HITTING k PRIMES BY DICE ROLLS

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Abstract. Let $S = (d_1, d_2, d_3, \ldots)$ be an infinite sequence of rolls of independent fair dice. For an integer $k \geq 1$, let $L_k = L_k(S)$ be the smallest i so that there are k integers $j \leq i$ for which $\sum_{t=1}^{j} d_t$ is a prime. Therefore, L_k is the random variable whose value is the number of dice rolls required until the accumulated sum equals a prime k times. It is known that the expected value of L_1 is close to 2.43. Here we show that for large k, the expected value of L_k is $(1+o(1))k \log_e k$, where the $o(1)$-term tends to zero as k tends to infinity. We also include some computational results about the distribution of L_k for $k \leq 100$.

Keywords: characteristic polynomial, Chernoff inequality, combinatorial probability, hitting time, Prime Number Theorem.

MSC2020 subject classifications: 60C05, 11A41, 60G40.

1. Results

Let $S = (d_1, d_2, d_3, \ldots)$ be an infinite sequence of rolls of independent fair dice. Thus the d_i are independent, identically distributed random variables, each uniformly distributed on the integers $\{1, 2, \ldots, 6\}$. For each $i \geq 1$ put $s_i = \sum_{j=1}^{i} d_j$. The sequence S *hits* a positive integer x if there exists an i so that $s_i = x$. In that case it hits x in step i.

For any positive integer k, let $L_k = L_k(S)$ be the random variable whose value is the smallest i so that the sequence S hits k primes during the first i steps (∞ if there is no such i, but it is easy to see that with probability 1 there is such i). The random variable L_1 is introduced and studied in [1], see also [4], [3] for several variants and generalizations.

Here we consider the random variable L_k for larger values of k, focusing on the estimate of its expectation.

1.1. Computational results.

This article is accompanied by a Maple package `PRIMESk`, available from

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/primesk.html` ,

where there are also numerous output files.

Using our Maple package, we computed the following values of the expectation of L_k for $k \leq 30$.

k	$E(L_k)$	k	$E(L_k)$	k	$E(L_k)$
1	2.428497914	11	48.14320555	21	106.3962997
2	5.712240468	12	53.61351459	22	112.5650207
3	9.498878119	13	59.16406655	23	118.7684092
4	13.65059271	14	64.79337350	24	125.0081994
5	18.05408931	15	70.50517127	25	131.2881683
6	22.64615402	16	76.30284161	26	137.6114097
7	27.42115902	17	82.18566213	27	143.9783110
8	32.37752852	18	88.14757626	28	150.3859881
9	37.50029903	19	94.17811256	29	156.8292462
10	42.76471868	20	100.2648068	30	163.3025173

The table suggests that the asymptotic value of this expectation is $(1+o(1))k \log k$, where the $o(1)$-term tends to zero as k tends to infinity, and the logarithm here and throughout the manuscript is in the natural basis. This is confirmed in the results stated in the next subsection and proved in Section 2.

The value of the standard deviation of L_k for $k \le 30$ is given in the following table.

k	$Std(L_k)$	k	$Std(L_k)$	k	$Std(L_k)$
1	2.4985553	11	14.9184147	21	23.3873070
2	4.2393979	12	15.8185435	22	24.0816339
3	5.7679076	13	16.7109840	23	24.7769981
4	7.1185391	14	17.6115574	24	25.4821834
5	8.3598784	15	18.5197678	25	26.1952166
6	9.5715571	16	19.4227324	26	26.9055430
7	10.7618046	17	20.3022748	27	27.5997195
8	11.9062438	18	21.1419697	28	28.2678482
9	12.9824596	19	21.9329240	29	28.9080719
10	13.9823359	20	22.6771846	30	29.5276021

The value of the skewness of L_k for $k \le 30$ is given in the following table.

