Piercing convex sets and the Hadwiger Debrunner (p, q)-problem

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Abstract

A family of sets has the (p, q) *property* if among any p members of the family some q have a nonempty intersection. It is shown that for every $p \geq q \geq d+1$ there is a $c = c(p, q, d) < \infty$ such that for every family $\mathcal{F}$ of compact, convex sets in R^d which has the (p, q) property there is a set of at most c points in R^d that intersects each member of $\mathcal{F}$. This settles an old problem of Hadwiger and Debrunner.

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1 Introduction

For two integers $p \geq q$, a family of sets $\mathcal{H}$ has the (p, q) *property* if among any p members of the family some q have a nonempty intersection. $\mathcal{H}$ is *k-pierceable* if it can be split into k or fewer subsets, each having a nonempty intersection. The *piercing number* of $\mathcal{H}$, denoted by $P(\mathcal{H})$, is the minimum value of k such that $\mathcal{H}$ is k-pierceable. (If no such finite k exists, then $P(\mathcal{H}) = \infty$.)

The classical theorem of Helly [15] states that any family of compact convex sets in R^d which satisfies the $(d+1, d+1)$-property is 1-pierceable. Hadwiger and Debrunner considered the more general problem of studying the piercing numbers of families $\mathcal{F}$ of compact, convex sets in R^d that satisfy the (p, q) property. By considering the intersections of hyperplanes in general position in R^d with an appropriate box one easily checks that for $q \leq d$ the piercing number can be infinite, even if $p = q$. Thus we may assume that $p \geq q \geq d + 1$.

Let $M(p, q; d)$ denote the maximum possible piercing number (which is possibly infinity) of a family of compact convex sets in R^d with the (p, q)-property. By Helly's Theorem,

$$M(d + 1, d + 1; d) = 1$$

for all d, and trivially $M(p, q; d) \geq p - q + 1$. Hadwiger and Debrunner [13] proved that for $p \geq q \geq d + 1$ that satisfy

$$p(d - 1) < (q - 1)d \tag{1}$$

this is tight, i.e., $M(p, q; d) = p - q + 1$. In all other cases, it is not even known if $M(p, q; d)$ is finite, and the question of deciding if this function is finite, raised by Hadwiger and Debrunner in 1957 in [13] remained open. This question, which is usually referred to as the (p, q)-problem, is considered in various survey articles and books, including [14], [6] and [9]. The smallest case in which finiteness is unknown, which is pointed out in all the above mentioned articles, is the special case $p = 4, q = 3, d = 2$. We note that in all the cases where finiteness is known, in fact $M(p, q; d) = p - q + 1$ and that there are examples of Danzer and Grünbaum (cf. [14]) that show that $M(4, 3; 2) \geq 3 > 4 - 3 + 1$.

The (p, q)-problem received a considerable amount of attention, and finiteness have been proved for various restricted classes of convex sets, including the family of parallelotopes with edges parallel to the coordinate axes in R^d ([14],[20], [7]), families of homothetes of a convex set ([20]), and, using a similar approach, families of convex sets with a certain "squareness" property ([10], see also [22]).

Despite these efforts, the problem of deciding if $M(p, q; d)$ is finite remained open for all values of $p \geq q \geq d+1$ which do not satisfy (1).

In the present paper we solve this problem and prove the following theorem.

Theorem 1.1 *For every $p \geq q \geq d+1$ there is a $c = c(p, q, d) < \infty$ such that $M(p, q; d) \leq c$. I.e., for every family $\mathcal{F}$ of compact, convex sets in R^d which has the (p, q) property there is a set of at most c points in R^d that intersects each member of $\mathcal{F}$.*

The proof is not long, and applies three tools; a fractional version of Helly's Theorem, first proved in [16], Farkas' Lemma (or Linear Programming Duality) and a recent result proved in [1]. Although the proof supplies finite upper bounds for $M(p, q; d)$ the bounds obtained are very large and the problem of determining this function precisely remains wide open.

The rest of the paper is organized as follows. In the next section we describe the proof of the the above theorem, without making any effort to optimize the contants $c(p, q, d)$. For completeness we describe a short proof of one of the result in [1], which we need here. In Section 3 we comment on the possibilities to improve the estimate for $c(p, q, d)$, focusing on obtaining a relatively small bound for $M(4, 3; 2)$. The final Section 4 contains some concluding remarks.

