On the product dimension of clique factors

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Abstract

The product dimension of a graph G is the minimum possible number of proper vertex colorings of G so that for every pair u, v of non-adjacent vertices there is at least one coloring in which u and v have the same color. What is the product dimension $Q(s, r)$ of the vertex disjoint union of r cliques, each of size s? Lovász, Nešetřil and Pultr proved in 1980 that for $s = 2$ it is $(1 + o(1)) \log_2 r$ and raised the problem of estimating this function for larger values of s. We show that for every fixed s, the answer is still $(1 + o(1)) \log_2 r$ where the $o(1)$ term tends to 0 as r tends to infinity, but the problem of determining the asymptotic behavior of $Q(s, r)$ when s and r grow together remains open. The proof combines linear algebraic tools with the method of Gargano, Körner, and Vaccaro on Sperner capacities of directed graphs.

1 Introduction

The product dimension of a graph $G = (V, E)$ is the minimum possible cardinality d of a collection of proper vertex colorings of G such that every pair of nonadjacent vertices have the same color in at least one of the colorings. Equivalently, this is the minimum d so that one can assign to every vertex v a vector in Z^d, so that two vertices are adjacent if and only if the corresponding vectors differ in all coordinates. This dimension is also the minimum number of complete graphs so that G is an induced subgraph of their tensor product, where the tensor product of graphs $H_1, \ldots, H_d$ is the graph whose vertex set is the cartesian product of the vertex sets of the graphs H_i, and two vertices $(u_1, u_2, \ldots, u_d)$ and $(v_1, v_2, \ldots, v_d)$ are adjacent iff u_i is adjacent (in H_i) to v_i for all $1 \leq i \leq d$. Yet another

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equivalent definition is the minimum number of subgraphs of the complement $\overline{G}$ of G so that each subgraph is a vertex disjoint union of cliques, and every edge of $\overline{G}$ belongs to at least one of the subgraphs.

For positive integers $s, r \geq 2$ let $K_s(r)$ denote the graph consisting of r pairwise vertex disjoint copies of the complete graph K_s, and let $Q(s, r)$ denote the product dimension of this graph. Lovász, Nešetřil and Pultr [10] (see also [1]) proved that $Q(2, r) = \lceil \log_2(2r) \rceil$. The proof of the upper bound is simple. If $q = \lceil \log_2(2r) \rceil$ then $2^q \geq 2r$. Hence one can assign distinct binary vectors of length q to the $2r$ vertices of $K_2(r)$ so that the vectors assigned to each pair of adjacent vertices are antipodal, that is, differ in all coordinates. It is easy to check that two vertices are adjacent if and only if the corresponding vectors differ in all coordinates, showing that $Q(2, r) \leq q$.

The lower bound is proved in [10] by a linear algebra argument, and the proof given in [1] applies exterior algebra. There is yet another (similar) short proof that proceeds by assigning to each vertex of $K_s(r)$ a multilinear polynomial in $x_1, x_2, \ldots, x_q$ that depends on the coloring used, and by showing that these polynomials are linearly independent. As mentioned in the abstract, Lovász, Nešetřil and Pultr [10] raised the problem of estimating $Q(s, r)$ for larger values of s. More recently, Kleinberg and Weinberg considered the same problem, motivated by the investigation of prophet inequalities for intersection of matroids [8]. In this paper, we determine the asymptotic behavior of $Q(s, r)$ for any fixed $s \geq 2$ and large r.

Theorem 1.1. *For every fixed s, $Q(s, r) = (1 + o(1)) \log_2 r$, where the $o(1)$-term tends to 0 as r tends to infinity.*

The main tool in the proof is the method of Gargano, Körner and Vaccaro in their work on Sperner capacities [7]. For completeness, and since we are interested in the behavior of the $o(1)$-term in the theorem above, we describe a variant of the method as needed here, in a combinatorial way that avoids any application of information theoretic techniques. The proof is based on what we call here Z_s-covering families of vectors.

Let Z_s denote the ring of integers modulo s. For a subset $A \subset Z_s$ and a vector $v = (v_1, v_2, \ldots, v_q) \in Z_s^q$, we say that v is A-covering if for every $a \in A$ there is an $1 \leq i \leq q$ so that $v_i = a$. The vector v is covering if it is Z_s-covering. A family $\mathcal{F} \subset Z_s^q$ is A-covering if for every ordered pair of distinct vectors $u, v \in \mathcal{F}$, the difference $u - v$ is A-covering. $\mathcal{F}$ is covering if it is Z_s-covering. Therefore, a family $\mathcal{F}$ of vectors in Z_s^q is covering if every element of Z_s appears in at least one coordinate of the difference between any two distinct vectors in the family. Let $R(s, q)$ denote the maximum possible cardinality of a covering family of vectors in Z_s^q. The following simple statement describes the connection between $Q(s,r)$ and $R(s,q)$.

Proposition 1.2. *If $R(s,q) \geq r$ then $Q(s,r) \leq q$.*

Note that by definition $R(s,q) = 1$ for all $q < s$. Our main result about $R(s,q)$ is the following.