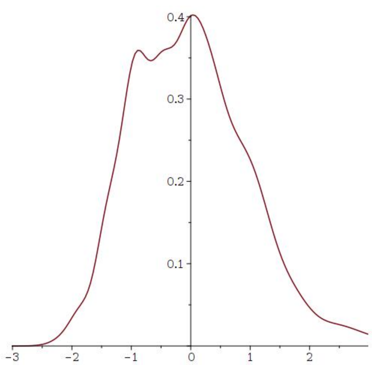
k	$Skew(L_k)$	k	$Skew(L_k)$	k	$Skew(L_k)$
1	3.3904247	11	0.7569428	21	0.5205173
2	2.1496468	12	0.7362263	22	0.5148284
3	1.6420771	13	0.7250716	23	0.5134409
4	1.3892778	14	0.7131387	24	0.5108048
5	1.2554076	15	0.6939289	25	0.5029053
6	1.1503502	16	0.6657344	26	0.4888319
7	1.0474628	17	0.6307374	27	0.4707841
8	0.9487703	18	0.5936550	28	0.4528198
9	0.8625227	19	0.5601812	29	0.4391145
10	0.7974496	20	0.5351098	30	0.4324204

The value of the kurtosis of L_k for $k \le 30$ is given in the following table.

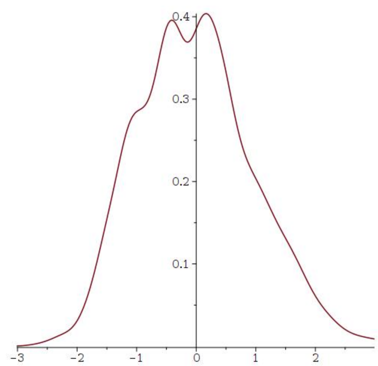
k	$Ku(L_k)$	k	$Ku(L_k)$	k	$Ku(L_k)$
1	20.6214485	11	3.9630489	21	3.4553514
2	10.0475452	12	3.9427896	22	3.4675149
3	7.2098904	13	3.9031803	23	3.4566369
4	6.1044828	14	3.8308431	24	3.4199435
5	5.5085380	15	3.7314241	25	3.3679599
6	5.0273441	16	3.6223695	26	3.3183350
7	4.6151697	17	3.5254483	27	3.2873677
8	4.2993763	18	3.4590869	28	3.2835481
9	4.0978890	19	3.4312823	29	3.3051186
10	3.9989275	20	3.4359883	30	3.3414988

We end this section with some figures and a table of the **scaled** probability density functions for the number of rolls of a fair die until visiting the primes k times for various k values. (Recall that the scaled version of a random variable X with expectation μ and variance σ^2 is $(X-\mu)/\sigma$).

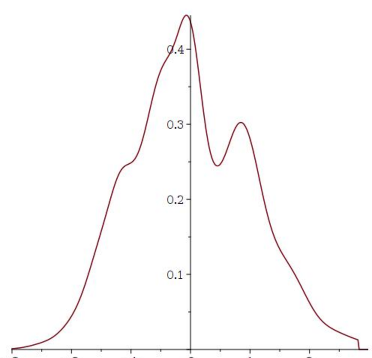
k	Expectation	Standard Deviation	Skewness	Kurtosis
20	100.2648068	22.6771846	0.5351098	3.4359883
40	229.8903783	36.1271902	0.3777949	3.1278526
60	370.5241578	46.0245135	0.1406763	2.6164507
80	520.2899340	57.8152360	0.2910580	2.9707515
100	676.3153763	65.2765933	0.2230411	3.0704308



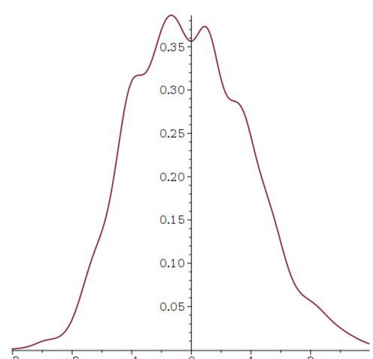
$k = 20$



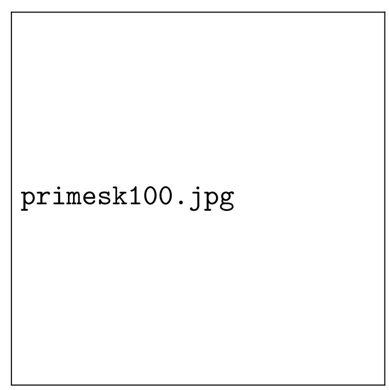
$k = 40$



$k = 60$



$k = 80$



$k = 100$

FIGURE 1. Scaled probability density function for the number of rolls of a fair die until visiting the primes k times.