2 The proof of the main result

In this section we prove Theorem 1.1. Since we do not try to optimize the constants here, and since obviously $M(p, q; d) \leq M(p, d+1; d)$ for all $p \geq q \geq d+1$ it suffices to prove an upper bound for $M(p, d+1; d)$. Another simple observation is that by compactness we can restrict our attention to finite families of convex sets.

Let $\mathcal{F}$ be a family of n convex sets in R^d, and suppose that $\mathcal{F}$ has the $(p, d+1)$ property. Our objective is to find an upper bound for the piercing number $P(\mathcal{F})$ of $\mathcal{F}$, where the bound depends only on p and d. For convenience, we split the proof into three subsections.

2.1 A fractional version of Helly's Theorem

Katchalski and Liu [16] proved the following result which can be viewd as a fractional version of Helly's Theorem.

Theorem 2.1 ([16]) *For every $0 < \alpha \leq 1$ and for every d there is a $\delta = \delta(\alpha, d) > 0$ such that for every $n \geq d+1$, every family of n convex sets in R^d which contains at least $\alpha\binom{n}{d+1}$ intersecting subfamilies of cardinality $d+1$ contains an intersecting subfamily of at least δn of the sets.*

Notice that Helly's Theorem is equivalent to the statement that in the above theorem $\delta(1, d) = 1$.

A sharp quantitative version of this theorem was proved by Kalai [17] and, independently, by Eckhoff [8]. See also [2] for a very short proof. All these proofs rely on Wegner's Theorem [21] that assrerts that the nerve of a family of convex sets in R^d is d-collapsible. This sharp quantitative result implies that for all $l \geq d+1$, the minimum possible number of intersecting subfamilies of cardinality l in a family of n convex sets in R^d no $s+1$ of which have a common intersection is obtained by a family consisting of $s-d$ copies of R^d together with $n-s+d$ hyperplanes in general position. In particular, it implies that the best possible value of $\delta(\alpha, d)$ in Theorem 2.1 is $\delta(\alpha, d) = 1 - (1-\alpha)^{1/(d+1)} \geq \frac{\alpha}{d+1}$.

Here we apply the above Theorem to prove the following lemma.

Lemma 2.2 *For every $p \geq d+1$ there is a positive constant $\beta = \beta(p, d)$ with the following property. Let $\mathcal{F} = \{A_1, \ldots, A_n\}$ be a family of n convex sets in R^d which has the $(p, d+1)$ property. Let a_i be nonnegative integers, define $m = \sum_{i=1}^{n} a_i$ and let $\mathcal{G}$ be the family of cardinality m consisting of a_i copies of A_i, for $1 \leq i \leq n$. Then there is a point x in R^d that belongs to at least βm members of $\mathcal{G}$.*

Proof We prove the lemma with

$$\beta = MIN\ \{\frac{1}{2p^2}, \frac{1}{2(d+1)\binom{p}{d+1}}\}. \tag{2}$$

This estimate can be easily improved, but we make no attempt here and in what follows to optimize the constants. If there exists an i such that $a_i \geq \beta m$ than simply choose an arbitrary point x that belongs to A_i to complete the proof. Thus we may assume that $a_i \leq \beta m$ for all i. Denote the members of $\mathcal{G}$ by $B_{i,j},\ 1 \leq i \leq n,\ 1 \leq j \leq a_i$, where for each fixed i, the sets $B_{i,j}$ are the a_i copies of A_i. Let $\mathcal{T}$ be the family of all subsets

$$\{B_{i_1,j_1}, \ldots, B_{i_p,j_p}\}$$

of cardinality p of $\mathcal{G}$ in which $i_u \neq i_v$ for all $1 \leq u < v \leq p$. Since $a_i \leq \beta m$ for all i we conclude that

$$|\mathcal{T}| \geq \frac{1}{p!} m(m-\beta m)(m-2\beta m)\dots(m-(p-1)\beta m) \geq \frac{1}{p!} m^p (1-p\beta)^p.$$