Theorem 1.3. *1. For every $q \geq s \geq 2$, $R(s,q) \leq 2^{q-1}$. Equality holds for $s = 2$.*

2. For every fixed s, $R(s,q) \geq (2 - o(1))^q$, where the $o(1)$-term tends to 0 as $q \to \infty$.

The rest of this paper is organized as follows. Section 2 contains the proof of Proposition 1.2 and that of a simple combinatorial lemma. In Section 3 we present the proof of Theorem 1.3 and note that in view of Proposition 1.2 it implies Theorem 1.1. The proof supplies better estimates for prime values of s, and we thus first present the proof for this special case (which suffices to deduce the assertion of Theorem 1.1 for every s) and then describe briefly the proof for general s. The final Section 4 contains some concluding remarks and open problems, including some (modest) estimates for $R(s,q)$ when q is not much larger than s.

To simplify the presentation we omit, throughout the paper, all floor and ceiling signs whenever these are not crucial.

2 Preliminaries

We first prove Proposition 1.2. If $R(s,q) \geq r$, then there exists a matrix of elements in Z_s with r rows and q columns so that the difference of any two rows is covering. We will use the q columns of this matrix to find q graphs, each being a disjoint union of cliques, that cover the complement of $K_s(r)$. This complement is a complete multipartite graph with r parts of size s. Label the vertices of this graph with elements of Z_s so that in each part all labels are used exactly once. We will associate each row of the matrix to a part and each column to a vertex disjoint union of cliques. For a column $(a_1, \cdots, a_r)^T$, consider the following graph. For each $0 \leq k < s$ take the $k + a_i$th vertex (taken modulo s) from the ith part of size s and take the union of the s cliques obtained as $0 \leq k < s$ ranges. This is clearly a union of s vertex disjoint cliques. Now, suppose we have some two vertices of the graph we are trying to cover in different parts, say the $\ell + m$th vertex of part i and the ℓth vertex of part j for some $1 \leq i < j \leq q$ and some $\ell, m \in Z_s$. Then the difference of the ith and jth rows contains an m in some column $(a_1, \cdots, a_r)^T$, so that $a_i - a_j \equiv m$

in Z_s, and then the disjoint union of cliques corresponding to this column will cover our desired edge.

Next, we need the following simple lemma.

Lemma 2.1. *Let H be a bipartite graph with classes of vertices A_1, A_2 where $|A_1| = n_1$, $|A_2| = n_2$, each vertex of A_1 has degree d_1, and each vertex of A_2 has degree d_2. Furthermore suppose that $d_2 \geq \log(2n_2)$. Then there is a union of vertex-disjoint stars with centers in A_1, each star having at least $\frac{d_1}{4\log(2n_2)}$ leaves, such that all vertices of A_2 are leaves.*

Proof. Define a random subset S of A_1 by choosing each vertex of A_1 to be in S with probability $p = \frac{\log(2n_2)}{d_2}$ uniformly and independently. We claim that with positive probability, each vertex of A_2 has between 1 and $4\log(2n_2)$ neighbors in S. The proof is a simple union bound; a fixed vertex $v \in A_2$ has probability

$$(1-p)^{d_2} < e^{-pd_2} = \frac{1}{2n_2}$$

of having no neighbors in S, and probability at most

$$\binom{d_2}{4\log(2n_2)} p^{4\log(2n_2)} \leq \left(\frac{ped_2}{4\log(2n_2)}\right)^{4\log(2n_2)} = \left(\frac{e}{4}\right)^{4\log(2n_2)} < \frac{1}{2n_2}$$

of having more than $4\log(2n_2)$ neighbors, proving the claim. Fix an S with this property.

We now finish the proof of the lemma by an application of Hall's theorem. For all $S' \subseteq S$, let $N(S')$ denote the set of all neighbors of S' and let $e(S', A_2)$ denote the number of edges from S' to A_2. Then $|N(S')| \geq \frac{e(S',A_2)}{4\log(2n_2)} = \frac{d_1}{4\log(2n_2)}|S'|$. Hence every subset of S expands by a factor of at least $\frac{d_1}{4\log(2n_2)}$. Thus by Hall's theorem, there is a union of disjoint stars whose centers are exactly the vertices of S, each having at least $\frac{d_1}{4\log(2n_2)}$ leaves. Every remaining vertex of A_2 is adjacent to some vertex in S, so we can simply add it to an existing star. $\square$

3 Covering families

3.1 The upper bound

The following proposition implies the assertion of Theorem 1.3, part 1.

Proposition 3.1. *Fix $s \geq 2$, and let $\mathcal{F} \subset Z_s^q$ be a $\{0, 1\}$-covering family of vectors. Then $|\mathcal{F}| \leq 2^{q-1}$. For $s = 2$ equality holds.*

Proof. Put $m = |\mathcal{F}|$. Let p be a prime divisor of s and consider the vectors in $\mathcal{F}$ as vectors in Z_p^q by reducing their coordinates modulo p. Note that these vectors form a $\{0,1\}$-covering family over Z_p. Let $v_i = (v_{i1}, v_{i2}, \ldots, v_{iq})$, $(1 \leq i \leq m)$ be the vectors in $\mathcal{F}$ (considered as elements of Z_p^q).

