The number of orientations having no fixed tournament

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Abstract

Let T be a fixed tournament on k vertices. Let $D(n,T)$ denote the maximum number of orientations of an n-vertex graph that have no copy of T. We prove that $D(n,T) = 2^{t_{k-1}(n)}$ for all sufficiently (very) large n, where $t_{k-1}(n)$ is the maximum possible number of edges of a graph on n vertices with no K_k, (determined by Turán's Theorem). The proof is based on a directed version of Szemerédi's regularity lemma together with some additional ideas and tools from Extremal Graph Theory, and provides an example of a precise result proved by applying this lemma. For the two possible tournaments with three vertices we obtain separate proofs that avoid the use of the regularity lemma and therefore show that in these cases $D(n,T) = 2^{\lfloor n^2/4 \rfloor}$ already holds for (relatively) small values of n.

1 Introduction

All graphs considered here are finite and simple. For standard terminology on undirected and directed graphs the reader is referred to [4]. Let T be some fixed tournament. An orientation of an undirected graph $G = (V,E)$ is called *T-free* if it does not contain T as a subgraph. Let $D(G,T)$ denote the number of orientations of G that are T-free. Let $D(n,T)$ denote the maximum possible value of $D(G,T)$ where G is an n-vertex graph. In this paper we determine $D(n,T)$ precisely for every fixed tournament T and all sufficiently large n. Problems of counting orientations and directed subgraphs of a given type have been studied by several researchers. Examples of such results appear in [1, 8].

The problem of determining $D(n,T)$ even for three-vertex tournaments is already quite complicated (it is trivial for the unique two-vertex tournament). If G has no k-clique and T is a k-vertex tournament, then, clearly, $D(G,T) = 2^{e(G)}$ where $e(G)$ denotes the number of edges of G. Thus,

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for a k-vertex tournament T we obtain the following easy lower bound:

$$D(n,T) \geq 2^{t_{k-1}(n)} \tag{1}$$

where $t_{k-1}(n)$ is the maximum possible number of edges of a graph on n vertices with no K_k. Turán's Theorem shows that $t_{k-1}(n)$ is the number of edges of the unique complete $(k-1)$-partite graph with n vertices whose vertex classes are as equal as possible. In some cases, the lower bound in (1) is not the correct answer. For example, Let $T = C_3$ denote the directed triangle. For $n = 7$, the graph $G = K_7$ has $7!$ orientations that have no directed triangle (all the acyclic orientations). Hence $D(7) \geq 7! = 5040 > 2^{t_2(7)} = 2^{12} = 4096$. Similar examples are true for other tournaments with more than three vertices. However, all examples have n relatively small as a function of the number of vertices of the tournament. This suggests that possibly for every tournament T, and all n sufficiently large (as a function of T), the lower bound in (1) is the correct value. Our main theorem shows that this, indeed, is the case.

Theorem 1.1 *Let T be a fixed tournament on k vertices. There exists $n_0 = n_0(T)$ such that for all $n \geq n_0$,*

$$D(n,T) = 2^{t_{k-1}(n)}.$$

The proof of Theorem 1.1 is presented in the next two sections. It is based on the basic approach in [2] with some additional ideas, and uses several tools from Extremal Graph Theory, including a (somewhat uncommon) directed version of the regularity lemma of Szemerédi. It provides a rare example in which this lemma is used to prove results on directed graphs, and an even more rare example of a precise result obtained with the lemma.

Unfortunately, the use of the regularity lemma forces the constant n_0 appearing in Theorem 1.1 to be horribly large even for the case $k = 3$. In section 4 we give a different proof for the special case $T = C_3$ that avoids using the regularity lemma, and obtain a moderate value for $n_0(C_3)$ (that can be optimized to less than 10000). Section 4 also contains a description of a simple reduction from the problem of counting the number or red-blue edge colorings of a graph G having no monochromatic K_k (solved in [11] for $k = 3$ and in [2] for $k > 3$) to the problem of counting the number of orientations of a graph G that do not contain the transitive tournament on k vertices, denoted T_k. Using this reduction we show, in particular, that $n_0(T_3) = 1$. The final section contains some concluding remarks and open problems.

In the rest of this paper, if x and y are vertices then xy refers to an edge between x and y in an undirected graph and (x, y) refers to a directed edge from x to y. If X and Y are disjoint subsets of vertices then $e(XY)$ denotes the number of edges between X and Y in an undirected graph, while $e(X, Y)$ denotes the number of edges from X to Y in a directed graph.

2 Graphs with many T-free orientations

Throughout the next two sections we assume that T is a fixed tournament on $k+1$ vertices and $k \geq 2$. Let G be an n-vertex graph with at least $2^{t_k(n)}$ distinct T-free orientations. Our aim in this section is to show that such graphs must be close to a k-partite graph. More precisely we prove the following.

Lemma 2.1 *For all $\delta > 0$ there exists $n_0 = n_0(k, \delta)$, such that if G is a graph of order $n \geq n_0$ which has at least $2^{t_k(n)}$ distinct T-free orientations then there is a partition of the vertex set $V(G) = V_1 \cup \cdots \cup V_k$ such that $\sum_i e(V_i) < \delta n^2$.*

Our approach in the proof of Lemma 2.1 is similar to the one from [2] and [3], which is based on two important tools, the Simonovits stability theorem and the Szemerédi regularity lemma. However, we shall require a (somewhat uncommon) version of the regularity lemma for directed graphs and a few other additional ideas. We now introduce the necessary tools and lemmas needed for the proof of Lemma 2.1.

