Multicolored matchings in hypergraphs

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Abstract

For a collection of (not necessarily distinct) matchings $\mathcal{M} = (M_1, M_2, \ldots, M_q)$ in a hypergraph, where each matching is of size t, a matching M of size t contained in the union $\cup_{i=1}^{t} M_i$ is called a *rainbow matching* if there is an injective mapping from M to $\mathcal{M}$ assigning to each edge e of M a matching $M_i \in \mathcal{M}$ containing e.

Let $f(r,t)$ denote the maximum k for which there exists a collection of k matchings, each of size t, in some r-partite r-uniform hypergraph, such that there is no rainbow matching of size t.

Aharoni and Berger showed that $f(r,t) \geq 2^{r-1}(t-1)$, proved that equality holds for $r = 2$ as well as for $t = 2$ and conjectured that equality holds for all r, t. We show that in fact $f(r,t)$ is much bigger for most values of r and t, establish an upper bound and point out a relation between the problem of estimating $f(r,t)$ and several results in additive number theory, which provides new insights on some such results.

1 Introduction

A *matching* in a hypergraph is a collection of pairwise disjoint edges. For a collection of (not necessarily distinct) matchings $\mathcal{M} = (M_1, M_2, \ldots, M_q)$ in a hypergraph, where each matching is of size t, a matching M of size t contained in the union $\cup_{i=1}^{t} M_i$ is called a *rainbow matching* if there is an injective mapping from M to $\mathcal{M}$ assigning to each edge e of M a matching $M_i \in \mathcal{M}$ containing e.

Let $f(r,t)$ denote the maximum k for which there exists a collection of k matchings, each of size t, in some r-partite r-uniform hypergraph, such that there is no rainbow matching of size t.

Aharoni and Berger [1] showed that $f(r,t) \geq 2^{r-1}(t-1)$ for all $r, t > 1$, proved that equality holds for $r = 2$ as well as for $t = 2$ and conjectured that equality holds for all $r, t > 1$.

Conjecture 1.1 ([1]) *For every integers $r, t > 1$, $f(r,t) = 2^{r-1}(t-1)$.*

In this note we observe that this question is closely related to a well studied problem in additive number theory. Using this relation we show that the conjecture is false for every pair (r,t) with $t \geq 3$ odd and $r \geq 4$ as well as for $t = 4, 6, 8$ and all sufficiently large r and for every even $t \geq 10$ and

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$r \geq 4$. In addition, we describe a probabilistic lower bound for $f(r,t)$ showing that for all sufficiently large t and all r, $f(r,t) > 2.71828^{r-1}$, and prove a (much bigger) upper bound: $f(r,t) \leq \frac{t^{rt}(t-1)}{t!}$. We conclude by pointing out that the known value of $f(2,t)$ provides a new graph theoretic proof of an old result of Erdős, Ginzburg and Ziv, and by discussing several extensions and open problems.

2 The lower bound

In this section we describe two methods that provide lower bounds for $f(r,t)$. The first is based on a simple connection of the problem to a question in additive number theory, and the second is probabilistic. Both methods suffice to provide counter-examples to Conjecture 1.1.

2.1 The first construction

Let $g(n,t)$ denote the least integer p so that any sequence of at least p (not necessarily distinct) elements of the abelian group Z_t^n contains a sub-sequence of exactly t elements whose sum (in Z_t^n) is zero. Equivalently, this is the minimum number p so that any set of at least p lattice points in Z^n contains a subset of exactly t points whose centroid is also a lattice point.

The problem of determining or estimating $g(n,t)$, suggested by Harborth in [17] received a considerable amount of attention. In particular it is known that $2^n(t-1)+1 \leq g(n,t) \leq (t-1)t^n+1$ ([17]), $g(3,3)=19$ ([17], [8]), $g(4,3)=41$ ([20], [9], [8], [18]), $g(5,3)=91$ ([12], [11]), $g(n,3) > 2.217^n$ for all sufficiently large n ([11], improving [15]), $g(n,t) \geq 1.125^{\lfloor n/3 \rfloor}(t-1)2^n+1$ for every odd $t \geq 3$ and every n ([13]), $g(n,3) \leq 2 \cdot \frac{3^n}{n}$ ([19]), $g(n,t)=o(n^t)$ for any fixed t, as n tends to infinity ([6]), and $g(n,t) \leq c(n)t$ ([6]).