For each $1 \leq i \leq m$ define two polynomials P_i, Q_i in the variables $x_1, x_2, \ldots, x_q$ over Z_p as follows.

$$P_i(x_1, \ldots, x_q) = \prod_{j=1}^{q}(x_j - v_{ij}), \quad Q_i(x_1, \ldots, x_q) = \prod_{j=1}^{q}(x_j - v_{ij} - 1).$$

It is not difficult to check that for every i, $Q_i(v_i) \neq 0$ and $P_i(v_i) = 0$. In addition, for every $1 \leq i \neq i' \leq m$, $P_{i'}(v_i) = 0$ (as there is a coordinate j for which $v_{ij} - v_{i'j} = 0$) and $Q_{i'}(v_i) = 0$ (as there is a j so that $v_{ij} - v_{i'j} = 1$).

Similar reasoning gives that for the vectors $v_i + J$, where J is the all 1-vector of length q, $P_i(v_i + J) \neq 0$, $Q_i(v_i + J) = 0$, and for every $i' \neq i$, $P_{i'}(v_i + J) = Q_{i'}(v_i + J) = 0$. Therefore, for each member of the collection of $2m$ polynomials $\{P_i, Q_i : 1 \leq i \leq m\}$ there is an assignment of values of the variables in which this member is nonzero and all others vanish. This easily implies that the set of $2m$ polynomials P_i, Q_i is linearly independent in Z_p, and as each of its members lies in the space of multilinear polynomials with the m variables x_j, the number, $2m$, of these polynomials is at most the dimension of this space which is 2^q. It follows that $|\mathcal{F}| = m \leq 2^{q-1}$, as needed. For $s = 2$ the family of all binary vectors in which the first coordinate is 1 is $\{0,1\}$-covering, showing that $R(2,q) \geq 2^{q-1}$ and completing the proof. □

□

3.2 Prime s

For prime $s \geq 3$, we will prove that $R(s,q) \geq (2 - o_s(1))^q$ where the $o(1)$-term tends to 0 as $q \to \infty$. The crux of the proof is a Markov chain argument from [7], which we will iterate $O(\log s)$ times.

A *balanced word* over Z_s^q is a word containing the letters 1 through $s-1$ an equal number of times. A *special balanced word* is such a balanced word so that the first $\frac{q}{(s-1)/2}$ letters are 1 and 2 in some order, the next $\frac{q}{(s-1)/2}$ letters are 3 and 4 in some order, and so on. Construct a bipartite graph between the set A_2 of balanced words w over Z_s^q and the set A_1 of permutations π on q elements defined as follows: w and π are adjacent if and only if $\pi(w)$ is a special balanced word. By symmetry all vertices in A_1 have the same degree d_1, and all vertices in A_2 have the same degree d_2. We have

$$n_2 = |A_2| = \binom{q}{q/(s-1), \cdots, q/(s-1)} \le (s-1)^q$$

and d_1 is the total number of special balanced words, so

$$d_1 = \binom{2q/(s-1)}{q/(s-1)}^{(s-1)/2} > \frac{2^q}{q^{s/2}}.$$

Furthermore, $d_2 = \left((\frac{q}{s-1})!\right)^{s-1} d_1 \ge \log(2n_2)$ so by Lemma 2.1 there exists a way to map balanced words to some set T of permutations π of q elements, so that each balanced word is associated to exactly one permutation, and each permutation in T is associated to at least $\frac{d_1}{4\log(2n_2)} > \frac{2^q/q^{s/2}}{4\log(2(s-1)^q)} > \frac{2^q}{q^s}$ balanced words. Thus, we can partition the balanced words into sets $S_1, S_2, \cdots$ so that for each S_i we have $|S_i| > \frac{2^q}{q^s}$ and for all i there exists π_i so that $\pi_i(s_i)$ is a special balanced word for all $s_i \in S_i$.

Given all of the special balanced words of length q, any two of them have a difference vector which covers $\{\pm 1\}$. The idea of the proof will be to amplify this set $\{-1, 1\}$, first to $\{-\alpha, -1, 1, \alpha\}$ for a primitive root α modulo s, and after r steps to $\{\pm\alpha^b\}$ for $0 \le b < 2^r$. At each stage, the number of vectors will be $(2 - o(1))^L$ where L is the length of the vectors. Thus after $O(\log s)$ steps we will have a set of vectors that is Z_s^* covering. We can then add an extra coordinate of 0 to all of the vectors to make them Z_s-covering.

