t,d)$ $\big(R_{FD}(s,t,d)\big)$ is the minimum integer m such that for any graph G on m vertices the following holds: either G contains an induced s-vertex subgraph isomorphic to a (faithful) distance graph in $\mathbb{R}^d$ or its complement $\bar{G}$ contains an induced t-vertex subgraph isomorphic to a (faithful) distance graph in $\mathbb{R}^d$.*

The quantity $R_D(s,s,d)$ is studied in [6], where the following theorem is proved:

Theorem 1.5 (A. Kupavskii, A. Raigorodskii, M. Titova [6])**.** *1. For every fixed $d \in \mathbb{N}$ greater than 2 we have*

$$R_D(s,s,d) \geqslant 2^{\left(\frac{1}{2[d/2]}+o(1)\right)s}.$$

2. For any $d = d(s)$, where $2 \leqslant d \leqslant s/2$, we have

$$R_D(s,s,d) \leqslant d \cdot R\left(\left\lceil \frac{s}{[d/2]} \right\rceil, \left\lceil \frac{s}{[d/2]} \right\rceil\right),$$

where $R(k,\ell)$ is the classical Ramsey number, which is the minimum number n so that any graph on n vertices contains either a clique of size k or an independent set of size ℓ.

Note that it is in fact not difficult to improve the upper bound to

$$R\left(\left\lceil \frac{s}{[d/2]} \right\rceil, \left\lceil \frac{s}{[d/2]} \right\rceil\right) + 2s,$$

but for our purpose here this improvement is not essential and we thus do not include its proof.

By the previous theorem the bounds for $R_D(s,s,d)$ are roughly the same as for the classical Ramsey number $R\left(\left\lceil \frac{s}{[d/2]} \right\rceil, \left\lceil \frac{s}{[d/2]} \right\rceil\right)$:

$$\frac{s}{2[d/2]}(1+o(1)) \leqslant \log R_D(s,s,d) \leqslant \frac{2s}{[d/2]}(1+o(1)),$$

where the $o(1)$-terms tend to zero as s tends to infinity.

What can we say about $R_{FD}(s,s,d)$? It turns out that $R_{FD}(s,s,d)$ is far larger than $R_D(s,s,d)$. Using Theorem 1.2 we can prove the following result:

Proposition 1.6. *1. For any $d = o(s)$ we have $R_{FD}(s,s,d) \geqslant 2^{(1+o(1))s/2}$.*

2. For $d \leqslant cs$, where $c < 1/2$ and $H(c) < 1/2$, there exists a constant $\alpha = \alpha(c) > 0$ such that $R_{FD}(s,s,d) \geqslant 2^{(1+o(1))\alpha s}$.

It is worth mentioning that, if $d = cs$ for a sufficiently small $c > 0$, the quantity $R_{FD}(s,s,d)$ grows exponentially, while the quantity $R_D(s,s,d)$ grows linearly (this follows from part 2 of Theorem 1.5).

The final (possible) difference between $\mathcal{D}(d)$ and $\mathcal{FD}(d)$ we point out is the following. Fix an $l \in \mathbb{N}$ and consider the distance graphs from $\mathcal{D}(d)$ and $\mathcal{FD}(d)$ that have girth greater than l. What can we say about the chromatic number of such graphs?

Theorem 1.7 (A. Kupavskii, [5])**.** *For any $g \in \mathbb{N}$ there exists a sequence of distance graphs in $\mathbb{R}^d$, $d = 1, 2, \dots$ with girth greater than g, such that the chromatic number of the graphs in the sequence grows exponentially in d.*

In the analogous problem for faithful distance graphs the situation is less understood. All we can prove here is the following

Proposition 1.8. *For any $g \in \mathbb{N}$ there exists a sequence of faithful distance graphs in $\mathbb{R}^d$, $d = 1, 2, \dots,$ with girth greater than g such that the chromatic number of the graphs in the sequence grows as $\Omega_g\left(\frac{d}{\log d}\right)$.*

2 Proofs

2.1 Proof of Theorem 1.1

1. Two circles are orthogonal if they lie in orthogonal two-dimensional planes. Choose d pairwise orthogonal circles of radius $1/\sqrt{2}$ with a common center. The distance between any two points from different circles equals 1. Embed each color class into one circle.

2. Suppose the graph K' can be realized as a faithful distance graph in $\mathbb{R}^d$. Denote both the vertices of K' and the points in the space that correspond to them by the same letters. Let $A = \{a_i\}, B = \{b_i\}$ be the parts of K', where $|A| = |B| = d$, and the edges (a_i, b_i), where $i \in \{4, \dots, d\}$, are not present in the graph. Then a_1, a_2, a_3 are the vertices that are connected to all vertices of B. In any faithful distance realization of K' the vertices a_1, a_2, a_3 must be affinely independent. Indeed, they cannot lie on the same line since otherwise there will be no points at unit distance from all of them.

The points b_j must lie on the $(d-3)$-dimensional sphere S in the $(d-2)$-dimensional subspace, orthogonal to the plane containing a_1, a_2, a_3. The sphere S has the same center as the circle, circumscribed around the triangle a_1, a_2, a_3.