The stability theorem ([9], see also [4], p. 340) asserts that a K_{k+1}-free graph with almost as many edges as the Turán graph is essentially k-partite. The precise statement follows.

Theorem 2.2 *For every $\alpha > 0$ there exists $\beta > 0$ (where $\beta \ll \alpha$), such that any K_{k+1}-free graph on m vertices with at least $t_k(m) - \beta m^2$ edges has a partition of the vertex set $V = V_1 \cup \cdots \cup V_k$ with $\sum_i e(V_i) < \alpha m^2$.* □

We also need the following lemma:

Lemma 2.3 *Let $\gamma > 0$ and let H be a k-partite graph with at least $t_k(m) - \gamma m^2$ edges. If we add to H at least $(2k+1)\gamma m^2$ new edges then the new graph contains a K_{k+1} with exactly one new edge connecting two vertices in the same vertex class of H.*

Proof: Let H' denote the new graph obtained from H by adding at least $(2k+1)\gamma m^2$ new edges. Since H is a k-partite graph, at least $(2k+1)\gamma m^2 - \gamma m^2 = 2k\gamma m^2$ new edges connect vertices in the same vertex class of H. Hence, some vertex class X contains at least $2\gamma m^2$ new edges. Since every graph contains a bipartite spanning subgraph with more than half the number of edges, we have that the induced subgraph of H' on X has a bipartite spanning subgraph with more than γm^2 edges. These edges, denoted F, together with the original edges of H define a subgraph of H' with more than $t_k(m)$ edges, which therefore contains a K_{k+1}. Such a K_{k+1} must contain exactly one edge of F and all other edges are original ones, as required. □

Next, we introduce the directed version of Szemerédi's regularity lemma. Although never published, it is a relatively easy consequence of the standard regularity lemma proved in [10] and its proof. For more details on the regularity lemma we refer the reader to the excellent survey of

Komlós and Simonovits [7], which discusses various applications of this powerful result. We now give the definitions necessary in order to state the directed regularity lemma.

Let $G = (V, E)$ be a directed graph, and let A and B be two disjoint subsets of $V(G)$. If A and B are non-empty, define the *density of edges* from A to B as

$$d(A, B) = \frac{e(A, B)}{|A||B|}.$$

For $\epsilon > 0$ the pair (A, B) is called *ϵ-regular* if for every $X \subset A$ and $Y \subset B$ satisfying $|X| > \epsilon|A|$ and $|Y| > \epsilon|B|$ we have

$$|d(X, Y) - d(A, B)| < \epsilon \qquad\qquad |d(Y, X) - d(B, A)| < \epsilon.$$

An *equitable partition* of a set V is a partition of V into pairwise disjoint classes $V_1, \ldots, V_m$ whose sizes are as equal as possible. An equitable partition of the set of vertices V of a directed graph G into the classes $V_1, \ldots, V_m$ is called *ϵ-regular* if $|V_i| \le \epsilon|V|$ for every i and all but at most $\epsilon\binom{m}{2}$ of the pairs (V_i, V_j) are ϵ-regular.

The directed regularity lemma states the following:

Lemma 2.4 *For every $\epsilon > 0$, there is an integer $M(\epsilon) > 0$ such that for every directed graph G of order $n > M$ there is an ϵ-regular partition of the vertex set of G into m classes, for some $1/\epsilon \le m \le M$.* □

A useful notion associated with an ϵ-regular partition is that of a *cluster graph*. Suppose that G is a directed graph with an ϵ-regular partition $V = V_1 \cup \cdots \cup V_m$, and $\eta > 0$ is some fixed constant (to be thought of as small, but much larger than ϵ). The *undirected* cluster graph $C(\eta)$ is defined on the vertex set $\{1, \ldots, m\}$ by declaring ij to be an edge if (V_i, V_j) is an ϵ-regular pair with $d(V_i, V_j) \ge \eta$ and also $d(V_j, V_i) \ge \eta$. From the definition, one might expect that if a cluster graph contains a copy of K_{k+1} then the original directed graph contains T (assuming ϵ was chosen small enough with respect to η and k). This is indeed the case, as established in the following slightly more general lemma whose proof is similar to an analogous lemma for the undirected case (see [7]).

Lemma 2.5 *Let $\eta > 0$ and suppose that $\epsilon < (\eta/2)^k/k$. Let G be a directed graph with an ϵ-regular partition $V = V_1 \cup \cdots \cup V_m$ and let $C(\eta)$ be the cluster graph of the partition.*

1. *If $C(\eta)$ contains K_{k+1} then G contains T.*

2. *If $C(\eta)$ does not have K_{k+1} and (V_s, V_t) is an ϵ-regular pair with $d(V_s, V_t) \ge \eta$ but $st \notin C(\eta)$, and the addition of st to $C(\eta)$ forms a K_{k+1}, then G contains T.*

Proof: It clearly suffices to prove the second statement. Without loss of generality assume $s = 1$ and $t = 2$. Label the vertices of T with $\{1, \ldots, k+1\}$ such that $(1, 2) \in T$ (namely, there is an edge directed from 1 to 2). We may assume that the addition of $(1, 2)$ to $C(\eta)$ forms a K_{k+1} whose vertices are $1, \ldots, k+1$. We will find a copy of T in G where vertex i of T corresponds to a vertex of G belonging to V_i, for $i = 1, \ldots, k+1$.