Since $\mathcal{F}$ has the $(p, d+1)$-property, for each member $T = \{B_{i_1,j_1}, \dots, B_{i_p,j_p}\}$ of $\mathcal{T}$ there is a subset $S \subset T$ of cardinlity $d+1$ which is intersecting. Moreover, the same subset S is contained in at most $\binom{m-d-1}{p-d-1}$ members of $\mathcal{T}$. It thus follows that the number of intersecting subsets of cardinality $d+1$ of $\mathcal{G}$ is at least

$$\frac{|\mathcal{T}|}{\binom{m-d-1}{p-d-1}} \geq \frac{(p-d-1)!}{p!}(1-p\beta)^p m^{d+1}$$

$$\geq \frac{1}{\binom{p}{d+1}}(1-p\beta)^p \binom{m}{d+1}.$$

By Theorem 2.1 (with the estimate for $\delta(\alpha, d)$ stated after it), this implies that there is a point x that belongs to at least

$$\frac{1}{(d+1)\binom{p}{d+1}}(1-p\beta)^p m \geq \frac{1}{(d+1)\binom{p}{d+1}}(1-p^2\beta)m \geq \frac{1}{2(d+1)\binom{p}{d+1}} m \geq \beta m$$

of the members of $\mathcal{G}$, where here we used equation (2). This completes the proof of the lemma. □

2.2 Farkas' Lemma and a Lemma on Hypergraphs

The following is a known variant of the well known lemma of Farkas (cf. [18], page 90).

Lemma 2.3 *Let A be a real matrix and b a real (column) vector. Then the system $Ax \leq b$ has a solution $x \geq 0$ if and only if for every (row) vector $y \geq 0$ which satisfies $yA \geq 0$ the inequality $yb \geq 0$ holds.*

Corollary 2.4 *Let $H = (V, E)$ be a hypergraph and let $0 \leq \gamma \leq 1$ be a real. Then the following two conditions are equivalent.*
(i) There exists a weight function $f : V \mapsto R^+$ satisfying $\sum_{v \in V} f(v) = 1$ and $\sum_{v \in e} f(v) \geq \gamma$ for all $e \in E$.
(ii) For every function $g : E \mapsto R^+$ there is a vertex $v \in V$ such that $\sum_{e;\ v \in e} g(e) \geq \gamma \sum_{e \in E} g(e)$.

Proof Let A be the $(|E|+1)$ by $|V|$ matrix whose first $|E|$ rows are indexed by the edges of H and whose columns are indexed by the vertices of H defined as follows. All the entries in the last row of A are 1, and for an edge $e \in E$ and a vertex $v \in V$, $A_{e,v}$ is -1 if $v \in e$ and is 0 otherwise. Let b be a (column) vector of length $|E|+1$ in which each of the first $|E|$ coordinates is $-\gamma$ and the last coordinate is 1. One can easily check that the matrix $Ax \leq b$ has a solution $x \geq 0$ iff condition (i) holds. Similarly, the inequality $yb \geq 0$ holds for all (row) vectors $y \geq 0$ satisfying $yA \geq 0$ iff condition (ii) holds. The result thus follows from Lemma 2.3. □

Corollary 2.5 *Suppose $p \geq d+1$ and let $\beta = \beta(p,d)$ be the constant from Lemma 2.2. Then for every family $\mathcal{F} = \{A_1, \ldots, A_n\}$ of n convex sets in R^d with the $(p, d+1)$ property there is a finite (multi)-set $Y \subset R^d$ such that $|Y \cap A_i| \geq \beta |Y|$ for all $1 \leq i \leq n$.*

Proof Let V be a finite subset of R^d containing at least one point in each nonempty intersection of members of $\mathcal{F}$. Let $H = (V, E)$ be the hypergraph on the set of vertices V whose set of edges is $\{V \cap A_i : 1 \leq i \leq n\}$. By Lemma 2.2 for every function $g : E \mapsto R^+$ for which $g(e)$ is rational for all e there is a vertex $v \in V$ such that $\sum_{e; v \in e} g(e) \geq \beta \sum_{e \in E} g(e)$. By continuity this holds without the rationality assumption. Therefore, by Corollary 2.4 there is a weight function $f : V \mapsto R^+$ satisfying $\sum_{v \in V} f(v) = 1$ and $\sum_{v \in e} f(v) \geq \beta$ for all $e \in E$. Since such a function is a solution of a Linear Program with rational constraints there is such a function f for which $f(v)$ is rational for all v. Let l be an integer so that $lf(v)$ is an integer for all v, and let Y consist of $lf(v)$ copies of v for all $v \in V$. The multiset Y clearly satisfies the conclusion of the corollary. □

2.3 Weak ϵ-nets for convex sets

The follwing result is proved in [1].