Now we show that b_i is affinely independent of the points $b_j, j < i$. For $i \leqslant 3$ this is clear for the same reason as for the points a_1, a_2, a_3. For $i \geqslant 4$, consider the sphere $S_i = S \cap \text{aff}\{b_j, j < i\}$, where by $\text{aff}\{x_1, \dots, x_l\}$ we denote the affine hull of the points $x_1, \dots, x_l$. The sphere S_i is contained in the sphere with the center in a_i and unit radius, because all points $b_j, j < i$, are connected to a_i. But if the point b_i lies in $\text{aff}\{b_j, j < i\}$, then $b_i \in S_i$. Thus, we are forced to draw an edge (a_i, b_i), which is forbidden.

Since each b_i is affinely independent of $b_j, j < i$, we obtain d affinely independent points in $\mathbb{R}^{d-2}$ — a contradiction.

3. Let K be an arbitrary bipartite graph with parts $A = \{a_i\}, B = \{b_j\}$ satisfying the condition $\max_i \deg(a_i) \leqslant d$, and such that no three vertices from A of degree d have the same set of neighbors. We introduce the following notation: a sphere S' is *complimentary* to the sphere S^f of dimension $f \leqslant (d-2)$ in the space $\mathbb{R}^d$ if S' is formed by all points of $\mathbb{R}^d$ that are at unit distance apart from the points of S^f. For a set of points X we use the notation $S(X)$ for the sphere of minimal dimension that contains all points from X (if one exists), and $S'(X)$ for the sphere, complimentary to $S(X)$ (again, if one exists).

We realize K as a faithful distance graph in $\mathbb{R}^d$. First embed all points of B in the space $\mathbb{R}^d$ so that the diameter of B is smaller than 1 and all points lie in a sufficiently general position:
(a) No k points of B lie in a $(k-2)$-dimensional plane , $k = 1, \dots, d$.
(b) No $d+1$ points lie on a unit sphere.
(c) There are no two subsets B_1, B_2 of B, both of size d, such that the distance between some of the points of $S'(B_1), S'(B_2)$ is 1 (note that $S'(B_i)$ consists of two points).
(d) There are no two subsets B_1, B_2 of B, such that $S'(B_1) \subset S(B_2)$. Moreover, if B_1 is of size d, then $S'(B_1) \cap S(B_2) = \emptyset$.
All the forbidden positions of the points from B may be expressed as zero sets of certain polynomials, so we can avoid all of them.

Next we embed the set A. Each point a_i is connected to $n_i \leqslant d$ points from B. Denote this set by B_i. By (a) the points from B_i form a $(n_i - 1)$-dimensional simplex with circumscribed sphere S of radius $r < 1/\sqrt{2}$. Consider a sphere $S'(B_i)$. The dimension of $S'(B_i)$ is $d - n_i \geqslant 0$.

First we embed all the points of A that have d neighbors in B one by one. For each such point a_i there are two possible points in $\mathbb{R}^d$ with which it can coincide and at most one of them is already occupied. Condition (b) guarantees that we do not get any extra edges between a and points not from B_i. Condition (c) guarantees that we cannot get an edge between the vertices $a_i, a_j \in A$ of degree d.

The remaining points of A can now be embedded, one by one, in the following way. We embed the point a_i onto S' in such a way that the distance between a_i and the points from $B \backslash B_i$ is not unit, a_i does not coincide or at unit distance apart from any previously placed a_j and a_i does not fall into $S(B_l)$ for any l. This can be done since any sphere of unit radius with center in any of the points from $B \backslash B_i$ can intersect $S'(B_i)$ only by a sphere $S'' \subset S'(B_i)$ of smaller dimension due to (a), and the same holds for spheres with centers in a_j due to the fact that no a_j fall into $S(B_l)$. This is, in turn, possible due to (d), out of which we get that for any k, l the sphere $S'(B_k) \cap S(B_l)$ is a sphere of strictly smaller dimension than $S'(B_k)$.

Remark. Condition (b) can be satisfied just by choosing points on a sphere of radius smaller than 1, and conditions (c), (d) can be satisfied by additionally requiring that for some small ε all the points are ε-flat with respect to some hyperplane γ, that is, all the hyperplanes determined by points of the set B form an angle with γ which is smaller than ε and all the pairwise distances between the points are at most εr, where r is the radius of the sphere on which the points lie. For (c) we additionally require that r is not close to $1/2$ in terms of ε. This will be used use in the proof of Theorem 1.4.

2.2 Proof of Theorem 1.2

1. Let $P_1, \dots, P_m$ be m real polynomials in l real variables. For a point $x \in \mathbb{R}^l$ the *zero pattern* of the P_j's at x is the tuple $(\varepsilon_1, \dots, \varepsilon_m) \in \{0,1\}^m$, where $\varepsilon_j = 0$, if $P_j(x) = 0$ and $\varepsilon_j = 1$ if $P_j(x) \neq 0$. Denote by $z(P_1, \dots, P_m)$ the total number of zero patterns of the polynomials $P_1, \dots, P_m$.