We prove that for every p, $0 \le p \le k+1$ there are subsets $B_i \subset V_i$, $1 \le i \le k+1$, and a set of vertices $\{a_1, \ldots, a_p\}$ where $a_i \in B_i$ with the following properties.
(i) $|B_i| \ge (\frac{\eta}{2})^{i-1}|V_i|$ for all $1 \le i \le p$ and $|B_i| \ge (\frac{\eta}{2})^p|V_i|$ for all $p < i \le k+1$.
(ii) For all $i = 1, \ldots, p$ and for all $i < j \le k+1$, if $(i, j) \in T$ then $(a_i, v) \in G$ for all $v \in B_j$ and if $(j, i) \in T$ then $(v, a_i) \in G$ for all $v \in B_j$.

The assertion of the lemma clearly follows from the above statement for $p = k+1$ since the vertices $\{a_1, \ldots, a_{k+1}\}$ induce T in G.

To prove (i) and (ii) we use induction on p. For $p = 0$ simply take $B_i = V_i$ for all i. Given the sets B_i and $\{a_1, \ldots, a_{p-1}\}$ satisfying (i), (ii) for $p-1$ we show how to modify them to hold for p. Observe that by assumption the cardinality of each B_j, for $p < j \le k+1$, is bigger than $(\eta/2)^k|V_j| \ge \epsilon|V_j|$. For each such j if $(p, j) \in T$ ($(j, p) \in T$) let B_p^j denote the set of all vertices in B_p that have outdegree (indegree) less than $(\eta - \epsilon)|B_j|$ into (from) B_j. We claim that $|B_p^j| \le \epsilon|V_p|$ for each j. This is because otherwise the two sets $X = B_p^j$ and $Y = B_j$ would contradict the ϵ-regularity of the pair (V_p, V_j), since $d(B_p^j, B_j) < \eta - \epsilon$, whereas $d(V_p, V_j) \ge \eta$, by assumption. Therefore, the cardinality of the set $B_p \setminus (B_p^{p+1} \cup \ldots \cup B_p^{k+1})$ is at least

$$|B_p| - (k+1-p)\epsilon|V_p| \ge \left(\frac{\eta}{2}\right)^{p-1}|V_p| - k\epsilon|V_p| > 0.$$

We can now choose arbitrarily a vertex a_p in $B_p \setminus (B_p^{p+1} \cup \cdots \cup B_p^{k+1})$ and replace each B_j for $p < j \le k+1$ by the set of outgoing (resp. incoming) neighbors of a_p in B_j. Since $\eta - \epsilon > \eta/2$ this will not decrease the cardinality of each B_j by more than a factor of $\eta/2$ and it is easily seen that the new sets B_i, and the set $\{a_1, \ldots, a_p\}$ defined in this manner satisfy the conditions (i), (ii) for p. $\square$

Proof of Lemma 2.1. Let $\delta > 0$ and let $\alpha < \delta/(4k+7)$. Whenever necessary we shall assume n is sufficiently large as a function of δ and k. Let $\beta = \beta(\alpha, k)$ be chosen as in Theorem 2.2. Recall that $\beta < \alpha$. Let $\eta < \beta$ be a positive constant to be chosen later. Let $\epsilon < (\eta/2)^k/k$ and notice that η and ϵ satisfy the conditions of lemma 2.5. Let $M = M(\epsilon)$ be as in Lemma 2.4.

Let $G = (V, E)$ be an undirected graph with n vertices and at least $2^{t_k(n)}$ distinct T-free orientations.

Let $\vec{G}$ be a T-free orientation of G. By applying Lemma 2.4 to $\vec{G}$ we get a partition $V = V_1 \cup \cdots \cup V_m$ satisfying the conditions of the lemma. In particular, $1/\epsilon \le m \le M$. Let $C = C(\eta)$ be the corresponding cluster graph on the vertex set $\{1, \ldots, m\}$. By Lemma 2.5, $C(\eta)$ is K_{k+1}-free and thus by Turán's theorem $C(\eta)$ has at most $t_k(m)$ edges.

Our first goal is to show that for some orientation of G the resulting cluster graph has *more than* $t_k(m) - \beta m^2$ edges. Assume this is false. In order to derive a contradiction we first bound the number of orientations of G that could give rise to a particular partition and a particular cluster graph $C = C(\eta)$. We therefore fix the partition (that is, the vertex sets $V_1, \dots, V_m$ and the non regular pairs) and a cluster graph agreeing with the partition.

Note that by definition, there are at most $m\binom{\lceil n/m\rceil}{2} < \epsilon n^2$ edges of G with both endpoints in the same part of the partition. Hence, there are at most $2^{\epsilon n^2}$ ways to orient such edges. Similarly, there are at most $\epsilon\binom{m}{2} \cdot (\lceil n/m\rceil)^2 < \epsilon n^2$ edges of G that belong to non ϵ-regular pairs. There are at most $2^{\epsilon n^2}$ ways to orient such edges.