Based on the available data above, the argument described in the next section, and the known results about the function $\pi(n)$ which is the number of primes that do not exceed n, a possible guess for a more precise expression for $E(L_k)$ may be $k(\log k + \log\log k + c_1) + c_2$. This is also roughly consistent with the computational evidence.

1.2. **Asymptotic results.** In the next section we prove the following two results.

Theorem 1.1. *For any fixed positive reals ε, δ there exists $k_0 = k_0(\varepsilon, \delta)$ so that for all $k > k_0$ the probability that $|L_k - k \log k| > \varepsilon k \log k$ is smaller than δ.*

Theorem 1.2. *For any fixed $\varepsilon > 0$ and any $k > k_0(\varepsilon)$, the expected value of the random variable L_k satisfies $|E(L_k) - k \log k| < \varepsilon k \log k$.*

2. Proofs

In all proofs we omit all floor and ceiling signs whenever these are not crucial, in order to simplify the presentation.

Lemma 2.1. *There are fixed positive C and μ, $0 < \mu < 1$ so that the following holds. Let $S = (d_1, d_2, \dots)$ be a random sequence of independent rolls of fair dice. For any positive integer x, let $p(x)$ denote the probability that S hits x. Then $|p(x) - 2/7| \le C(1-\mu)^x$, that is, as x grows, $p(x)$ converges to the constant $2/7$ with an exponential rate.*

Proof. Define $p(-5) = p(-4) = p(-3) = p(-2) = p(-1) = 0$, $p(0) = 1$ and note that for every $i \ge 1$,

$$p(i) = \frac{1}{6}\sum_{j=1}^{6} p(i-j).$$

Indeed, S hits i if and only if the last number it hits before i is $i-j$ for some $j \in \{1, \dots, 6\}$, and the die rolled after that gives the value j. The probability of this event for each specific value of j is $p(i-j) \cdot (1/6)$, providing the equation above. (Note that the definition of the initial values is consistent with this reasoning, as before any dice rolls the initial sum is 0). Thus, the sequence $(p(i))$ satisfies the homogeneous linear recurrence relation given above. The characteristic polynomial of that is

$$P(z) = z^6 - \frac{1}{6}(z^5 + z^4 + z^3 + z^2 + z + 1).$$

One of the roots of this polynomial is $z = 1$, and its multiplicity is 1 as the derivative of $P(z)$ does not vanish at 1. It is also easy to check that the absolute value of each of the other roots λ_j, $2 \le j \le 6$ of $P(z)$ is at most $1-\mu$ for some absolute positive constant μ, $0 < \mu < 1$. Therefore, there are constants c_j so that

$$p(i) = c_1 \cdot 1^i + \sum_{j=2}^{6} c_j \lambda_j^i,$$

implying that

$$|p(i) - c_1| \le C(1-\mu)^i$$

for some absolute constant C. It remains to compute the value of c_1. By the last estimate, for any positive n,

$$|\sum_{i=1}^{n} p(i) - c_1 n| \le C/(1-\mu).$$

Note that the sum $\sum_{i=1}^{n} p(i)$ is the expected number of integers in $[n] = \{1, 2, \dots, n\}$ hit by the sequence S.

For each fixed f, $d_1 + d_2 + \cdots + d_f$ is a sum of f independent identically distributed random variables, each uniform on $\{1, 2, \dots, 6\}$. By the standard estimates for the distribution of sums of independent bounded random variables, see., e.g., [2], Theorem A.1.16, this sum is very close to $7f/2$ with high probability. Therefore for large n the expectation considered above is $(1 + o(1))(2/7)n$. Dividing by n and taking the limit as n tends to infinity shows that $c_1 = 2/7$, completing the proof. □

Note that the lemma above implies that there exists an absolute positive constant c so that for any (large) integer g the following holds:

(1) For any $x \geq c\log g - 5$, $p(x) = \frac{2}{7}e^{\varepsilon_1(x)}, 1 - p(x) = \frac{5}{7}e^{\varepsilon_2(x)}$ where $|\varepsilon_1(x)| < 1/g, |\varepsilon_2(x)| \leq 1/g$.