The upper bound in part 1 of the theorem is a corollary of the following proposition ([10, Theorem 1.3]):

Proposition 2.1 (L. Rónyai, L. Babai, M.K. Ganapathy [10])**.** *Let $P_1, \dots, P_m$ be m real polynomials in l real variables, $m \geqslant l$, and suppose the degree of each P_j does not exceed k. Then $z(P_1, \dots, P_m) \leqslant \binom{km-(k-2)\ell}{\ell}$.*

Associate a family of $n(n-1)/2$ polynomials P_{ij} in dn real variables with an arbitrary labelled distance graph G of order n in $\mathbb{R}^d$ as follows. Denote by $(v_1^i, \dots, v_d^i)$ the coordinates of the vertex v_i in the distance graph. For each unordered pair $\{i, j\}$ of vertices of the graph define a polynomial P_{ij} that corresponds to the square of the distance between the pair v_i, v_j minus 1:

$$P_{ij} = -1 + \sum_{r=1}^{d} (v_r^i - v_r^j)^2.$$

It is easy to see that each labelled distance graph in $\mathbb{R}^d$ corresponds to a zero pattern of the polynomials $P_{12}, \dots, P_{(n-1)n}$. It is also clear that different distance graphs correspond to different zero patterns, since this pattern specifies the set of (labelled) edges in the graph. Thus the number of labelled faithful distance graphs of order n in $\mathbb{R}^d$ is at most the number of zero patterns of the above polynomials. All we are left to do in order to establish the upper bound in part 1 is to substitute $k = 2, l = dn, m = \frac{n(n-1)}{2}$ in Proposition 2.1.

The lower bound follows from part 3 of Theorem 1.1 by taking a bipartite graph with classes of vertices B of size, say, $n/\log n$ and A of size $n - |B|$, so that any vertex of A has exactly d neighbors in B and no two vertices of A have exactly the same set of neighbors. This implies that

$$|\mathcal{FD}_n(d)| \geqslant |A|! \binom{\binom{|B|}{d}}{|A|},$$

supplying the desired asymptotic bound.

2. This was proved in [6]. Here we present a short proof of this fact using the following proposition from [6] and a theorem from [3]:

Proposition 2.2 (A. Kupavskii, A. Raigorodskii, M. Titova [6])**.** *The graph $K_{\underbrace{3, \dots, 3}_{[d/2]+1}}$ is not realizable as a distance graph in $\mathbb{R}^d$.*

Theorem 2.3 (P. Erdős, P. Frankl, V. Rödl, [3])**.** *Let G be a graph, $\chi(G) = r \geqslant 3$. Then the number $F_n(G)$ of labelled graphs of order n, not containing a copy of G, satisfies: $F_n(G) = 2^{\left(1-\frac{1}{r-1}+o(1)\right)\frac{n^2}{2}}$.*

Applying the above we get

$$|D_n(d)| \leqslant F_n\Bigg(K_{\underbrace{3,\ldots,3}_{[d/2]+1}}\Bigg) = 2^{\left(1-\frac{1}{[d/2]}+o(1)\right)\frac{n^2}{2}}.$$

On the other hand, by part 1 of Theorem 1.1

$$|D_n(d)| \geqslant 2^{\left(1-\frac{1}{[d/2]}+o(1)\right)\frac{n^2}{2}},$$

since any $[d/2]$-partite graph of order n is realizable as a distance graph in $\mathbb{R}^d$.

2.3 Proof of Theorem 1.4

We first prove the upper bound. It is easy to check that in the proof of part 2 of Theorem 1.1 the edges $(a_i, b_j), i, j > 3, i < j$ are not used, and thus their presence or absence does not affect the validity of the proof. In particular, the bipartite graph K'' with parts $A = \{a_1, \ldots, a_d\}, B = \{b_1, \ldots, b_d\}$ and the set of edges $E = \{(a_i, b_j) : i > j\} \cup \{(a_i, b_j) : i \leqslant 3\}$ is not realizable as a faithful distance graph in $\mathbb{R}^d$. The number of edges in this graph is $\binom{d+3}{2} - 6$, establishing the bound $g_2(d) \leqslant \binom{d+3}{2} - 6$.

Remark. The following bipartite graph is also not realizable as a faithful distance graph in $\mathbb{R}^d$: $H = (A \cup B, E),\ A = \{a_1, \ldots, a_{d+2}\}, B = \{b_1, \ldots, b_{d+2}\},\ E = \{(a_i, b_j) : i \geqslant j\}$. This graph has $\binom{d+3}{2}$ edges.

We proceed with the proof of the lower bound. Consider a bipartite graph $G = (A \cup B, E)$, where $A = \{a_1, \ldots, a_{n+s}\},\ B = \{b_1, \ldots, b_m\}$. Suppose that the vertices of A are ordered in such a way that $\deg(a_i) \leqslant \deg(a_j)$ if $i < j$. Suppose also that $\deg(a_{n+1}) = \deg(a_{n+s}) = m, \deg(a_n) < m$. We provide sufficient conditions for G to be realizable as a faithful distance graph in $\mathbb{R}^d$. Recall the following notation: a sphere S' is *complimentary* to a sphere S^f of dimension $f \leqslant (d-2)$ in the space $\mathbb{R}^d$, if S' is formed by all points of $\mathbb{R}^d$ that are at unit distance apart from the points of S^f.