Next, consider an ϵ-regular pair (V_i, V_j) such that $ij \notin C(\eta)$. Thus, either $e(V_i, V_j) \le |V_i||V_j|\eta$ or else $e(V_j, V_i) \le |V_i||V_j|\eta$. In either case, if $e(V_iV_j)$ is the number of undirected edges of G between V_i and V_j then there are at most

$$2\left(\sum_{q=0}^{\lfloor |V_i||V_j|\eta\rfloor} \binom{e(V_iV_j)}{q}\right) < 2\frac{n^2}{m^2}\eta \cdot 2^{H(\eta)n^2/m^2} \ll 2^{H(2\eta)n^2/m^2}$$

orientations of the edges of G belonging to this pair. Here we use the well known estimate $\binom{a}{xa} \le 2^{H(x)a}$ for $0 < x < 1$, where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the entropy function.

Finally, consider a pair corresponding to an edge of $C(\eta)$. Trivially there are at most $2^{(\lceil n/m\rceil)^2}$ possible orientations of the edges belonging to this pair.

Altogether, the total number of orientations of G giving rise to a fixed partition and a fixed cluster graph with $r \le t_k(m) - \beta m^2$ edges is at most

$$2^{\epsilon n^2} \cdot 2^{\epsilon n^2} \cdot 2^{H(2\eta)(n^2/m^2)\binom{m}{2}} \cdot 2^{(\lceil n/m\rceil)^2 r} < 2^{2\epsilon n^2} 2^{H(2\eta)n^2} 2^{(n^2/m^2)(t_k(m)-\beta m^2)} 2^{nm} <$$

$$2^{2\epsilon n^2} 2^{H(2\eta)n^2} 2^{(t_k(n)-\beta n^2)} 2^{nM} 2^k$$

where the last inequality follows from the well known fact that for every x,

$$\frac{k-1}{k}\frac{x^2}{2} - k < t_k(x) \le \frac{k-1}{k}\frac{x^2}{2}.$$

Note that M is a constant and there are at most n^{M+1} partitions of the vertex set of G into at most M parts. Also, for every such partition there are at most $2^{M^2/2}$ choices for the cluster graph $C(\eta)$ and (significantly) less than $2^{M^2/2}$ choices for the non-regular pairs.

Thus, the total number of T-free orientations of G is at most

$$n^{M+1} 2^{M^2} 2^{2\epsilon n^2} 2^{H(2\eta)n^2} 2^{nM} 2^k 2^{-\beta n^2} 2^{t_k(n)}.$$

Since $\epsilon < \eta$ and since $H(2\eta)$ tends to zero with η we have that for η sufficiently small as a function of β, the number of T-free orientations of G is less than $2^{t_k(n)}$, a contradiction.

Fix an orientation $\vec{G}$ of G for which $C = C(\eta)$ has at least $t_k(m) - \beta m^2$ edges. Let $V_1, \dots, V_m$ denote the parts in the ϵ-regular partition. According to Theorem 2.2, C has a vertex partition $W = W_1 \cup \dots \cup W_k$ with $\sum_i e(W_i) < \alpha m^2$. Thus, let C^* be the spanning subgraph of C from which the edges with both endpoints in W_i have been removed, for $i = 1, \dots, k$. Notice that C^* is a k-partite graph with at least $t_k(m) - (\beta + \alpha)m^2 = t_k(m) - \gamma m^2$ edges where $\gamma = \alpha + \beta$. We call a pair (V_i, V_j) a *one-sided dense pair* if it is an ϵ-regular pair and ij is not an edge of C but either $d(V_i, V_j) > \eta$ or $d(V_j, V_i) > \eta$. We claim that there are at most $(2k+1)\gamma m^2$ one-sided dense pairs. Assume this is false, adding to C^* the edges corresponding to one-sided dense pairs we get, by Lemma 2.3, that there are $k+1$ vertices of C (w.l.o.g. assume they are $\{1, \dots, k+1\}$) such that $ij \in C$ for all $1 \le i < j \le k+1$ except for the edge $1\,2$ which is not in C but corresponds to the one-sided dense pair (V_1, V_2) where $d(V_1, V_2) > \eta$. By Lemma 2.5, $\vec{G}$ has T, yielding the contradiction.

We now delete from G the following edges:

1. The edges with both endpoints in V_i for $i = 1, \dots, m$. We have shown that there are at most ϵn^2 such edges.

2. The edges belonging to non ϵ-regular pairs. We have shown that there are at most ϵn^2 such edges.

3. The edges belonging to non-dense pairs or one-sided dense pairs. There are at most $(2\eta + (2k+1)\gamma)n^2$ such edges.

4. The edges belonging to pairs (V_i, V_j) such that $ij \in W_s$ for $s = 1, \dots, k$. Since there are at most αm^2 such pairs, there are at most αn^2 such edges.

In other words, we keep only edges belonging to pairs (V_i, V_j) such that $ij \in C^*$. Denote this subgraph of G by G'. Then, G' is k-partite and, recalling that $\epsilon < \eta < \beta < \alpha$ and $\gamma = \alpha + \beta$, the number of edges deleted from G is at most

$$(\alpha + 2\eta + (2k+1)\gamma + 2\epsilon)n^2 < (4\eta + (4k+3)\alpha)n^2 \le (4k+7)\alpha n^2 < \delta n^2.$$

This concludes the proof of Lemma 2.1. $\square$

3 Proof of Theorem 1.1

In this section we complete the proof of our main theorem. The proof follows along the lines of [2] with several essential modifications required to deal with directed graphs. We start by recalling some notation and facts. $T_k(n)$ denotes the Turán graph, which is a complete k-partite graph on n

vertices with class sizes as equal as possible, and $t_k(n)$ is the number of edges in $T_k(n)$. Let $\delta_k(n)$ denote the minimum degree of $T_k(n)$. The following equalities are well known simple observations.

$$t_k(n) = t_k(n-1) + \delta_k(n), \qquad \delta_k(n) = n - \lceil n/k \rceil, \qquad \frac{k-1}{k}n^2/2 - k < t_k(n) \le \frac{k-1}{k}n^2/2. \quad (2)$$

We also need one additional easy lemma, before we present the proof of Theorem 1.1.