Theorem 2.1 *For all $r,t > 1$, $f(r,t) \geq g(r-1,t)-1$.*

Proof. By the definition of g, there is a sequence S of $|S| = g(r-1,t)-1$ members of Z_t^{r-1} containing no sub-sequence of t terms that sum to zero. Using this sequence, we define a collection of $|S|$ matchings, each of size t, in an r-uniform r-partite hypergraph on the vertex classes $A_1, A_2, \ldots, A_r$, where each A_i is a copy of Z_t. Note that each matching will be a perfect matching.

For each element $s=(s_1,s_2,\ldots,s_{r-1}) \in S$ let M_s be the matching whose i-th edge , for $0 \leq i < t$, is $(s_1+i, s_2+i, \ldots, s_{r-1}+i, i)$, where the addition is in Z_t, and where for each j, $1 \leq j \leq r$, the j-th coordinate of the vector is interpreted as an element of A_j. This defines a family of $|S|$ perfect matchings in our hypergraph. A rainbow matching here corresponds to a choice of t distinct members $s^{(1)}, s^{(2)}, \ldots, s^{(t)}$ of the sequence S, and an edge from each matching $M_{s^{(i)}}$ such that these t edges form a perfect matching. As these edges have to cover the last vertex class A_r, it follows that there is a permutation $\sigma \in S_t$ so that the rainbow matching consists of the edges $(s_1^{(i)}+\sigma(i), s_2^{(i)}+\sigma(i), \ldots, s_{r-1}^{(i)}+\sigma(i), \sigma(i))$, $1 \leq i \leq t$.

This implies that for every j, $1 \leq j \leq r-1$, the t numbers $s_j^{(i)}+\sigma(i)$ form a permutation of Z_t, and hence in Z_t the two sums $\sum_{i=1}^t (s_j^{(i)}+\sigma(i))$ and $\sum_{i=1}^t \sigma(i)$ are equal. Thus $\sum_{i=1}^t s_j^{(i)} = 0$ for all

$1 \leq j \leq r-1$, and the sum of the sub-sequence $s^{(1)}, \ldots, s^{(t)}$ is zero in Z_t^{r-1}, contradicting the choice of the sequence S. It follows that there is no rainbow matching, completing the proof. $\square$

The above proposition, together with the known lower bounds for the function $g(n,t)$ imply that $f(4,3) \geq 18$, $f(5,3) \geq 40$, $f(6,3) \geq 90$, and $f(r,3) > 2.216^r$ for all sufficiently large r, showing that the assertion of Conjecture 1.1 fails for these values of the parameters. The known bounds also show that for every fixed odd t, $f(r,t) \geq 1.125^{\lfloor (r-1)/3 \rfloor}(t-1)2^{r-1}$ which is strictly larger than $(t-1)2^{r-1}$ for all $r \geq 4$. It is easy to check that for every $t > 2$ and every r, $f(r,t) \geq f(r,t-1)$, as one can simply take a large collection of matchings, each of size t, with no rainbow matching of size t, and add the same edge, disjoint from all existing edges, to each of the matchings. This suffices to show that $f(r,t)$ exceeds $(t-1)2^{r-1}$ for all even values of $t \geq 10$ and $r \geq 4$ as well as for $t = 4, 6, 8$ and all large values of r.

The function $f(r,t)$ is likely to be much bigger than $g(r-1,t)-1$, and indeed it is known, for example, that for every $t = 2^a$ which is a power of 2 $g(r-1, 2^a) = 2^{r-1}(2^a - 1) + 1$ (see [17]) while as mentioned above $f(r,t)$ is bigger by an exponential factor for all such $t \geq 4$ and large r.

2.2 A probabilistic construction

In this subsection we describe a simple probabilistic lower bound for $f(r,t)$, using the so-called alteration method (c.f., e.g., [7], chapter 3). For fixed large t and $r > b \log t$ for an appropriate absolute constant b this bound is better than the ones given in the previous subsection.