Here is an outline of the proof. The general goal is to find a good realization for the set B. Having such a realization, the vertices of A will be placed on the corresponding complimentary spheres using a general position argument, in a similar way to that described in the proof of part 3 of Theorem 1.1. To find the realization, we want the points of B to satisfy the analogues of the conditions (a), (b), (c), (d). Conditions (b), (c), (d) are technical and can be satisfied without difficulties in this case (see the remark following the proof of Theorem 1.1). The main difference is concerning condition (a): we cannot simply place all vertices of B in a sufficiently general position, since there may be vertices in A of a very high degree, and we may get unexpected edges.

In most cases we try to place the vertices of B on a sphere, and, as we have already seen in the proof of part 2 of Theorem 1.1, if the vertices of a part of a bipartite graph lie on a sphere then there is a tight connection between the presence of certain edges and the affine independence of certain vertices.

We begin with all points of B on the circle and start to modify the realization so that it is getting closer and closer to the desired one. To be more precise, we treat each vertex v of A as a condition on vertices of B, which states that the vertices not connected to v must be affinely independent from the vertices that are connected to v. We consider the vertices one by one in an increasing order according to the degree. Suppose that at some step the degree of the vertex considered v from A is D, and the dimension of the sphere on which B currently lies is d. If $D > d$, then we add one dimension and move points not connected to v into that new direction. If $D \leqslant d$, then we do not add the dimension and rearrange all the points of B, so that they now lie in a general position on the sphere. In both cases the condition is satisfied. We also keep track of the "ε-flat" condition from the remark, moving vertices into the new direction just a little. This approach allows us to estimate the total number of edges in the graph needed so that this algorithm ends with a sphere of dimension at least d. At the last step we embed all the vertices from A, and the fact that all the conditions are satisfied allows us to get exactly the edges needed.

We proceed with the detailed proof. We treat vertices $a_{n+1}, \ldots, a_{n+s}$ separately, so we have cases depending on s. First we find a specific realization of the set B on a k-dimensional sphere S^k_r of radius r. The dimension k and radius r will be defined later. Define the set system $\mathcal{H}$, $\mathcal{H} = \{H_1, \ldots, H_n\}$, where the set H_i is the subset of indices of vertices from B that are connected to the vertex a_i. Note that some sets may coincide, and that $|H_i| \leqslant |H_j|$, $i \leqslant j$.

Let $X = \{x_1, \ldots, x_m\}$ be a set of points in $\mathbb{R}^d$. We will denote by l-condition the condition $x_i \notin \operatorname{aff}\{x_j : j \in H_l\}$ for all $i \notin H_l$. By $\mathcal{H}_l$ we denote the set of all i-conditions, $i \leqslant l$. We use the notation $(B, \mathcal{H})$ for the set B of vertices and the set $\mathcal{H}$ of conditions (here we are slightly abusing notation, identifying sets of indices with the conditions they impose). We say that $(B, \mathcal{H})$ is realizable in S^d, if there is a set of distinct points $X = \{x_1, \ldots, x_m\} \subset S^d$ such that X satisfies all the conditions from $\mathcal{H}$ and such that X is ε-flat with respect to some plane that passes through the center of S^d for some small fixed ε, say, $\varepsilon = 0.01$.

Choose k to be the minimal dimension such that $(B, \mathcal{H})$ is realizable in S^k.

Next, we find a faithful distance realization for G. We consider several cases.

$\boldsymbol{s \geqslant 3}$ **:** Fix some $0 < r < 1$ and find a realization of $(B, \mathcal{H})$ on the sphere S^k_r described above. We have to find a proper point y_i for each vertex a_i from A. Embed the points $y_{n+1}, \ldots, y_{n+s}$ on the complimentary sphere S of S^k_r. Next, choose $y_i, i \leqslant n$, on the complimentary sphere S_i to the minimal sphere that contains the points y_j, $j \in H_i$. It is clear that if the dimensions of S, S_i are at least 1 (they are at least circles), then, using a standard general position argument, we can find distinct points y_i that are at unit distance apart precisely from the points $x_j, j \in H_i$. Indeed, in this case we do not need conditions (b), (c) at all since all the vertices of A have at least a one-dimensional sphere as a possible position, and condition (d) is satisfied due to the ε-flatness. It follows that if $d \geqslant k + 3$, then we can find the desired realization.

On the other hand, if $d \leqslant k + 2$ then it is clear that there is no faithful distance realization of G in $\mathbb{R}^d$. Indeed, the points $y_{n+1}, y_{n+2}, y_{n+3}$ are in general position, and all the points of X lie on the $(d-3)$-dimensional sphere, complimentary to the circle circumscribed around $y_{n+1}, y_{n+2}, y_{n+3}$. But there is no such realization of $(B, \mathcal{H})$, since $d - 3 < k$.

$\boldsymbol{s = 2}$ **:** This case is similar to the case $s \geqslant 3$, with the only difference that we need S to be zero-dimensional. We additionally require $r < 1/2$, so that the diameter of S is bigger than

1, and the condition (c) is satisfied. Condition (b) is again redundant, since we may need it only for the vertices $a_1, \dots, a_n$, and they have an at least one-dimensional sphere as a possible position. As a result, we need the following inequality: $d \geqslant k+2$. This bound is tight for the same reasons.