It will be convenient to apply this estimate later.

Let $Y_m(S)$ denote the number of primes in $[m] = \{1, 2, \ldots, m\}$ hit by S. In the next lemma we use the letters H and N to represent "hit" and "not-hit", respectively.

Lemma 2.2. *For any sequence of integers $1 \leq x_1 < x_2 < \cdots < x_g$ that satisfy $x_1 \geq c\log g$ and $x_{i+1} - x_i \geq c\log g$ for all $1 \leq i \leq g-1$, where c is the constant from (1), and for every $\nu \in \{H, N\}^g$ the following holds. Let h be the number of H coordinates of ν. Then,*

$$P\left(S \text{ hits } x_i \text{ iff } \nu_i = H\right) = \left(\frac{2}{7}\right)^h \left(\frac{5}{7}\right)^{g-h} e^{\varepsilon(\nu)},$$

where $|\varepsilon(\nu)| \leq 1$.

Proof. The probability of the event (S hits x_i iff $\nu_i = H$) is a product of g terms. The first term is the probability that S hits x_1 (if $\nu_1 = H$) or the probability that S does not hit x_1 (if $\nu_1 = N$). Note that since $x_1 > c\log g$ this probability is $\frac{2}{7}e^{\varepsilon_1}$ in the first case and $\frac{5}{7}e^{\varepsilon_2}$ in the second case, where both $|\varepsilon_1|$ and $|\varepsilon_2|$ are at most $1/g$.

The second term in the product is the conditional probability that S hits x_2 (if $\nu_2 = H$), or that it does not hit x_2 (if $\nu_2 = N$), given the first value it hit in the interval $x_1, x_1+1, \ldots, x_1+5$. If $\nu_1 = H$, this first value is x_1 itself, and then the probability to hit x_2 is exactly $p(x_2 - x_1)$. If $\nu_1 = N$, then this first value is one of the 5 possibilities $x_1 + j$ for some $1 \leq j \leq 5$. Subject to hitting $x_1 + j$, the conditional probability to hit x_2 is exactly $p(x_2 - x_1 - j)$, which by the assumption on the difference $x_2 - x_1$, is very close to $\frac{2}{7}$. By the law of total probability it follows that in any case the conditional probability to hit x_2 is $\frac{2}{7}e^{\varepsilon'}$ and the conditional probability not to hit it is $\frac{5}{7}e^{\varepsilon''}$ where the absolute value of ε' and of ε'' is at most $1/g$. Continuing in this manner we get a product of g terms, h of which are very close to $2/7$ and $g-h$ are very close to $5/7$, where the product of all error terms $e^{\varepsilon'''}$ is of the form e^{ε} for some $|\varepsilon| \leq g \cdot (1/g) = 1$. This completes the proof of the lemma. ☐

Proposition 2.3. *For any sequence $x_1 < x_2 < \cdots < x_n$ of positive integers and any $a \geq \sqrt{n}\log(n)$*

$$P\left(\left|\#x_i \text{ hit } - \frac{2}{7}n\right| \geq a\right) \leq e^{-c'\frac{a^2}{n\log(n)}},$$

for some absolute positive constant c'.

Proof. Split $x_1, \ldots, x_n$ into $c\log(n)$ subsequences, where subsequence number j consists of all x_i with index $i \equiv j \bmod (c\log n)$ where c is the constant from (1). Note that the difference between any two distinct elements in the same subsequences is at least $c\log n$ and that each of these subsequences can contain at most one element smaller than $c\log n$. Each one of the subsequences contains $r := \frac{n}{c\log(n)}$ elements x_i. In each subsequence, the probability to deviate in absolute value from $\frac{2}{7}r$ hits by more than $\frac{a}{c\log(n)}$ can be bounded by the Chernoff's bound for binomial distributions, up to a factor of e. Indeed, Lemma 2.2 shows that the contribution of each term does not exceed the contribution of the corresponding term for the binomial random variable with parameters r and $2/7$ by more than a factor of e. Note that although each subsequence may contain one element smaller than $c\log n$, the contribution of this single element to the deviation is negligible and can be ignored. Plugging in the standard bound, see, e.g. [2], Theorem A.1.16, we get that the probability of the event considered is at most