Theorem 2.6 ([1]) *For every real $0 < \epsilon < 1$ and for every integer d there exists a constant $b = b(\epsilon, d)$ such that the following holds.*
For every m and for every multiset Y of m points in R^d, there is a subset X of at most b points in R^d such that the convex hull of any subset of ϵm members of Y contains at least one point of X.

Several arguments that supply various upper bounds for $b(\epsilon, d)$ are given in [1]. For completenss we present here the simplest one, which is based on the following Theorem of Bárány [3] (see also [4] for a more exact statement for the special case $d = 2$).

Theorem 2.7 ([3]) *For every integer $d \geq 1$ there exists a constant $c(d) > 0$ such that for every multiset Y of s points in R^d there is a point in R^d which lies in at least $c(d)\binom{s}{d+1}$ of the simplices determined by subsets of cardinality $d+1$ of Y.*

The proof of this theorem, which is based on a deep result of Tverberg [19] shows that for large values of s the above statement holds with $c(d) = \frac{1}{(d+1)^{d+1}}$.

Proof of Theorem 2.6 We construct the set X as follows. Starting with $X = \emptyset$, we keep adding to X points as long as there is a point $x \in R^d$ which lies in at least $c(d)\binom{\epsilon s}{d+1}$ simplices determined by subsets of cardinality $d+1$ of Y which contain no previously chosen point of X. Observe that by Theorem 2.7 this process does not terminate as long as there is a subset of ϵs members of Y whose convex hull contains no point of X. On the other hand, the above process must terminate after at most

$$\frac{\binom{s}{d+1}}{c(d)\binom{\epsilon s}{d+1}}$$

steps, since the total number of simplices determined by points of Y is $\binom{s}{d+1}$. Since the last quantity can be bounded by a function of d and ϵ, this completes the proof. $\square$

Proof of Theorem 1.1

Let $\mathcal{F} = \{A_1, \ldots, A_n\}$ be a family of n convex sets in R^d with the $(p, d+1)$ property , where $p \geq d+1$. By Corollary 2.5 there is a finite (multi)-set $Y \subset R^d$ such that $|Y \cap A_i| \geq \beta|Y|$ for all $1 \leq i \leq n$, where $\beta = \beta(p,d)$ is as in Lemma 2.2. By Theorem 2.6 there is a set X of at most $b(\beta, d)$ points in R^d such that the convex hull of any set of $\beta|Y|$ members of Y contains at least one point of X. Since each member of $\mathcal{F}$ contains at least $\beta|Y|$ points in Y it must contain at least one point of X. Therefore, $P(\mathcal{F}) \leq |X| \leq b(\beta(p,d), d)$, completing the proof. $\square$

3 Improved estimates

By Theorem 1.1, proved in the previous section, $M(p,q;d) \leq c(p,q,d)$ for all $p \geq q \geq d+1$, where $c(p,q,d)$ is some (finite but huge) number depending on p, q and d. The estimates for $c(p,q,d)$ can be improved in several ways. In this section we describe briefly some of these ways by obtaining a relatively small upper bound for $M(4,3;2)$-the smallest case for which finiteness was not known before. Some of the arguments here apply only for this special case (or only for the case $d = 2$) and some can be used for the general case as well. Our objective is mainly to present the arguments, without trying to optimize the constants obtained in this manner, since it seems clear that these arguments do not suffice for determining the correct value of $M(4,3;2)$ (which is probably close to 3-the known lower bound for it).

Let $\mathcal{F}$ be a finite family of convex sets in R^2 which satisfies the $(4,3)$ property. Our objective is to bound the piercing number $P(\mathcal{F})$. We first observe that if A and B are two non-intersecting sets in $\mathcal{F}$ then any two members of $\mathcal{F} \setminus \{A,B\}$ must intersect. Therefore, $M(4,3;2) \leq 2 + m(4,3;2)$, where here $m(4,3;2)$ denotes the maximum possible piercing number of a finite family of planar convex sets in which each pair intersects, and which has the $(4,3)$ property.