$\boldsymbol{s = 0}$ **or** $\mathbf{1}$ **:** We find a realization of $(B, \mathcal{H})$ on the sphere S_1^k. If $s = 1$, then we place y_{n+1} in the center of the sphere S_1^k. The rest of the points y_i are placed almost as in the case $s \geqslant 3$. We have to make sure that no plane $\operatorname{aff}\{x_j : x_j \in H_i\}$ contains the center of the sphere S_1^k, otherwise there may be no room for y_i. The existence of such realization again follows from the general position argument. Then the conditions (c), (d) are satisfied. As for the condition (b), all the points from X lie on the unit sphere S_1^k, and any other unit sphere intersects S_1^k in a hypersphere, and since the affine independence conditions are satisfied, we do not get any extra edges between vertices a_i and b_j.

One would expect that in this case we can find a faithful distance realization of G if $d \geqslant k+1$. This is not exactly the case. If $d = k+1$ and for some i, j, l we have $H_i = H_j = H_l$ and $S_i = S_j = S_l$ consists of two points, then we cannot find room for all of a_i, a_j, a_l.

In this case we have to modify slightly the construction of X. We choose k to be a minimum dimension such that there are points $X = \{x_1, \dots, x_m\} \subset S_1^k$ that satisfy the conditions from $\mathcal{H}$ and are ε-flat. Suppose there exists a configuration X such that for all triples i_1, i_2, i_3, for which we have $H_{i_1} = H_{i_2} = H_{i_3}$, the dimension of the complimentary sphere to the sphere circumscribed around $x_j,\ j \in H_{i_1}$, is at least one. Then this X is the desired construction, and G is realizable as a faithful distance graph in $\mathbb{R}^{k+1}$. If not, then G is realizable in $\mathbb{R}^{k+2}$. This is tight for $s = 1$. It is unclear whether this is is tight for $s = 0$ or not, since the points of B need not lie on the sphere.

It seems hard to find the minimum dimension k in which we can realize $(B, \mathcal{H})$. But, nevertheless, we can use a simple realization algorithm that provides a relatively good upper bound on k, as describe next. Consider the conditions one by one and modify the set X so that it satisfies the conditions that were already considered. Next we describe the realization of $(B, \mathcal{H})$ on the k-dimensional sphere. Note that if we find a realization on the sphere of some radius, then, using homothety, we can change the radius to any desired prescribed positive value.

- In the zero step we take points $x_1^0, \dots, x_m^0$ in general position on the circle. No conditions are considered at this step.
- In step l we find such $X^l = \{x_j^l,\ j = 1, \dots, m\} \subset S^{k_l}$ that satisfies the conditions $\mathcal{H}_l$. In this step we get one additional condition (l-condition). We have two possibilities.

  $\mathbf{|H_l| \geqslant k_{l-1} + 1}$ If $|H_l| \geqslant k_{l-1} + 1$, then we put $k_l = k_{l-1} + 1$ and modify the set $X^{l-1} = \{x_1^{l-1}, \dots, x_m^{l-1}\}$ in the following way. Initially, the first $(k_l - 1)$ coordinates of x_i^l are just the coordinates of x_i^{l-1}, and we put the last coordinate of x_i^l to be equal to 0. If $i \in H_l$, then we rotate the point by the angle equal to $f(l, \varepsilon)$ into that new direction, and if $i \notin H_l$, then we rotate the point by the same angle into the opposite direction. In that case the l-condition is satisfied, and if we choose $|f(l, \varepsilon)|$ decreasing rapidly enough, then the ε-flatness condition is also satisfied.

$|\mathbf{H_l}| \leqslant \mathbf{k_{l-1}}$ Recall that $|H_i| \leqslant |H_j|$ if $i \leqslant j$. If $|H_l| \leqslant k_{l-1}$, then $|H_i| \leqslant k_{l-1}$, where $i \leqslant l$. We put $k_l = k_{l-1}$ and find a set of m points in S^{k_l} in general position that are $\varepsilon/2$-flat. Then the conditions from $\mathcal{H}_l$ are satisfied.

Using this algorithm we can estimate how many edges the graph G should have so that $(B, \mathcal{H})$ cannot be realized in S^k.

Lemma 2.4. *If $(B, \mathcal{H})$, $\mathcal{H} = \{H_1, \ldots, H_n\}$, cannot be realized on the sphere S^k then $\sum_{i=1}^{n} |H_i| \geqslant \binom{k+3}{2} - 3$.*

Proof. If $(B, \mathcal{H})$ cannot be realized in the sphere S^k, then it cannot be realized in S^k using the described algorithm. The proof is by induction. For $k = 1$ we need at least one set H_i to be of cardinality at least three, so the condition is satisfied. Consider a pair $(B, \mathcal{H})$ that cannot be realized in S^k using the described algorithm. Find the minimum l, $l < n$, such that $(B, \mathcal{H}_l)$ cannot be realized in S^{k-1}. Such l exists since at each step of the algorithm we increase the dimension by at most 1. By induction, $\sum_{i=1}^{l} |H_i| \geqslant \binom{k+2}{2} - 3$. Then, surely, $(B, \mathcal{H}_l)$ can be realized in S^k. Consequently, $|H_n| \geqslant k + 2$, otherwise $|H_i| \leqslant k + 1$, $i = 1, \ldots, n$, and $(B, \mathcal{H}_l)$ can be realized in S^k. Then $\sum_{i=1}^{n} |H_i| \geqslant \binom{k+2}{2} - 3 + (k + 2) \geqslant \binom{k+3}{2} - 3$. □