Lemma 3.1 *Let S be a tournament with the vertices $\{1, \ldots, k\}$. Let G be a directed graph and let $W_1, \ldots, W_k$ be subsets of vertices of G such that for every $i \neq j$ and every pair of subsets $X_i \subseteq W_i, |X_i| \ge 10^{-k}|W_i|$ and $X_j \subseteq W_j, |X_j| \ge 10^{-k}|W_j|$ there are at least $\frac{1}{10}|X_i||X_j|$ edges of G from X_i to X_j if $(i,j) \in S$ or at least $\frac{1}{10}|X_i||X_j|$ edges of G from X_j to X_i if $(j,i) \in S$. Then G contains a copy of S where the vertex playing the role of $i \in S$ belongs to W_i.*

Proof. We use induction on k. For $k = 1$ and $k = 2$ the statement is obviously true. Suppose it is true for $k-1$ and let $W_1, \ldots, W_k$ be the subsets of vertices of G which satisfy the conditions of the lemma for the fixed tournament S.

For every $1 \le i \le k-1$ denote by W_k^i the subset of vertices in W_k defined as follows. If $(i,k) \in S$ then $v \in W_k^i$ if v has less than $|W_i|/10$ incoming edges from W_i. If $(k,i) \in S$ then $v \in W_k^i$ if v has less than $|W_i|/10$ outgoing edges to W_i. By definition, if $(i,k) \in S$, we have $e(W_i, W_k^i) < |W_k^i||W_i|/10$ and if $(k,i) \in S$, we have $e(W_k^i, W_i) < |W_k^i||W_i|/10$ and therefore, in any case, $|W_k^i| < 10^{-k}|W_k|$. Thus we deduce that $\left|\bigcup_{i=1}^{k-1} W_k^i\right| < (k-1)10^{-k}|W_k| < |W_k|/2$. So in particular there exists a vertex v in W_k which does not belong to $\bigcup_{i=1}^{k-1} W_k^i$. For every $1 \le i \le k-1$ if $(i,k) \in S$ let W_i' be the set of incoming neighbors of v in W_i, and if $(k,i) \in S$ let W_i' be the set of outgoing neighbors of v in W_i. By definition, W_i' has size at least $|W_i|/10$. Note that for every pair of subsets $X_i \subseteq W_i'$ and $X_j \subseteq W_j'$ with sizes $|X_i| \ge 10^{-(k-1)}|W_i'| \ge 10^{-k}|W_i|$ and $|X_j| \ge 10^{-(k-1)}|W_j'| \ge 10^{-k}|W_j|$, G contains at least $\frac{1}{10}|X_i||X_j|$ edges between X_i and X_j in the appropriate direction. By the induction hypothesis there exists a copy of $S-k$ with one vertex in each W_i', playing the role of $i \in S$ in this copy, for $1 \le i \le k-1$. This copy, induced together with the vertex v, forms a copy of S where v plays the role of k. $\Box$

Proof of Theorem 1.1. Let n_0 be large enough to guarantee that the assertion of Lemma 2.1 holds for $\delta = 10^{-8k}$. Suppose that G is a graph on $n > n_0^2$ vertices with at least $2^{t_k(n)+m}$ distinct T-free orientations, for some $m \ge 0$. Our argument is by induction with an improvement at every step. More precisely, we will show that if G is not the corresponding Turán graph then it contains a vertex x such that $G-x$ has at least $2^{t_k(n-1)+m+1}$ distinct T-free orientations. Iterating, we obtain a graph on n_0 vertices with at least $2^{t_k(n_0)+m+n-n_0} > 2^{n_0^2}$ distinct T-free orientations. But a graph on n_0 vertices has at most $n_0^2/2$ edges and hence at most $2^{n_0^2/2}$ orientations. This contradiction will prove the theorem for $n > n_0^2$.

Recall from (2) that $\delta_k(n)$ denotes the minimum degree of $T_k(n)$, and $t_k(n) = t_k(n-1)+\delta_k(n)$. If G contains a vertex x of degree less than $\delta_k(n)$, then the edges incident with x can have, together, at most $2^{\delta_k(n)-1}$ orientations. Thus $G-x$ should have at least $2^{t_k(n-1)+m+1}$ distinct T-free orientations and we are done. Hence we may and will assume that all the vertices of G have degree at least $\delta_k(n)$.

Consider a partition $V_1 \cup \cdots \cup V_k$ of the vertex set of G which minimizes $\sum_i e(V_i)$. By our choice of n_0 in Lemma 2.1, we have that $\sum_i e(V_i) < 10^{-8k}n^2$. Note that if $|V_i| > \left(1/k + 10^{-6k}\right)n$, for some i, then every vertex in V_i has at least $\delta_k(n) - \left(\frac{k-1}{k}n - 10^{-6k}n\right) \geq 10^{-6k}n - 1$ neighbors in V_i. Thus $\sum_i e(V_i) > (10^{-6k}n - 1)(1/k + 10^{-6k})n/2 > 10^{-8k}n^2$, a contradiction. Therefore, $|V_i| - n/k \leq 10^{-6k}n$ for every i and also $|V_i| = n - \sum_{j \neq i} |V_j| \geq n/k - (k-1)10^{-6k}n$. So for every i we have $\big||V_i| - n/k\big| < 10^{-5k}n$. Let $\mathcal{D}$ denote the set of all possible T-free orientations of G.