Theorem 2.2 *For any real number $p \in (0,1)$, $f(r,t) \geq p \cdot t^{r-1} - (t!)^{r-1} p^t$. Therefore, for every $\epsilon > 0$ and $t > t_0(\epsilon)$, $f(r,t) > (e-\epsilon)^{r-1}$, where $e = 2.718281828..$ is the basis of the natural logarithm.*

Proof. As before, all our matchings are perfect matchings in an r-partite r-uniform hypergraph on the classes of vertices $A_1, A_2, \ldots, A_r$, where each A_r is a copy of Z_t. Every edge of this hypergraph is represented by a vector $s = (s_1, s_2, \ldots, s_r) \in Z_t^r$, where the jth coordinate is an element of A_j. For each vector $s = (s_1, s_2, \ldots, s_{r-1}, 0) \in Z_t^r$ whose last coordinate is 0, let M_s denote the matching consisting of the t edges $(s_1 + i, s_2 + i, \ldots, s_{r-1} + i, i)$, $(0 \leq i < t)$, where the addition is in Z_t. Let $\mathcal{M}$ be a random collection of matchings obtained by picking each matching M_s for s as above, randomly and independently, with probability p, to be a member of $\mathcal{M}$. Let $X = X(\mathcal{M})$ be the random variable counting the number of matchings in $\mathcal{M}$, and let $Y = Y(\mathcal{M})$ be the random variable counting the number of rainbow matchings in the union of all edges of $\mathcal{M}$. The expectation of X is clearly pt^{r-1}.

We claim that the expectation of Y is at most $(t!)^{r-1}p^t$. Indeed, the total number of perfect matchings in the complete r-partite r-uniform hypergraph on the sets A_i is exactly $(t!)^{r-1}$. Some of these matchings cannot be rainbow matchings in the union of our randomly selected matchings, as they contain two edges that belong to the same matching M_s for some s. Note, crucially, that for each matching that may become a rainbow matching, the probability that it lies in the union of the chosen matchings is precisely p^t, as each edge of it belongs to a different M_s and hence the choices are independent. This proves, by linearity of expectation, that the expected value of Y is at most $(t!)^{r-1}p^t$. Applying linearity of expectation again we conclude that the expectation of the difference $X - Y$ is at least $pt^{r-1} - (t!)^{r-1}p^t$. Thus, there is a choice of the collection $\mathcal{M}$ for which $X(\mathcal{M}) - Y(\mathcal{M}) \geq pt^{r-1} - (t!)^{r-1}p^t$. Fix such an $\mathcal{M}$, and for each rainbow matching M it contains omit from the collection an arbitrary matching that contributes an edge to M. This gives a collection of at least $pt^{r-1} - (t!)^{r-1}p^t$ matchings with no rainbow one, as needed.

We can now choose p optimally in order to maximize the bound obtained. This is given by $p = \left(\frac{1}{t[(t-1)!]^{r-1}}\right)^{1/(t-1)}$ (but in fact even choosing $p = (t!)^{-(r-1)/t}$ gives the same asymptotic result.) Plugging this value of p and using Stirling's formula we conclude that as t tends to infinity the bound obtained is at least $e^{(1+o(1))(r-1)} - 1$. As for $r \leq \frac{1}{2}\log t$, say, the lower bound $f(r,t) \geq 2^{r-1}(t-1)$, proved in [1], exceeds e^{r-1}, the desired estimate follow for all sufficiently large t and all r. This completes the proof. □

3 The upper bound

In this section we prove an upper bound for $f(r,t)$. The proof is probabilistic and applies to matchings in general, not necessarily r-partite, r-uniform hypergraphs. Let $F(r,t)$ denote the maximum k for which there exists a collection of k matchings, each of size t, in some r-uniform hypergraph, such that there is no rainbow matching of size t. Obviously $F(r,t) \geq f(r,t)$ for all r and t, and it is not difficult to see that $F(r,t) \leq (\frac{r^r}{r!})^t f(r,t) \leq e^{rt} f(r,t)$. Indeed, given a collection $\mathcal{M}$ of matchings, each of size t, in an arbitrary r-uniform hypergraph $H = (V,E)$, take a random partition $V = V_1 \cup V_2 \cup \ldots \cup V_r$ of V into r pairwise disjoint sets, and let $\mathcal{M}'$ consist of all matchings in $\mathcal{M}$ in which every edge intersects each V_i exactly once. Let H' be the hypergraph consisting of all edges in all matchings in $\mathcal{M}'$. Then H' is r-partite and the expected number of matchings in $\mathcal{M}'$ is exactly $(\frac{r!}{r^r})^t|\mathcal{M}|$. If there is no rainbow matching in $\cup_{M\in\mathcal{M}}M$, then there is no rainbow matching in $\cup_{M'\in\mathcal{M}'}M'$, implying that $f(r,t) \leq (\frac{r!}{r^r})^t F(r,t)$.