We describe the first step of this iteration in detail. Fix a primitive root α modulo s, which will be constant throughout the steps. Also fix $n = \frac{100}{\epsilon}(\log(s))^2$, which will again be constant throughout the steps. Initially we set $q = q_0 = (1 + o(1))100s^2$, ensuring it is divisible by $s - 1$. We will construct words of length $q_1 = qn$ by stringing together balanced words of length q in a specific way. If $x_i \in S_j$, then force $x_{i+1} \in \alpha S_j$; here we mean that if we take x_{i+1} and multiply its letters by α^{-1} pointwise, the result will be in S_j. Consider all vectors of length qn constructed according to this rule, by concatenating n balanced words of length q in this way. There are more than $n_2(\frac{2^q}{q^s})^{n-1}$ such words, because at each stage other than the first we must pick x_{i+1} so that $x_i \in \alpha S_j$ for some j, and thus there are more than $\frac{2^q}{q^s}$ choices for x_{i+1}. We will make x_1, x_n uniform over all such words; this costs us a factor of n_2^2 and thus we now have more than $\frac{1}{n_2}(\frac{2^q}{q^s})^{n-1}$ such words. Using that $n_2 < s^q$, we can find more than $\frac{2^{q(n-1)}}{q^{s(n-1)}s^q} = (2 - o(1))^{qn}$ words of length qn of the form $x_1 x_2 \cdots x_n$ so that x_1, x_n are fixed. We now note that for any two different words of this form $x_1 \cdots x_n$ and $x'_1 \cdots x'_n$, with $x_1 = x'_1, x_n = x'_n$, there is some minimal $i \ge 2$ so that $x_i \ne x'_i$. But then there is some k so that $x_{i-1} = x'_{i-1} \in S_k$, and that means $x_i, x'_i \in \alpha S_k$. Because $x_i \ne x'_i$, we have $\alpha^{-1}x_i \ne \alpha^{-1}x'_i$, and both $\alpha^{-1}x_i$ and $\alpha^{-1}x'_i$ are in

S_k. It follows that $\alpha^{-1}x_i - \alpha^{-1}x_i'$ covers $\{\pm 1\}$ and so $x_i - x_i'$ covers $\{\pm\alpha\}$. Similarly, there is some maximal $i < n$ so that $x_i \neq x_i'$. Then $x_{i+1} = x_{i+1}' \in \alpha S_j$ for some j, so $x_i \in S_j$, $x_i' \in S_j$. As $x_i \neq x_i'$, it follows that $x_i - x_i'$ must cover $\{\pm 1\}$ as coordinates. When $s = 5$, setting $\alpha = 2$ and applying this construction already gives a covering family for $\mathbb{Z}_s^* = \mathbb{Z}_5^*$ with $(2 - o(1))^N$ vectors of length N. In the general case we iterate this argument to find $(2 - o(1))^N$ vectors of length N, so that after the rth iteration the vectors we get cover $\{\pm\alpha^b\}$ for all $0 \leq b < 2^r$. We describe how to do this inductively.

After r iterations, we find for some $q = q_r$ (that depends on s) a family of $M^q = (2 - o_q(1))^q$ (balanced) vectors over Z_s^q which covers $\pm\alpha^b$ for all $0 \leq b < 2^r$. We call these vectors $y_1, \cdots, y_{M^q}$ and let $Y = \{y_1, \cdots, y_{M^q}\}$. We repeat the above argument. The y_i play the role of the special balanced words. Again we construct a bipartite graph. On one side there is a set $A_2 = Y$ and on the other side there is A_1, permutations of q elements. We have π is adjacent to a (balanced) vector y if and only if $\pi(y) \in Y$. Again $n_2 = |A_2| \leq (s-1)^q$ but now $d_1 = M^q$. It can easily be verified that $d_2 \geq \log(2n_2)$. Thus by Lemma 2.1 the balanced words of length q will be split into sets $S_1, S_2, \cdots$ so that all S_i satisfy $|S_i| \geq \frac{d_1}{4\log(2n_2)} \geq \frac{M^q}{4+4q\log(s-1)} = (2 - o_q(1))^q$, and furthermore for each S_i, there exists a permutation π_i so that for all $s_i \in S_i$, $\pi_i(S_i) \in Y$. Now we will again construct a Markov chain. Consider all words of length $qn = q_{r+1}$ consisting of n balanced words of length q of the form $x_1 x_2 \cdots x_n$, so that if $x_i \in S_j$, then $x_{i+1} \in \alpha^{2^r} S_j$. If we have two such words $x_1 x_2 \cdots x_n$ and $x_1' x_2' \cdots x_n'$ so that $x_1 = x_1'$, $x_n = x_n'$, let $i > 1$ be minimal so that $x_i \neq x_i'$. Then if $x_{i-1} = x_{i-1}' \in S_j$, $x_i, x_i' \in \alpha^{2^r} S_j$. This means that $x_i - x_i'$ covers $\{\pm\alpha^b\}$ modulo s for $2^r \leq b < 2^{r+1}$. Now, if we let x_i, $i < n$, be maximal so that $x_i \neq x_i'$, then $x_{i+1} = x_{i+1}' \in \alpha^{2^r} S_j$ for some j and so x_i, x_i' are not equal but are both in S_j. This means $x_i - x_i'$ covers $\{\pm\alpha^b\}$ modulo s for $0 \leq b < 2^r$. So indeed the family covers $\{\pm\alpha^b\}$ modulo s for $0 \leq b < 2^{r+1}$, as long as x_1 and x_n are fixed over the family. We can always find

$$\frac{\min_i |S_i|^{n-1}}{\binom{q}{q/(s-1), \cdots, q/(s-1)}} \geq \frac{(2 - o(1))^{qn}}{(s-1)^q} = (2 - o(1))^{qn}$$

such words if n is large enough, where the $o(1)$ terms tend to 0 as $q \to \infty$ and n is sufficiently large. We will soon see that our choice of $n = \frac{100}{\epsilon}(\log(s))^2$ is sufficient. Iterating the argument $\log_2(s)$ times allows us to find $(2 - o(1))^q$ vectors of some length q which cover Z_s^*. Adding a single coordinate where all vectors are 0 gives us vectors of length $q + 1$ that cover Z_s, without changing the asymptotic analysis.