First consider the case when there is some vertex with many neighbors in its own class of the partition, say $x \in V_1$ with $|N(x) \cap V_1| > n/(400k)$. Our choice of partition guarantees that in this case $|N(x) \cap V_i| > n/(400k)$ also for all $2 \leq i \leq k$, or by moving x to another part we could reduce $\sum_i e(V_i)$. Consider a permutation σ of $\{1, \ldots, k+1\}$. Let $\mathcal{D}_\sigma \subset \mathcal{D}$ be a subset of orientations defined as follows: An orientation belongs to $\mathcal{D}_\sigma$ if for all $i = 1, \ldots, k$ there exist $W_i \subset V_i$ with $|W_i| \geq n/(900k)$ such that if $(\sigma(i), \sigma(k+1)) \in T$ then x has an incoming edge from each $v \in W_i$ and if $(\sigma(k+1), \sigma(i)) \in T$ then x has an outgoing edge to each $v \in W_i$. Let $\mathcal{D}^* = \mathcal{D} \setminus \left(\cup_{\sigma \in S(k+1)} \mathcal{D}_\sigma\right)$.

Consider an orientation of G belonging to $\mathcal{D}_\sigma$. Since the orientation is T-free we have by Lemma 3.1 that there is some ordered pair (i,j) (corresponding to $(\sigma(i), \sigma(j)) \in T$) and subsets $X_i \subset W_i$, $X_j \subset W_j$ with $|X_i| \geq 10^{-k}|W_i|$ and $|X_j| \geq 10^{-k}|W_j|$ with at most $\frac{1}{10}|X_i||X_j|$ edges from X_i to X_j. There are at most $\binom{k}{2} 2^{|V_i|} 2^{|V_j|} < 2^{2n}$ ways to choose such an ordered pair (i,j) and to choose X_i and X_j and at most

$$\frac{1}{10}|X_i||X_j| \binom{|X_i||X_j|}{\lfloor |X_i||X_j|/10 \rfloor} < \frac{1}{10}|X_i||X_j| 2^{H(0.1)|X_i||X_j|} < 2^{H(0.11)|X_i||X_j|}$$

ways to orient at most $\frac{1}{10}|X_i||X_j|$ edges from X_i to X_j. In addition, from the structure of G we know that there are at most $t_k(n) + 10^{-8k}n^2 - |X_i||X_j|$ other edges in this graph, so the number of orientations in $\mathcal{D}_\sigma$ can be bounded as follows

$$\begin{aligned}
|\mathcal{D}_\sigma| &\leq 2^{t_k(n)+10^{-8k}n^2 - |X_i||X_j|}\, 2^{2n} 2^{H(0.11)|X_i||X_j|} \\
&\leq 2^{t_k(n)+10^{-8k}n^2}\, 2^{2n}\, (\sqrt{2}/2)^{|X_i||X_j|} \leq 2^{t_k(n)+10^{-8k}n^2}\, 2^{2n}\, (\sqrt{2}/2)^{10^{-2k-6}k^{-2}n^2} \\
&< 2^{t_k(n)+10^{-8k}n^2}\, 2^{2n} \left(2^{-0.01}\right)^{10^{-2k-6}k^{-2}n^2} = 2^{t_k(n)}\, 2^{2n}\, 2^{-(10^{-2k-8}k^{-2} - 10^{-8k})n^2} \\
&\ll \frac{2^{t_k(n)}}{2(k+1)!}.
\end{aligned}$$

In this estimate we used the facts that $H(0.11) < 1/2$, $|X_i|, |X_j| \geq n/(k10^{k+3})$, $\sqrt{2}/2 < 2^{-0.01}$ and that $10^{-2k-8}k^{-2} - 10^{-8k} > 0$ for all $k \geq 2$.

By the above discussion, $|\mathcal{D}^*|$ contains at least $2^{t_k(n)+m} - 2^{t_k(n)}/2 \geq 2^{t_k(n)+m-1}$ distinct T-free orientations of G. Let $\vec{G}$ be one of them. Since $\vec{G} \notin \mathcal{D}_\sigma$ for no $\sigma \in S(k+1)$ we must have some i such that there are at most $n/(900k)$ edges from x to V_i or at most $n/(900k)$ edges from V_i to x. Assume w.l.o.g. that there are at most $n/(900k)$ edges from x to V_i. Thus, there are at least $n/(400k) - n/(900k) > n/(900k)$ edges from V_i to x. Let $\sigma \in S(k+1)$ be a permutation for which $(\sigma(i), \sigma(k+1)) \in T$. Since $\vec{G} \notin \mathcal{D}_\sigma$ we must have some $j \neq i$ for which there are at most $n/(900k)$ edges from x to V_j or at most $n/(900k)$ edges from V_j to x.

We have shown that for every element of $\mathcal{D}^*$ there are (at least) two distinct indices i, j such that there are at most $n/(900k)$ edges connecting x to V_i in at least one of the two possible directions and the same hold for V_j (although not necessarily in the same direction). We call the direction with less than $n/(900k)$ edges the *sparse* direction.