Theorem 3.1 *For every r and t,*

$$f(r,t) \leq F(r,t) \leq \frac{t^{rt}(t-1)}{t!}.$$

Proof. Let $\mathcal{M}$ be a collection of matchings in an r uniform hypergraph, where each matching $M \in \mathcal{M}$ is of size t. Let $H = (V,E)$ be the hypergraph consisting of all edges in all matchings $M \in \mathcal{M}$, and let $c : V \to [t] = \{1, 2, \ldots, t\}$ be a random function, assigning to each vertex $v \in V$, randomly and independently, a uniformly chosen color $c(e) \in [t]$. Call a matching $M \in \mathcal{M}$ *multicolored* if for every $i \in [t]$ it contains exactly one edge in which all r vertices are colored i. Note that for each $M \in \mathcal{M}$, the probability that M is multicolored is exactly $\frac{t!}{t^{rt}}$, as there are $t!$ ways to distribute the colors among the edges, and once this is done, the probability that each vertex saturated by the matching gets the color assigned to its edge is $\frac{1}{t^{rt}}$.

By linearity of expectation, the expected number of multicolored matchings in $\mathcal{M}$ is $|\mathcal{M}| \cdot \frac{t!}{t^{rt}}$. If $|\mathcal{M}| > \frac{t^{rt}(t-1)}{t!}$ then this expectation exceeds $t-1$, and hence there exists a coloring in which there are at least t multicolored matchings $M_1, M_2, \ldots, M_t \in \mathcal{M}$. Fix such a coloring, and let $e_i \in M_i$ be the edge of M_i in which all vertices are colored i, $(1 \le i \le t)$. Then the matching $\{e_1, e_2, \ldots, e_t\}$ is a rainbow matching. This shows that any collection of more than $\frac{t^{rt}(t-1)}{t!}$ matchings contains a rainbow matching, implying that $F(r,t) \le \frac{t^{rt}(t-1)}{t!}$, as needed. □

It is worth noting that the upper bound in the above Theorem can be slightly improved by coloring the vertices randomly by some $t' > t$ colors, calling a matching multicolored if there is a set T of t of the colors, so that for each $i \in T$ there is an edge of the matching in which all vertices are colored i. It is easy to check that here, too, a collection of t multicolored matchings must contain a rainbow matching, and one can choose the optimal value of t' to (slightly) improve the upper bound.

4 Concluding remarks and open problems

- The authors of [1] defined, for r and $t \ge s$, $f(r,s,t)$ to be the maximum k for which there exists a collection of k matchings, each of size t, in some r-partite r-uniform hypergraph, such that there is no matching of size t in which there are at least s edges that belong to distinct matchings. Thus $f(r,t,t)$ is exactly the function $f(r,t)$ considered in the previous sections. They showed that $f(r,s,t) \ge 2^{r-1}(s-1)$, and that equality holds for $r = 2$ and for $s = t = 2$. The upper bound proved in Section 3 can be modified to yield improved upper bounds for this function when $s < t$ (although in general these are still far from the lower bound). Indeed, given a collection $\mathcal{M}$ of matchings, each of size t, in an r-uniform hypergraph $H = (V,E)$, consider a random coloring of V by s colors, where each vertex, randomly and independently, is colored i with probability p_i, where $\sum_{i=1}^{s} p_i = 1$, $p_i \ge 0$. Let $q_1, q_2, \ldots, q_s$ be positive numbers whose sum is t. Call a matching $M \in \mathcal{M}$ q-multicolored if it contains exactly q_i monochromatic edges of color $i \in [s] = \{1,2,\ldots,s\}$ for every $i \in [s]$. Note that a collection S of s q-multicolored matchings always contains a matching of size t using at least one edge of each matching in S. By choosing the numbers q_i and the probabilities p_i optimally and by computing the expected number of q-multicolored matchings we get an upper bound for $f(r,s,t)$ (and in fact for $F(r,s,t)$ which is defined in the obvious way.) Here, too, one can use more than s colors and more than one vector $(q_1, \ldots, q_s)$ to improve the estimate in some cases. As an example, for $t > s = 2$ and any r one can take $p_1 = 1/t, p_2 = (t-1)/t$, $q_1 = 1, q_2 = t-1$ and conclude that $f(r,2,t) \le F(r,2,t) \le (\frac{t^t}{(t-1)^{t-1}})^r$.
- The connection between the function $f(r,t)$ and problems in additive number theory, described in Section 2, leads to some new insights about problems in additive combinatorics using the known results about $f(r,t)$. In particular, one can get a new proof of an old theorem of Erdős, Ginzburg and Ziv [14], that asserts that any sequence of $2t-1$ elements of Z_t contains a subsequence of exactly t terms whose sum in Z_t is zero. (In the notation of Section 2, this is the known fact that $g(1,t) = 2t-1$.) There are several known proofs of this result, see [5] for five such proofs. A common feature of all these proofs is that they first establish the result for all prime values of t