With care, we can extract quantitative bounds. We assume $\epsilon > 0$ is a fixed constant and show that we only require $q = \left(\frac{O(1)}{\epsilon}(\log(s))^2\right)^{\log(s)}$ to have a $(2 - \epsilon)^q$ size covering system over Z_s^q for large s. Say that after r iterations, we have $M^q = M_r^q$ vectors of length $q = q_r$ for some $M \leq 2$ (which is nearly 2). Then $\min_i S_i \geq \frac{M^q}{5q\log(s)}$ and thus we find at least

$$\frac{\min_i |S_i|^{n-1}}{\binom{q}{q/(s-1), \cdots, q/(s-1)}} \geq \frac{M^{qn}}{M^q(5q\log(s))^n s^q} \geq \frac{M^{qn}}{(2s)^q(5q\log(s))^n}$$
$$= \left(\frac{M}{(2s)^{1/n}q^{1/q}(5\log(s))^{1/q}}\right)^{qn} = M_{r+1}^{q_{r+1}}$$

vectors of length $qn = q_{r+1}$. Recall that $n = \frac{100}{\epsilon}(\log(s))^2$ and $q_0 = (1+o(1))100s^2$, so when we iterate the Markov chain argument $\log_2(s)$ times, we lose a factor of at most

$$\frac{M_0}{M_{\log_2(s)}} \leq (2s)^{2\log(s)/n}\left(\prod_{j=0}^{\infty}(q_0 n^j)^{1/q_0 n^j}\right)(5\log(s))^{1/q_0}(5\log(s))^{2\log(s)/n}$$
$$\leq (10s\log(s))^{2\log(s)/n}2^{10\log(s)/s^2}(5\log(s))^{1/100s^2}.$$

In the last inequality here the bound on the second term holds because the relevant infinite product is at most $(q_0^{1/q_0})^{100} < 2^{10\log s/s^2}$. For large s this is easily seen to be smaller than, say, $1+\epsilon/4$. Since for large s, $M_0 > 2-\epsilon/2$, for $s > s_0(\epsilon)$ we get $M_{\log_2(s)} > 2-\epsilon$. Thus at the end we have $q = 100s^2\left(\frac{O(1)}{\epsilon}(\log(s))^2\right)^{\log_2(s)}$ and at least $(2-\epsilon)^q$ vectors of length q which cover Z_s. One can easily modify the argument to work for any larger q, or simply use the super-multiplicative property of $R(s,q)$ (see the beginning of Section 4) to conclude, taking $\epsilon = 1/\log s$, that for every large s and for all $q > s^{(3+o(1))\log\log s}$, $R(s,q) > (2-\frac{1}{\log s})^q$. This completes the proof of Theorem 1.3, part 2, for prime s.

3.3 General s

Given an arbitrary fixed integer $s > 2$, we now show how to find $(2-o(1))^q$ vectors in Z_s^q which form a Z_s-covering family; this shows $R(s,q) \geq (2-o(1))^q$ even for composite s. The general strategy is similar to our strategy in the previous section, except now we work over Z, and the set we are covering does not grow so quickly. As before, we are not concerned with the vectors covering 0, because we can simply add an extra coordinate to deal with it. Since the argument is very similar to the one described in the previous subsection, we only provide a brief description omitting some of the formal details.

We prove the stronger statement that for any fixed s, we can find a $(2-o_q(1))^q$ size family over Z that covers $[-s,s]$, i.e. the difference of any two vectors contains all integers between $-s$ and s as coordinates. Let S be the least common multiple of the first s positive integers. We will assume without loss of generality that $2S \mid q$.

At the first step of our iteration, we consider vectors over Z^q with an equal number of each element of $[2S]$ as coordinates so that the first $\frac{q}{S}$ coordinates are 1s and 2s in some order, the next $\frac{q}{S}$ are an equal number of 3s and 4s, and so on. We can find $(2 - o_q(1))^q$ of these and they cover $\{\pm 1\}$. Furthermore, they have the property that for any ordered pair of these vectors, there is a coordinate in which the first has an even integer $2k$ and the second has the odd integer $2k-1$ for some $1 \leq k \leq S$. This property is crucial and maintained throughout our iterations.

Now we describe the mth step of our iteration, for $2 \leq m \leq s$. Define a bijection $f = f_m$ on $[2S]$ so that for all integers $1 \leq k \leq S$, $f(2k) = f(2k-1)+m$. We can do this for instance by setting $f(1) = 1, f(2) = m+1, f(3) = 2, f(4) = m+2, \cdots, f(2m-1) = m, f(2m) = 2m$ and then set $f(x) = f(x-2m) + 2m$ for $x > 2m$ as long as $x \leq 2S$.