$$2e \cdot e^{-c'\left(\frac{a}{c\log(n)}\right)^2/\left(\frac{n}{c\log(n)}\right)} \leq e^{-c''\frac{a^2}{n\log(n)}}$$

for appropriate absolute constants c', c''. Here we used the fact that since a is large the constant $2e$ can be swallowed by the choice of c''. Therefore, the probability to deviate in at least one of the subsequences by more than $a/(c\log n)$ is at most

$$c\log(n)e^{-c''\frac{a^2}{n\log(n)}} \leq e^{-c'''\frac{a^2}{n\log(n)}},$$

where in the last inequality we used again the fact that $a \geq \sqrt{n}\log(n)$. □

Recall that L_k is the minimum i so that S hits k primes in the first i steps.

Corollary 2.4. *(1) If* $\frac{2}{7}\pi(m_1) \leq k - a$ *and* $a \geq \sqrt{\pi(m_1)}\log(\pi(m_1))$, *then*

$$P\left(Y_{m_1} \geq k\right) \leq e^{-c'''\frac{a^2}{\pi(m_1)\log(\pi(m_1))}}.$$

(2) If $\frac{2}{7}\pi(m_2) \geq k + a$ *and* $a \geq \sqrt{\pi(m_2)}\log(\pi(m_2))$ *then*

$$P\left(Y_{m_2} \leq k\right) \leq e^{-c'''\frac{a^2}{\pi(m_2)\log(\pi(m_2))}}.$$

Proof. (1) The event $\{Y_{m_1} \geq k\}$ means that the number of primes that are at most m_1 and are hit by the infinite sequence of the initial sums of dice rolls is a least k. Therefore, if $\frac{2}{7}\pi(m_1) \leq k - a$, we have

$$P\left(Y_{m_1} \geq k\right) = P\left(Y_{m_1} - \frac{2}{7}\pi(m_1) \geq k - \frac{2}{7}\pi(m_1)\right) \leq P\left(\left|Y_{m_1} - \frac{2}{7}\pi(m_1)\right| \geq a\right) \leq e^{-c'''\frac{a^2}{\pi(m_1)\log(\pi(m_1))}},$$

where the last inequality follows from Proposition 2.3.

(2) Similarly, if $\frac{2}{7}\pi(m_2) \geq k + a$, we have

$$\begin{aligned} P\left(Y_{m_2} \leq k\right) &= P\left(Y_{m_2} - \frac{2}{7}\pi(m_2) \leq k - \frac{2}{7}\pi(m_2)\right) \leq P\left(Y_{m_2} - \frac{2}{7}\pi(m_2) \leq -a\right) \\ &\leq P\left(\left|Y_{m_2} - \frac{2}{7}\pi(m_2)\right| \geq a\right) \leq e^{-c'''\frac{a^2}{\pi(m_2)\log(\pi(m_2))}}, \end{aligned}$$

where the last inequality follows from Proposition 2.3. □

Corollary 2.5. *(1) For a given (large)* k, *let* m_1 *be the smallest integer so that*

$$\pi(m_1) = \lfloor \frac{7}{2}(k - 2\sqrt{k}\log k \rfloor.$$

Then for any i *satisfying* $\frac{7}{2}i \leq m_1 - a$, *where* $a \geq 2\sqrt{k}\log(k)$,

$$P\left(L_k \leq i\right) \leq P\left(d_1 + \cdots + d_i \geq m_1\right) + P\left(Y_{m_1} \geq k\right) \leq e^{-c''''\frac{a^2}{i}} + e^{-c'''\frac{k\log^2 k}{\pi(m_1)\log(\pi(m_1))}} \leq k^{-\alpha}$$

for some absolute constant $\alpha > 0$.