Modifying the proof slightly, we can get the following generalization:

Lemma 2.5. *Consider a sequence $|H_1|, \ldots, |H_n|$. Choose a subsequence $i_1 < \ldots < i_s$ of $1, \ldots, n$ of maximal length with the following properties: $|H_{i_j}| \geqslant j + 2, j = 1, \ldots, s$ and each i_j is the minimal number that satisfies this property. Then, if $s \leqslant k - 1$, $(B, \mathcal{H})$ is realizable in S^k.*

Next we return to the bipartite graph G. We want to estimate the number of edges G should have so that G is not realizable as a faithful distance graph in $\mathbb{R}^d$. We again consider several cases depending on s:

$\boldsymbol{s \geqslant 3}$ **:** In this case $(B, \mathcal{H})$ cannot be realized on S_r^{d-3}, so by Lemma 2.4 we have $\sum_{i=1}^{n} |H_i| \geqslant \binom{d}{2} - 3$. Moreover, $|H_{n+1}| = \ldots = |H_{n+s}| = m$, so $\sum_{i=n+1}^{n+s} |H_i| = sm \geqslant 3m$. But, on the other hand, $m \geqslant d$, since otherwise the graph G is realizable in $\mathbb{R}^d$ by part 3 of Theorem 1.1. So

$$\sum_{i=1}^{n+s} |H_i| \geqslant \binom{d}{2} - 3 + 3d = \binom{d+3}{2} - 6.$$

$\boldsymbol{s = 2}$ **:** In this case $(B, \mathcal{H})$ cannot be realized on S_r^{d-2}, so by Lemma 2.4 $\sum_{i=1}^{n} |H_i| \geqslant \binom{d+1}{2} - 3$. We have $m \geqslant d + 1$ since otherwise there are m points on S_r^{d-2} forming a simplex, and, consequently, satisfying the conditions $\mathcal{H}$. Similarly to the previous case we obtain

$$\sum_{i=1}^{n+2} |H_i| \geqslant \binom{d+1}{2} - 3 + 2(d + 1) = \binom{d+3}{2} - 4.$$

$\boldsymbol{s = 1}$ **:** In this case we have two possibilities. Assume first that for any triple i_1, i_2, i_3, for which we have $H_{i_1} = H_{i_2} = H_{i_3}$, we also have $|H_{i_3}| < d$. Then for any such i_1, i_2, i_3 and in any realization of the set B in the space $\mathbb{R}^d$ the dimension of the complimentary sphere to the sphere, circumscribed around $x_j,\ j \in H_{i_1}$ is at least 1. Consequently, $(B, \mathcal{H})$ cannot be realized on S_1^{d-1}, and by Lemma 2.4 we have $\sum_{i=1}^n |H_i| \geqslant \binom{d+2}{2} - 3$. Similarly to the previous case we obtain that $m \geqslant d+2$. So $\sum_{i=1}^{n+1} |H_i| \geqslant \binom{d+3}{2} - 3$.

Next we assume that there is a triple i_1, i_2, i_3 for which we have $H_{i_1} = H_{i_2} = H_{i_3}$ and $|H_{i_3}| \geqslant d$. Note that the pair $(B, \mathcal{H})$ can be realized on S_r^f if and only if the pair $(B, \mathcal{H}')$ can be realized on S_r^f, where $\mathcal{H}' = \{H_i,\ i = 1, \ldots, n,\ i \neq i_2, i \neq i_3\}$. On the other hand, $(B, \mathcal{H})$ cannot be realized on S_r^{d-2}, so $\sum_{i=1}^n |H_i| \geqslant \binom{d+1}{2} - 3 + 2|H_{i_2}|$. Again, $m \geqslant d+1$. So $\sum_{i=1}^{n+1} |H_i| \geqslant \binom{d+4}{2} - 8$.

$\boldsymbol{s = 0:}$ We again have two possibilities. If there is a triple i_1, i_2, i_3 for which we have $H_{i_1} = H_{i_2} = H_{i_3}$ and $|H_{i_3}| \geqslant d$, then we obtain $\sum_{i=1}^n |H_i| \geqslant \binom{d+1}{2} - 3 + 2|H_{i_2}| \geqslant \binom{d+3}{2} - 6$.

Suppose that for any triple i_1, i_2, i_3, for which we have $H_{i_1} = H_{i_2} = H_{i_3}$, we also have $|H_{i_3}| < d$. If we apply the previous technique directly, we obtain the bound $\sum_{i=1}^n |H_i| \geqslant \binom{d+2}{2} - 3$. To get a better bound we modify the realization algorithm. Note that in this case we have additional flexibility which is not taken into account by the algorithm: the algorithm produces a set X that lies on the sphere, and we do not need it in this case.