We then apply the same Markov chain argument as before, defining sets S_i and constructing words $x_1x_2\cdots$ starting and ending at the same vectors. Now, however, when x_i is in some set S_j, instead of demanding $x_{i+1} \in \alpha S_j$ we require $x_{i+1} \in f(S_j)$, meaning that if we apply f^{-1} to each coordinate of x_{i+1}, the result will be in S_j. Looking at the first place where two vectors differ gives us a difference of $\pm m$. Looking at the last place where they differ shows that the crucial property is preserved, and also that the differences $\pm 1, \cdots, \pm(m-1)$ are retained.

Thus after $s = O(1)$ iterations, this algorithm produces $(2 - o_q(1))^q$ vectors of length q which cover all of $[-s, s]$, after we add an extra coordinate to deal with covering 0. This completes the proof of Theorem 1.3.

4 Concluding remarks and open problems

A natural open problem is to study the functions $R(s, q)$ and $Q(s, r)$ in general. There are several simple properties that $R(s, q)$ satisfies. We know that $R(s, q)$ is (weakly) increasing in q, because to create r covering vectors for Z_s of length $q' > q$, we can take r vectors for Z_s of length q and pad them with $q' - q$ zeroes at the end. Furthermore, we know that $R(s, q)$ is super-multiplicative, i.e. $R(s, q_1+q_2) \geq R(s, q_1)R(s, q_2)$, because if $m = R(s, q_1)$, $n = R(s, q_2)$ then we can find vectors $v_1, \cdots, v_m$ of length q_1 and $w_1, \cdots, w_n$ of length q_2 that form covering families. The mn vectors v_iw_j obtained by concatenating v_i and w_j are clearly a covering family of length $q_1 + q_2$.

For $q < s$, we have $R(s, q) = 1$, because of course we can take a single vector in Z_s^q, but if we take two then their difference can only cover a set of size q and cannot cover Z_s.

The next natural question is studying the value of $R(s,s)$.

Proposition 4.1. *$R(s,s) \leq s$, and $R(s,s) \geq p$ where p is the smallest prime factor of s. When $p = 2$ this is tight, that is, if s is even, then $R(s,s) = 2$.*

Proof. For the lower bound, for each $0 \leq a < p$ we have a vector $(0, a, 2a, \cdots, (s-1)a)$ reduced modulo s. This is a covering system for Z_s, since all positive integers smaller than p are relatively prime to s.

For the upper bound, assume there was a covering family in Z_s^s with $s+1$ vectors, so that the difference of any two of these vectors has all values of Z_s exactly once. By the pigeonhole principle, there exist two vectors v and w so that the difference of their first and second coordinates is the same. But then $v - w$ has the same value in its first and second coordinate, a contradiction.

When s is even, $R(s,s) \geq 2$ as 2 is the least prime factor of s. If $R(s,s) \geq 3$ then there exist 3 vectors in $\mathbb{Z}_s^s$, $(a_1, \cdots, a_s), (b_1, \cdots, b_s), (c_1, \cdots, c_s)$, which are covering. But then $\sum(a_i - b_i) \equiv \sum_{j=0}^{s-1} j \equiv (s-1)\frac{s}{2} \equiv \frac{s}{2}$ modulo s, and similarly $\sum(b_i - c_i) \equiv \sum(c_i - a_i) \equiv \frac{s}{2}$ modulo s. Hence, these three sums are all $\frac{s}{2}$ modulo s, so they must add up to $\frac{3s}{2} \equiv \frac{s}{2}$ modulo s. But they add up to 0, so this is a contradiction. $\square$

Nonetheless, the problem of determining $R(s,s)$ for all s remains open, and the lower bound in the last proposition is not tight. A computer search gives that $R(15,15) \geq 4$ ([9]). One example is the 4 vectors given by the rows of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 0 & 2 & 1 & 5 & 7 & 9 & 12 & 14 & 13 & 3 & 6 & 4 & 10 & 8 & 11 \\ 0 & 3 & 9 & 1 & 10 & 14 & 7 & 11 & 4 & 12 & 5 & 8 & 2 & 6 & 13 \end{pmatrix}$$

which are covering for Z_{15}.

For q which is only a little bigger than s, we can prove a reasonable upper bound, using essentially the same observation applied in the proof that $R(s,s) \leq s$.

Proposition 4.2. *If $2(q-s)^2 < s-1$, then $R(s,q) \leq s + 2(q-s) + 2(q-s)^2$.*

Proof. Assume a contradiction for some $q > s$. By the definition of $R(s,q)$, for any $r \leq R(s,q)$ there exists a matrix of r rows and q columns so that the difference of any two rows contains all values modulo s; in particular there exists such a matrix for $r = s+1+2(q-s)+2(q-s)^2$. We will double count the number of pairs of rows and columns

r_i, r_j, c_k, c_ℓ with the following property: the 2×2 submatrix formed by r_i, r_j, c_k, c_ℓ has sums of opposite corners equal.