and then use the fact that the validity of the result for t_1 and t_2 implies its validity for the product t_1t_2. Here is a new short proof, based on the result of [1] (following [10]), that $f(2,t) = 2t-2$ for all t. Note that the proof in [1], [10] is graph theoretic, based on an alternating path argument, and works directly for all (prime or non-prime) t.

Given a sequence $a_1, a_2, \ldots a_{2t-1}$ of elements of Z_t, define a family $\mathcal{M} = \{M_1, \ldots, M_{2t-1}\}$ of $2t-1$ perfect matchings in a bipartite graph on the color classes A_1, A_2 where each A_i is a copy of Z_t. The matching M_i is defined from a_i as in Section 2, that is, $M_i = \{(a_i+j, j) \in A_1 \times A_2 : j \in Z_t\}$, where addition is in Z_t. Since $f(2,t) = 2t-2$, there is a rainbow matching, implying that there is a set $I \subset \{1, 2, \ldots, 2t-1\}$, $|I| = t$, and a bijection $\sigma : I \to Z_t$ so that the edges $(a_i+\sigma(i), \sigma(i)), i \in I$ form a perfect matching. In particular, the elements $a_i + \sigma(i)$, $i \in I$, form a permutation of Z_t, implying that in Z_t, $\sum_{i \in I}(a_i + \sigma(i)) = \sum_{j \in Z_t} j$ and hence in Z_t, $\sum_{i \in I} a_i = 0$, as needed.

Note that the argument works for any abelian group of order t. It may seem that the above proof gives a stronger result than the Erdős-Ginzburg-Ziv Theorem, as it does not only supply a subsequence of t terms whose sum is zero, but in fact it provides a subsequence to which one can add a permutation of the elements of Z_t and get a permutation. However, by an old result of M. Hall [17], these two assertions are equivalent in any abelian group, that is, a sequence of t elements in an abelian group of order t can be expressed as the pointwise difference of two permutations if and only if the sum of its elements is zero. The proof in [17] is also based on an alternating path argument. Note that for prime t the assertion of Hall's Theorem can be easily deduced from a special case of Theorem 1.2 in [4].

- In [1] it is proved that $f(r,2) = 2^{r-1}$, using a special case of the main result of [2]. It is interesting to note that this is also equivalent to the assertion of Corollary 1.2 in [3]. This corollary asserts that the largest n for which the complement of a perfect matching of n edges can be covered by r subgraphs, each being a vertex disjoint union of complete graphs, is 2^{r-1}. To see the equivalence, let the edges of the missing matching be $\{a_i, b_i\}$, and let the subgraphs be $H_1, \ldots, H_r$, where H_i is the disjoint union of cliques $C_{i,1}, \ldots, C_{i,q_i}$. Put $V_i = \{C_{i,1}, \ldots, C_{i,q_i}\}$ and consider the sets V_i as the vertex classes of an r-partite, r-uniform hypergraph. For each edge $\{a_i, b_i\}$ as above, let $M_i = \{e_i, f_i\}$ be a matching of size 2 in this hypergraph, where e_i consists of all vertices C_{i,j_i} with $a_i \in C_{i,j_i}$ and f_i consists of all vertices C_{i,j_i} with $b_i \in C_{i,j_i}$. There is no rainbow matching here, since for every $i \neq j$, e_i and f_j intersect (as the edge a_ib_j has to belong to some subgraph H_i). This gives a correspondence between families of matchings of size 2 without a rainbow matching, and coverings of complements of graph-matchings by subgraphs that are disjoint unions of cliques, showing that indeed the fact that $f(r,2) = 2^{r-1}$ is equivalent to the covering result stated above. The proofs in [2], [3] apply linear algebra tools based on some simple properties of exterior algebra.

- The problem of determining $f(r,t)$ or obtaining better estimates for it remains open. In particular, it seems interesting to determine the asymptotic behaviour of $f(r,3)$ (which is exponential in r) more accurately, and to decide whether or not for any fixed r there is a constant $c(r)$ so that $f(r,t) \leq c(r)t$ for all t.

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