Suppose the set B contains a vertex, say b_1, of degree 3, which is connected to $a_{i_1}, a_{i_2}, a_{i_3}$. Then we exclude b_1 out of B, and apply the usual algorithm for $(B \backslash \{b_1\}, \mathcal{H})$, where $\mathcal{H}$ is modified in such a way that element $\{1\}$ is excluded out of its sets. Suppose this pair can be realized on S_r^{d-1}, where r is sufficiently small. Then we try to choose an appropriate position for the vertices $y_{i_1}, y_{i_2}, y_{i_3}$ so that $y_{i_1}, y_{i_2}, y_{i_3}$ do not lie on one line and form a triangle with a radius of a circumscribed circle less than one. We surely can guarantee that $y_{i_1}, y_{i_2}, y_{i_3}$ are in general position, if at least one of the spheres S_{i_j} (the geometric place of the point y_{i_j}), $j = 1, 2, 3$ is not zero-dimensional. Suppose all of them are zero-dimensional. It means that $|H_{i_j}| \geqslant d,\ j = 1, 2, 3$. We apply Lemma 2.5 and obtain that $\sum_{i=1}^n |H_i| \geqslant \binom{d+2}{2}$. Indeed, there are different cases, when some of $i_j, j = 1, 2, 3$, fall into the maximal sequence, and in any case it is clear provided that $d \geqslant 2$ (note that the optimal bound given in Lemma 2.4 can only be obtained when the sequence of $|H_i|$ is a progression $3, 4, \ldots, d+1$ and when all $|H_i|$ are present).

We choose points $y_{i_1}, y_{i_2}, y_{i_3}$ in general position and such that the plane $\operatorname{aff}\{y_{i_1}, y_{i_2}, y_{i_3}\}$ does not contain the center O of the sphere that contains all x_i. This is possible since the center of the sphere S_{i_j} coincides with the center of the sphere that contains $x_l, l \in H_{i_j}$, while the plane that contains $x_l, l \in H_{i_j}$, does not contain O. So the centers of the spheres S_{i_j} are different from O. Then it is not difficult to prove that we can choose $y_{i_1}, y_{i_2}, y_{i_3}$ so that the radius of the circumscribed circle around them is less than 1. If we view S_r^{d-1} as a point, then the points on S_{i_j} are just some unit vectors going out of S_r^{d-1}. We can choose a hyperplane π that passes through S_r^{d-1} with the following condition: there are affinely independent points $y_{i_j} \in S_{i_j}$ that lie at distance $\geqslant c$ apart from π and in the same half-space, where $c > 0$ is an absolute constant. Then, if we move a point u from the sphere S_r^{d-1} orthogonally to π inside the half-space that contains y_{i_j}, at some moment the distance between u and y_{i_j} will be equal to $1 - c'$, for any $c' \leqslant \sqrt{1 - c^2}$. Since we can choose r sufficiently small and c can be chosen independently of r, we can find $y_{i_1}, y_{i_2}, y_{i_3}$ with the desired properties.

Next we just choose the point x_1 in $\mathbb{R}^d$ in such a way that $|y_{i_j} x| = 1,\ j = 1, 2, 3$ and that the sphere of radius 1 with center in x_1 does not contain $x_2, \ldots, x_m$ and $S_i, i \neq i_1, i_2, i_3$. After that we choose appropriate points $y_i, i \neq i_1, i_2, i_3$. This means that G is realizable in $\mathbb{R}^d$, a contradiction.

Thus, the pair $(B\backslash\{b_1\},\mathcal{H})$ is not realizable in $\mathbb{R}^d$, and $\sum_{i=1}^n |H_i| \geqslant \binom{d+2}{2} - 3 + \deg(b_1) = \binom{d+2}{2}$.

Suppose next that the set B does not contain vertices of degree 3, but contains a vertex, say b_1, of degree 4, which is connected to $a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}$. We argue as in the previous case, and try to find appropriate $y_{i_1}, y_{i_2}, y_{i_3}, y_{i_4}$ that are in general position. If there are no such y_{i_j}, then we have two possible reasons for that. The first is that three out of the spheres S_{i_j} are zero-dimensional, and then we can interchange the roles of the parts A and B. In this case we get three sets of equal size, and can conclude as above that $\sum_{i=1}^n |H_i| \geqslant \binom{d+2}{2}$. The second is that none of the spheres S_{i_j} are two dimensional, so we have at least s one-dimensional spheres and $(4-s)$ zero-dimensional spheres, where $s \geqslant 2$. In any case, applying Lemma 2.5, we get that $\sum_{i=1}^n |H_i| \geqslant \binom{d+2}{2}$.

If there are $y_{i_1}, y_{i_2}, y_{i_3}, y_{i_4}$ in general position, then again one can show that they can be chosen in such a way that the radius of the circumscribed sphere around them is less than 1, and the reasoning goes as for the case of a vertex of degree 3. Finally, we get the estimate $\sum_{i=1}^n |H_i| \geqslant \binom{d+2}{2} - 3 + \deg(b_1) = \binom{d+2}{2} + 1$.