For any two rows r_i, r_j, we examine their difference. This is a vector over Z_s with q entries containing all elements modulo s, so at most $\binom{q-s+1}{2}$ pairs of its entries can be equal. This means that for the pair of rows r_i, r_j, there exists at most $\binom{q-s+1}{2}$ pairs of columns c_k, c_ℓ that satisfy our property. So the total number of such pairs as $1 \leq i, j \leq r$, $1 \leq k, \ell \leq q$ range is at most $\binom{r}{2}\binom{q-s+1}{2}$. However, if we instead fix c_k, c_ℓ we see that the difference of these two columns is a vector of length r over Z_s which has at least $r - s$ pairs of equal elements. This means that there are at least $\binom{q}{2}(r-s)$ r_i, r_j, c_k, c_ℓ with our property. Hence

$$\binom{q}{2}(r-s) \leq \binom{r}{2}\binom{q-s+1}{2}$$

and so $2q(q-1)(r-s) \leq r(r-1)(q-s+1)(q-s)$. By our assumption, $2q > r$ and so $2(q-1) \geq r-1$. Hence $2q(q-1)(r-s) \leq r(r-1)(q-s+1)(q-s) \leq 4q(q-1)(q-s+1)(q-s)$. Thus $r - s \leq 2(q-s+1)(q-s) = 2(q-s) + 2(q-s)^2$ and so $r \leq s + 2(q-s) + 2(q-s)^2$, a contradiction. $\square$

Note that this argument gives a bound only when $q = s + O(\sqrt{s})$, and the bound is $s + O((q-s)^2)$. We believe that this is not tight.

There is a nontrivial (though weak) bound which holds in the regime $q = s + \omega(\sqrt{s})$ as long as $q < Cs$ for some fixed constant $C < \log_2 e$. The natural open problem here is to study cases when q is larger but still fairly small, for instance if $q = 1.5s$, $q = s\log(s)$, or $q = s^2$. Furthermore, it would be interesting to figure out how large q has to be so that $R(s,q) > (2-\epsilon)^s$ for a small positive constant ϵ; the following argument shows that if ϵ is small enough, then $\frac{q}{s}$ must be bigger than an absolute constant which is above 1.44, but we believe this is far from tight.

Proposition 4.3. *If $1 \leq \frac{q}{s} < C < \log_2(e)$, then $R(s,q) \leq (2-\varepsilon)^q$ for $\varepsilon = \varepsilon(C) > 0$.*

Proof. Say that $R(s,q) \geq r$ so there are r vectors in Z_s^q which are covering. Place them in a matrix with r rows and q columns. Now, for each column select uniformly an element of Z_s, and add it to all the elements of that column, reducing modulo s. This does not change the covering property. After having done this, replace all entries of this matrix by their reduction modulo 2. Given two entries of a matrix in the same column and different rows, if initially they differed by k modulo s they now have some probability p_k of being equal that depends only on k, which is the probability that if $x \in Z_s$ is chosen randomly and uniformly then the reductions of $x, x+k$ modulo s have the same parity. One can easily check that $p_0 = 1$, $p_1 = 1/s$, $p_2 = (s-2)/s$, and so on, so that $\prod_{i \in Z_s} p_i = e^{-s(1+o(1))}$ by Stirling's formula. Furthermore, if we take this product over all $i \in Z_s$ except for a subset of size $2cs$ for a small constant c, we have a bound of $e^{-s(1-\epsilon)}$, where ϵ is a small positive constant that tends to 0 with c.

Let $K = cs$. For any pair of rows, the probability their colors match in all but at most $2K$ places is at most $e^{-s(1-\epsilon')}$ for some ϵ' that tends to 0 with c, by a union bound over all the at most $2\binom{q}{2K}$ possible sets of places where they do not match. Thus if we define a graph on our r vectors where two are adjacent iff their values upon reduction modulo 2 match in all but at most $2K$ places, this graph will in expectation have edge density $e^{-s(1-\epsilon')}$ and thus for some fixed choice of the random shifts will have at most $r^2 e^{-s(1-\epsilon')}$ edges. Fixing these shifts one can remove less than $r^2 e^{-s(1-\epsilon')}$ vertices from this graph and get an independent set. But this set must be of size at most $(2-\delta)^q$ for some $\delta = \delta(c) > 0$, because this size is bounded by the cardinality of a family of disjoint Hamming balls in $\{0,1\}^q$, each of radius $K = cs$. Thus we have $r \le r^2 e^{-s(1-\epsilon')} + (2-\delta)^q$, and the same inequality holds for any $r' < r$ by applying the same reasoning to a set of r' of our vectors.

Now if $q < Cs$ for some fixed $C < \log_2 e$ then setting $q > q_0(\epsilon', C)$, $r' = 3(2-\delta)^q$ violates the last inequality (with r replaced by r') since for this value of r' and large q, $(r')^2 e^{-s(1-\epsilon')} < r'/2$ and $(2-\delta)^q < r'/2$ so their sum is smaller than r'. This establishes the assertion of the proposition. $\square$

On the opposite extreme, one may ask what happens if $s \ge 3$ is a small fixed positive integer and q grows. We know by Proposition 3.1 that $R(2,q) = 2^{q-1}$, so it is natural to ask what happens when 2 is replaced by a larger positive integer.