(2) For a given (large) k *and for* $a \geq \sqrt{k}\log^2 k$ *let* m_2 *be the smallest integer so that*

$$\pi(m_2) = \lceil \frac{7}{2}(k + a) \rceil.$$

Then for any i satisfying $\frac{7}{2}i \geq m_2 + b$, where $b \geq a$

$$P\left(L_k \geq i\right) \leq P\left(d_1 + \cdots + d_i \leq m_2\right) + P\left(Y_{m_2} \leq k\right) \leq e^{-c'''' \frac{b^2}{i}} + e^{-c''' \frac{a^2}{\pi(m_2)\log(\pi(m_2))}}.$$

Proof. (1) If both events $\{d_1 + \cdots + d_i \geq m_1\}$ and $\{Y_{m_1} \geq k\}$ do not occur, then the event $\{L_k \leq i\}$ does not occur. Therefore, for $\frac{7}{2}i \leq m_1 - a$ we have

$$P\left(L_k \leq i\right) \leq P\left(d_1 + \cdots + d_i \geq m_1\right) + P\left(Y_{m_1} \geq k\right) \leq P\left(d_1 + \cdots + d_i - \frac{7}{2}i \geq a\right) + P\left(Y_{m_1} \geq k\right)$$

$$\leq e^{-c'''' \frac{a^2}{i}} + e^{-c''' \frac{k \log^2 k}{\pi(m_1)\log(\pi(m_1))}},$$

where the last inequality follows from Chernoff's bound and the first part of Corollary 2.4. Note that here $2\sqrt{k}\log k \geq \sqrt{\pi(m_1)}\log(\pi(m_1))$ and therefore the corollary can be applied.

(2) Similarly, if both events $\{d_1 + \cdots + d_i \leq m_2\}$ and $\{Y_{m_2} \leq k\}$ do not occur, then the event $\{L_k > i\}$ does not occur. Therefore, for $\frac{7}{2}i \geq m_2 + b$, we have

$$P\left(L_k \geq i\right) \leq P\left(d_1 + \cdots + d_i - \frac{7}{2}i \leq -b\right) + P\left(Y_{m_2} \leq k\right) \leq e^{-c'''' \frac{b^2}{i}} + e^{-c''' \frac{a^2}{\pi(m_2)\log(\pi(m_2))}},$$

where the last inequality follows again from Chernoff's bound and the second part of Corollary 2.4. Indeed the corollary can be applied since it is not difficult to check that for large k and any $a \geq \sqrt{k}\log^2 k$,

$$a \geq \sqrt{\pi(m_2)}\log(\pi(m_2)) = \sqrt{\lceil \frac{7}{2}(k+a)\rceil}\log(\lceil \frac{7}{2}(k+a)\rceil).$$

□

Proof of Theorem 1.1: Note that by the Prime Number Theorem in the first part of Corollary 2.5,

$$m_1 = (\frac{7}{2} + o(1))k\log k.$$

Taking $a = 2\sqrt{k}\log k$ and letting i_1 be the largest integer so that $\frac{7}{2}i \leq m_1 - a$ it follows from this first part that $i_1 = (1 + o(1))k\log k$ and that the probability that L_k is smaller than i_1 is smaller than some negative power of k, that is, tends to 0 as k tends to infinity.

Similarly, substituting in the second part of the corollary $a = b = \sqrt{k}\log^2 k$ and letting i_2 be the smallest integer so that $\frac{7}{2}i \geq m_2 + a$ it is easy to see that i_2 is also $(1 + o(1))k\log k$ (since

$$m_2 = (\frac{7}{2} + o(1))k\log k,$$

by the Prime Number Theorem). By the second part of the corollary the probability that L_k is larger than i_2 is smaller than any fixed negative power of k, and hence tends to 0 as k tends to infinity. Therefore L_k is $(1 + o(1))k\log k$ with probability tending to 1 as k tends to infinity, completing the proof of the theorem. □