Since the size of V_i is at most $(1/k + 10^{-5k})n$, we obtain that the number of orientations of edges between x and V_i, given the sparse direction, is bounded by

$$\frac{n}{900k}\binom{\lfloor(1/k+10^{-5k})n\rfloor}{\lfloor n/(900k)\rfloor} \leq 2^{H(0.002)(1/k+10^{-5k})n} \leq 2^{0.03(1/k+10^{-5k})n}, \tag{3}$$

since $H(0.002) < 0.03$. Clearly, this estimate is also valid for the number of orientations of edges between x and V_j, given the sparse direction between them. Note that in addition x is incident to at most $n - |V_i| - |V_j| \leq (\frac{k-2}{k} + 2 \cdot 10^{-5k})n$ other edges, which can have two possible directions. Using the above inequalities together with the facts that there are $\binom{k}{2}$ possible pairs i, j and four possible choices for the sparse directions between x and V_i and between x and V_j we obtain that the number of orientations of the edges incident with x is at most

$$4\binom{k}{2}\left(2^{0.03(1/k+10^{-5k})n}\right)^2 2^{\left(\frac{k-2}{k}+2\cdot 10^{-5k}\right)n} < 2^{\left(\frac{k-1}{k}-\frac{1}{100k}\right)n}.$$

But we had that $|\mathcal{D}^*| \geq 2^{t_k(n)+m-1}$. Hence the number of T-free orientations of $G - x$ is at least

$$2^{t_k(n)+m-1-(\frac{k-1}{k}-\frac{1}{100k})n} \gg 2^{t_k(n-1)+m+1}.$$

This completes the induction step in the first case.

Now we may assume that every vertex has degree at most $n/(400k)$ in its own class. We may suppose that G is not k-partite, or else by Turán's theorem $e(G) \leq t_k(n)$ and therefore $|\mathcal{D}| \leq 2^{t_k(n)}$ with equality only for $G = T_k(n)$. So, without loss of generality, we suppose that G contains an edge xy with $x, y \in V_k$. For $\sigma \in S(k+1)$ Let $\mathcal{D}_\sigma$ denote the set of all T-free orientations $\vec{G}$ of G in which $(x, y) \in \vec{G}$ if and only if $(\sigma(k), \sigma(k+1)) \in T$ and there are sets $W_i \subset V_i, |W_i| \geq n/(900k)$ for every $1 \leq i \leq k-1$ such that all the edges from x to W_i exist and are oriented from x to W_i if $(\sigma(k), \sigma(i)) \in T$ or oriented from W_i to x if $(\sigma(i), \sigma(k)) \in T$, and also all the edges from y to W_i exist and are oriented from y to W_i if $(\sigma(k+1), \sigma(i)) \in T$ or oriented from W_i to y if $(\sigma(i), \sigma(k+1)) \in T$. Let $\mathcal{D}^* = \mathcal{D} \setminus \left(\cup_{\sigma \in S(k+1)} \mathcal{D}_\sigma\right)$ denote the remaining orientations.

Consider an orientation $\vec{G} \in \mathcal{D}_\sigma$. Let T_σ denote the sub-tournament of T obtained by deleting the vertices $\sigma(k)$ and $\sigma(k+1)$. Since there is no T in $\vec{G}$, there is also no copy of T_σ in which the role of vertex $\sigma(i)$ is played by a vertex from W_i for $i = 1, \dots, k-1$. Thus, by Lemma 3.1, there is a pair (i,j) and subsets $X_i \subset W_i$, $X_j \subset W_j$ with $|X_i| \geq 10^{-(k-1)}|W_i|$ and $|X_j| \geq 10^{-(k-1)}|W_j|$ with at most $\frac{1}{10}|X_i||X_j|$ edges from X_i to X_j if $(\sigma(i),\sigma(j)) \in T$ or at most $\frac{1}{10}|X_i||X_j|$ edges from X_j to X_i if $(\sigma(j),\sigma(i)) \in T$. Arguing exactly as before in the first case we can prove that $|\mathcal{D}_\sigma| < \frac{2^{t_k(n)}}{2(k+1)!}$ and thus $|\mathcal{D}^*| \geq 2^{t_k(n)+m-1}$.

Next consider an orientation $\vec{G}$ of G from $\mathcal{D}^*$ and suppose, without loss of generality, that $(x,y) \in \vec{G}$. Let $\sigma \in S(k+1)$ be such that $(\sigma(k),\sigma(k+1)) \in T$. Since $\vec{G} \notin \mathcal{D}_\sigma$ there is some class V_i, $i \leq 1 \leq k-1$, in which x and y have at most $n/(900k)$ "common neighbors" in the sense that x has an outgoing edge to all these common neighbors in case $(\sigma(k),\sigma(i)) \in T$ or else x has an incoming edge from all these common neighbors in case $(\sigma(i),\sigma(k)) \in T$ and also y has an outgoing edge to all these common neighbors in case $(\sigma(k+1),\sigma(i)) \in T$ or else y has an incoming edge from all these common neighbors in case $(\sigma(i),\sigma(k+1)) \in T$. Note that for any other vertex z in V_i which is not such a common neighbor, we can only have at most three possible simultaneous orientations of the two edges xz and yz of G (assuming they exist). Since there are at most $(1/k + 10^{-5k})n$ vertices in V_i we have at most $3^{(1/k+10^{-5k})n}$ ways to orient such edges and, as in (3), at most