It thus suffices to bound $m(4,3;2)$. The advantage in assuming that every pair of subsets of $\mathcal{F}$ intersect is that with this assumption, if $\mathcal{G}$ is obtained from $\mathcal{F}$ by duplicating some of the members of $\mathcal{F}$ (as in Lemma 2.2), then $\mathcal{G}$ also has the $(4,3)$-property. This immediately implies that if $|\mathcal{G}| = m$ then at least $\frac{1}{4}\binom{m}{3}$ of the subsets of cardinality 3 of $\mathcal{G}$ are intersecting. However, this can be improved by applying the known bounds for Turán's problem for hypergraphs. This problem deals with the determination or estimation of the numbers $T(m,k,l)$- the minimum possible number of edges in an l-uniform hypergraph on m vertices in which each set of k vertices contains at least one edge. In our case, the number of intersecting subfamilies of size 3 is clearly at least $T(m,4,3)$, and it is known that this number is at least $\frac{7-\sqrt{21}}{6}\binom{m}{3}$ for all sufficiently large m, as proved by Giraud (cf., e.g., [5]). (In fact, it may be possible to improve this bound for the special case of hypergraphs obtained from planar convex sets in the above manner). We can now apply the Fractional Helly Theorem, as in the proof of Lemma 2.2, and conclude that here the assertion of the lemma holds with $\beta = 1 - (\frac{-1+\sqrt{21}}{6})^{1/3} > 1/7$. (Note that we can always assume that m is as large as we wish by duplicating each set as many times as needed).

Repeating the proof as in Section 2, we can now improve the estimate by applying another result of [1] which asserts that $b(\epsilon, 2) \leq 7/\epsilon^2$. This gives that $m(4,3;2) \leq 343$ and hence that $M(4,3;2) \leq 345$. As mentioned above we suspect that the correct value of $M(4,3;2)$ is much smaller.

4 Concluding remarks

1). It may seem that there are almost no interesting families of compact convex sets in R^d which satisfy the (p, q)-property, for some $p \geq q \geq d+1$. A large class of examples can be constructed as follows. Let μ be an arbitrary probability distribution on R^d, and let $\mathcal{F}$ be the family of all compact convex sets F in R^d satisfying $\mu(F) \geq \epsilon$. Since the sum of the measures of any set of more than d/ϵ such sets is greater than d it follows that if p is the smallest integer strictly larger than d/ϵ then $\mathcal{F}$ has the $(p, d+1)$ property. It follows that $P(\mathcal{F}) \leq M(p, d+1; d+1)$, i.e., there is a set X of at most $M(p, d+1; d+1)$ points such that any compact convex set in R^d whose measure exceeds ϵ intersects X. This result is, in fact, equivalent to Theorem 2.6, and our proof of Theorem 1.1 can be viewed as a reduction of the general case to a case of this form, by applying the methods in Subsections 2.1 and 2.2.

2). The following Theorem is an immediate consequence of Theorem 1.1.

Theorem 4.1 *Let $\mathcal{F}$ be a family of compact convex sets in R^d, and suppose that for every subfamily $\mathcal{F}'$ of cardinality x of $\mathcal{F}$ the inequality $P(\mathcal{F}') < \lceil x/d \rceil$ holds; i.e., $\mathcal{F}'$ can be pierced by less than x/d points. Then $P(\mathcal{F}) \leq M(x, d+1; d+1)$.*

Proof By the assumption $\mathcal{F}$ has the $(x, d+1)$ property. $\Box$

Observe that in order to deduce a finite upper bound for the piercing number of $\mathcal{F}$, the assumption that $P(\mathcal{F}') < \lceil x/d \rceil$ cannot be replaced by $P(\mathcal{F}') \leq \lceil x/d \rceil$ as shown by an infinite family of hyperplanes in general position (intersected with an appropriate box), whose piercing number is infinite.

3). It would be interesting to estimate the numbers $M(p, q; d)$ more precisely.

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