Proof of Theorem 1.2: The expectation of L_k is the sum over all positive integers i, of the probabilities $P(L_k \geq i)$. Taking $a = \sqrt{k}\log^2 k$ and defining m_1 and m_2 as before we break this sum into three parts,

$$S_1 = \sum_{i:\frac{7}{2}i \leq m_1 - a} P\left(L_k \geq i\right),$$

$$S_2 = \sum_{i:\frac{7}{2}i \geq m_2 + a} P\left(L_k \geq i\right),$$

and

$$S_3 = \sum_{i: m_1 - a < \frac{7}{2} i < m_2 + a} P\left(L_k \geq i\right).$$

By the first part of Corollary 2.5 each summand in the first sum S_1 is $1 - o(1)$ and therefore $S_1 = (1 + o(1))k \log k$, as the number of summands is $(1 + o(1))k \log k$, since $m_1 = (\frac{7}{2} + o(1))k \log k$. By the second part of the corollary (applied to an appropriately chosen sequence of a, m_2 and b) it is not difficult to check that the infinite sum S_2 is only $o(1)$. Indeed, it is possible, for example, to choose $a_0 = \sqrt{k} \log^2 k$ and $a_j = jk$ for all $j \geq 1$. The corresponding value $m_{2,j}$ of m_2 for each a_j is defined as the smallest integer satisfying $\pi(m_{2,j}) = \lceil \frac{7}{2}(k + a_j) \rceil$. Taking $b_j = a_j$ we can apply the estimate in the second part of the corollary to all values of i satisfying $m_{2,j} + a_j \leq \frac{7}{2} i < m_{2,j+1} + a_{j+1}$. The sum of the probabilities $P(L_k \geq i)$ for these values of i is thus at most $ke^{-\Omega(\log^3 k)}$ for $j = 0$, and at most $ke^{-\Omega(jk/\log(jk))}$ for each $j \geq 1$. The sum of all these quantities is smaller than any fixed negative power of k, and is therefore $o(1)$, as needed.

The sum S_3 is a sum of at most $m_2 - m_1 + 2a$ terms, and each of them is at most 1, implying that $0 \leq S_3 \leq m_2 - m_1 + 2a = o(k \log k)$, since both m_1 and m_2 are $(\frac{7}{2} + o(1))k \log k$, and $2a = O(\sqrt{k} \log^2 k)$. This completes the proof of the theorem. □

3. Concluding remarks and extensions

- **Extensions for biased r-sided dice and arbitrary subsets of the integers.** The proofs in the previous section use very little of the specific properties of the primes and the specific distribution of each d_i. It is easy to extend the result to any r-sided dice with an arbitrary discrete distribution on $[r]$ in which the values obtained with positive probabilities do not have any nontrivial common divisor. The constants 3.5 and 2/7 will then have to be replaced by the expectation of the random variable d_i and by its reciprocal, respectively. It is interesting to note that while for different dice the expectation of L_k for small values of k can be very different from the corresponding expectation for a standard fair die, once the die is fixed, for large k the expectation is always $(1 + o(1))k \log k$, where the $o(1)$-term tends to 0 as k tends to infinity.

  It is also possible to replace the primes by an arbitrary subset T of the positive integers, and repeat the arguments to analyze the corresponding random variable for this case, replacing the Prime Number Theorem by the counting function of T. We omit the details.
- **Heuristic suggestion for a more precise expression for $E(L_k)$.** If we substitute for $\pi(n)$ its approximation $n/\log n$ and repeat the analysis described here with this approximation, the more precise value for the expectation $E(L_k)$ that follows is $k(\log k + \log \log k + O(1))$. Since at the beginning there are some fluctuations, we tried to add another constant and consider an expression of the form $k(\log k + \log \log k + c_1) + c_2$. Choosing c_1 and c_2 that provide the best fit for our (limited and therefore maybe overfitted) computational evidence we obtained the heuristic expression $f(k) = k(\log k + \log \log k + 0.543) + 8.953$. For the record, here are the ratios of $E[L_k]/f(k)$ for $k = 20, 40, 60, 80, 100$, respectively:

  $0.9861651120, 0.9976101939, 0.9966486957, 0.998338113, 0.9997448512.$

  One can also replace $n/\log n$ by the more precise approximation $Li(n)$ for $\pi(n)$, but the difference between these two estimates does not change the expression obtained for $E(L_k)$ in a significant way.

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