Suppose now that the smallest degree of a vertex in B is 5. Then we interchange the roles of parts A and B, form a set system $\mathcal{H}^B = \{H_1^B, \ldots, H_m^B\}$ analogous to the way we formed the set system $\mathcal{H}$ and apply Lemma 2.5. As the first two elements of the increasing sequence we get $|H_1^B|, |H_2^B| \geqslant 5$ instead of $|H_1^B| = 3, |H_2^B| = 4$, and we finally get $\sum_{i=1}^n |H_i| = \sum_{i=1}^m |H_i^B| \geqslant \binom{d+2}{2}$. This completes the proof of Theorem 1.4. □

2.4 Proof of Proposition 1.6

Having the statement of Theorem 1.2, the proof of both parts is merely a slight modification of the proof of the lower bound for the classical Ramsey number. Indeed, by a simple probabilistic argument one can show that if $\binom{m}{s} 2^{1-\binom{s}{2}} |\mathcal{FD}_s(d)| < 1$, then $R_{FD}(s,s,d) > m$.

For the proof of part 1 of Theorem 1.6 we use part 2 of Theorem 1.2, and obtain the inequality $m^s 2^{-(1+o(1))s^2/2} < 1$, which holds for $m = 2^{(1+o(1))s/2}$. For the proof of part 2 of Theorem 1.6 we use part 3 of Theorem 1.2, and obtain the inequality $m^s 2^{-(1/2-c'+o(1))s^2} < 1$. Thus, we can choose $\alpha = 1/2 - c'$.

2.5 Proof of Proposition 1.8

We use the following theorem from [1]:

Theorem 2.6 (D. Achlioptas, C. Moore, [1]). *Given any integer $l \geqslant 3$, let k be the smallest integer such that $l \leqslant 2k \log k$. Then with high probability the chromatic number of the random l-regular graph is $k, k+1$ or $k+2$.*

We also need a theorem from [9]:

Theorem 2.7 (B.D. McKay, N.C. Wormald, B. Wysocka, [9]). *For $(l-1)^{2g-1} = o(n)$, the probability that a random l-regular graph has girth greater than g is*

$$\exp\left(-\sum_{r=3}^{g} \frac{(l-1)^r}{2r} + o(1)\right)$$

By Theorem 2.7 we get that for any fixed $l, g \in \mathbb{N}$ the random l-regular graph has girth $\geqslant g$ with probability bounded away from 0. Thus, by Theorem 2.6, a random l-regular graph satisfies, with positive probability, the condition on the chromatic number from Theorem 2.6 and also has girth greater than g. Consider such a graph G with $l = [d/2]$. Then $\chi(G) = \frac{d}{4\log d}(1 + o(1))$. Finally, we use the following theorem from [8]:

Theorem 2.8 (H. Maehara, V. Rödl, [8])**.** *Any graph with maximum degree k can be realized as a faithful distance graph in $\mathbb{R}^{2k}$.*

Applying Theorem 2.8 to the graph G, we obtain the statement of Proposition 1.8.

3 Additional Problems

Problem 2 seems to be quite challenging, and is probably the most interesting question among the ones stated in this paper.

Theorem 1.4 supplies relatively tight bounds on $g_2(d)$, but it will be interesting to determine the exact value. We believe that the graph that provides the upper bound in Theorem 1.4 is optimal.

More generally, we suggest the following problem:

Problem 3. *Determine the minimum possible number $g_k(d)$ of edges of a k-colorable graph G which is not realizable as a faithful distance graph in $\mathbb{R}^d$.*

It seems interesting to find any non-trivial examples of graphs that are not faithful distance graphs in $\mathbb{R}^d$ and have a small number of edges. We do not know any example except for bipartite graphs similar to the one that gives the upper bound in Theorem 1.4. Is there any non-trivial example whose number of edges is between that of K_{d+2} and this upper bound ?

Recall the distance Ramsey numbers discussed in Section 1. Can we determine the minimum $f_D = f_D(s)$, such that $R_D(s, s, f_D) = s$? In other words, $f_D(s)$ is the smallest possible d, such that for any graph G on s vertices either G or its complement $\bar{G}$ can be realized as a distance graph in $\mathbb{R}^d$.

We can show that $f_D(s) = (\frac{1}{2} + o(1))s$. The lower bound follows by considering the graph G which is a clique on $\lceil s/2 \rceil$ vertices (and $\lfloor s/2 \rfloor$ isolated ones.) The upper bound follows from the fact (proved by an iterative application of the classical Ramsey theorem) that the vertices of any graph G on s vertices can be partitioned into $O(s/\log s) = o(s)$ pairwise disjoint sets, each spanning either a clique or an independent set of G. This implies that either G or $\bar{G}$ can be colored properly by $(\frac{1}{2} + o(1))s$ colors so that at least $s/2$ color classes are of size 1. The argument in the proof of Theorem 1.1, part 1 can be easily modified to show that any graph that has a proper coloring with a color classes of size 1 and b bigger color classes can be realized as a distance graph in $\mathbb{R}^{a+2b}$, implying the desired upper bound.

A similar question can be asked for the function $f_{FD}(s)$ whose definition is obtained from that of f_D by replacing distance Ramsey numbers by faithful distance Ramsey numbers. It seems harder to determine the asymptotic behaviour of $f_{FD}(s)$. We suggest the following

Problem 4. *Determine $f_D(s)$ and $f_{FD}(s)$.*

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