Conjecture 4.4. *For any fixed $s \ge 3$, $R(s,q) = o(2^q)$, and furthermore $R(s,q) = \Theta(2^q/q^c)$ where $c = c(s) > 0$ is a constant that depends only on s.*

Note that the vectors over Z_3^q with a zero in the first coordinate and with exactly $\lfloor q/2 \rfloor$ ones and $\lceil q/2 \rceil$ zeroes are a covering system, so $R(3,q) = \Omega(2^q/\sqrt{q})$. When s is odd the best upper bound known for $R(s,q)$ is $O(2^q)$, as shown in Proposition 3.1. When s is even, if the vectors of a covering system for Z_s^q are reduced modulo 2, any two differ in at least $s/2$ places, so there are at most $O(2^q/q^{\lfloor \frac{s-1}{4} \rfloor})$ of them, as the Hamming balls of radius $\lfloor \frac{s-1}{4} \rfloor$ centered at these reduced vectors are pairwise disjoint. It would be interesting to establish the above conjecture and to find the relevant constants c if they indeed exist. In particular it seems plausible that $c = 1/2$ when $s = 3$, and if so then the lower bound for $R(3, q)$ is essentially optimal. We conjecture that this is indeed the case.

Conjecture 4.5. $R(3, q) = \Theta(2^q/\sqrt{q})$.

In [4] it is shown that $R(3, q) \leq (\frac{1}{2} + o(1))2^q$ when q is even and $R(3, q) \leq (\frac{1}{3} + o(1))2^q$ when q is odd, leaving this problem open. Note that we do not even know how to prove the weaker claim that a $\{-1, 0, 1\}$ covering system (or equivalently just a $\{\pm 1\}$ covering system) over Z must have size $o(2^q)$.

Our original motivation for studying the function $R(s, q)$ here is its connection to the product dimension $Q(s, r)$ of the disjoint union of r cliques, each of size s. This connection is described in Proposition 1.2. The results here suffice to determine the asymptotic behaviour of $Q(s, r)$ for every fixed s as r tends to infinity, but do not provide tight bounds when r is not much bigger than s. We conclude this short paper with several simple comments about this range of the parameters. The first remark is that the product dimension $Q(s, r)$ is at least s for every $r \geq 2$. To see this fix a vertex u of the first clique and observe that in every proper coloring of the graph $K_s(r)$ of r disjoint cliques, each of size s, there is at most one pair uv with a vertex v of the second clique so that u and v have the same color. As altogether there are s such pairs, and each one has to be monochromatic in at least one vertex coloring in a collection exhibiting an upper bound for the product dimension, the number of such colorings is at least s. Another comment is that $Q(s, r)$ is clearly monotone non-decreasing in both r and s. Therefore, for every $r, s \geq 2$, $Q(s, r) \geq Q(2, r) = \lceil \log_2(2r) \rceil$.

A *transversal design* $TD(r, s)$ of order s and block size r (with multiplicity $\lambda = 1$) is a set V of sr elements partitioned into r pairwise disjoint groups, each of size s, and a collection of blocks, each containing exactly one element of each group, so that every pair of elements from distinct groups is contained in exactly one block. A transeversal design is *resolvable* if its blocks can be partitioned into parallel classes where the blocks in any parallel class partition the set V. There is a substantial amount of literature about transversal designs, see [5]. It is not difficult to check that $Q(s, r) = s$ if and only if a resolvable $TD(r, s)$ exists and hence the known results about resolvable transversal designs supply nearly precise information for the range $r \leq s$ (it is easy to see that such a design cannot exist for $r > s$). In particular, for every prime power s, $Q(s, s) = s$ and therefore by the obvious monotonicity, for any prime power s, $Q(s, r) = s$ for every $2 \leq r \leq s$, and if p is a prime power then for every $s, r \leq p$, $Q(s, r) \leq p$. (Note that Propositions 1.2 and

4.1 also imply that $Q(s,s)=s$ when s is a prime.)

It is not difficult to prove that for every s, r_1, r_2,

$$Q(s, r_1 r_2) \leq Q(s, r_1) + Q(s, r_2). \tag{1}$$

Indeed, given the graph $K_s(r_1 r_2)$ consisting of $r_1 r_2$ disjoint cliques, each of size s, we can split the cliques into r_1 disjoint groups, each consisting of r_2 cliques. Define $Q(s, r_1)$ proper colorings in which the cliques in every group are colored the same, based on the system of colorings that shows that the product dimension of $K_s(r_1)$ is $Q(s, r_1)$. Add to these $Q(s, r_2)$ additional colorings, whose restrictions to the r_2 cliques in each group are exactly the colorings showing that the product dimension of $K_s(r_2)$ is $Q(s, r_2)$. The resulting $Q(s, r_1) + Q(s, r_2)$ colorings establish (1).

The above comments together with the results in the previous sections provide upper and lower bounds for $Q(s, r)$ for all s and r, but these bounds are quite far from each other when r is much bigger than s but much smaller than $2^{s^{3 \log\log s}}$. In particular, for $r = 2^s$ the bounds we have are

$$s \leq Q(s, 2^s) \leq (1 + o(1)) \frac{s^2}{\log s}.$$

It would be interesting to close this gap.

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