$$\frac{n}{900k}\binom{\lfloor (1/k+10^{-5k})n \rfloor}{\lfloor n/(900k) \rfloor} \leq 2^{H(0.002)(1/k+10^{-5k})n} \leq 2^{0.03(1/k+10^{-5k})n}$$

possibilities to choose a set of common neighbors of x and y in V_i. Thus, there are at most

$$2^{0.03\left(1/k+10^{-5k}\right)n}\,3^{\left(1/k+10^{-5k}\right)n} < 2^{1.7\left(1/k+10^{-5k}\right)n}$$

ways to orient edges from x,y to V_i. Note that, since the degree of x and y in V_k is at most $n/(400k)$ we have that the number of edges from x,y to $\bigcup_{j\neq i} V_j$ is bounded by $n(2(\frac{k-2}{k}+2\cdot 10^{-5k})+2/(400k))$. Even if all these edges can be oriented arbitrarily, since we have $k-1$ choices for the index i, and four possible combinations for the direction between x and y to their common neighbors in V_i, we can bound the number of orientations of the edges incident at x and y by

$$4(k-1)\,2^{1.7\left(1/k+10^{-5k}\right)n}\,2^{2\left(\frac{k-2}{k}+\frac{1}{400k}+2\cdot 10^{-5k}\right)n} < 2^{2\left(\frac{k-1}{k}-\frac{1}{100k}\right)n}.$$

But we know that $|\mathcal{D}^*| \geq 2^{t_k(n)+m-1}$. Thus the number of T-free orientations of $G - \{x,y\}$ is at least

$$2^{t_k(n)+m-1-2(\frac{k-1}{k}-\frac{1}{100k})n} \gg 2^{t_k(n-2)+m+2}.$$

This completes two induction steps for the second case and proves the theorem. $\square$

4 Directed triangles and transitive tournaments

In this section we consider two special cases of Theorem 1.1. We first show an easy proof of Theorem 1.1 in case $T = T_k$ is the transitive tournament with k vertices. Next, we consider the smallest non-transitive tournament, namely $T = C_3$ and obtain a proof for C_3 that avoids using the regularity lemma. Indeed, the proof for C_3 is more complicated than the proof for T_3. The proof for T_3 follows rather easily from a result of the second author in [11] concerning the number of red-blue edge colorings of a graph that avoid monochromatic triangles and the result for T_k ($k > 3$) follows from a recent result of [2] that generalizes the result of [11] to larger cliques. The proof for C_3 does not follow from these coloring results and requires an ad-hoc proof (although some arguments are similar to those appearing in the proof of [11]). To see the difficulty consider the following argument. Let $F(G)$ denote the number of red-blue edge colorings of G with no monochromatic triangle and let $D(G)$ denote the number of orientations of G with no C_3. Since the Ramsey number $R(3) = 6$, we have $F(G) = 0$ whenever G has a K_6. In particular, $F(K_n) = 0$ for $n \geq 6$. On the other hand, $D(K_n) = n!$, and $D(G) > 0$ always. Thus, it is more difficult to show that dense graphs have a relatively small $D(G)$ than it is to show that dense graphs have a relatively small $F(G)$. In fact, our proof for $T = C_3$ uses some powerful decomposition results that are not needed in the coloring case.

4.1 Orientations with no transitive tournaments

Let $F(G, k)$ denote the number of red-blue edge colorings of a graph G that have no monochromatic K_k. Let $F(n, k)$ denote the maximum possible value of $F(G, k)$ where G has n vertices. The following result is proved in [11] for $k = 3$ and in [2] for all $k > 3$ (the result in [2] also considers colorings with more than two colors).

Lemma 4.1 *Let $k \geq 3$. There exists $n_0 = n_0(k)$ such that for all $n \geq n_0$, $F(n, k) = 2^{t_{k-1}(n)}$.* □

In fact, in [11] it is shown that $n_0(3) = 6$ (and this is tight) while the $n_0(k)$ obtained in [2] is a huge number already for $k = 4$, as their proof uses the regularity lemma.
Lemma 4.1 and (1) enable us to prove the following:

Proposition 4.2 *Let $k \geq 3$. Then, $F(n, k) \geq D(n, T_k)$. Consequently, $D(n, T_k) = 2^{t_{k-1}(n)}$ for all $n \geq n_0(k)$ where $n_0(k)$ is the constant appearing in Lemma 4.1.*

Proof: Consider a graph G on n vertices. Label its vertices with the numbers $1, \ldots, n$. There is a bijection between red-blue edge colorings of G and orientations of G as follows: An edge is colored blue if and only if in the associated orientation the edge is oriented from the smaller vertex to the larger. Now assume that G has an orientation with no T_k. We show that the associated coloring has no monochromatic K_k. Consider a K_k of G. It must contain a directed cycle in the orientation.

In the associated coloring, we cannot have all the edges of such a cycle colored with the same color. We have shown that $F(G,k) \geq D(G,T_k)$. Hence, $F(n,k) \geq D(n,T_k)$. $\square$.

Notice that although $n_0(3)=6$ (in fact, $F(5,3)=82>2^6$) it is easy to check that $D(n,T_3)=2^{\lfloor n^2/4 \rfloor}$ for all $n \geq 1$ (one needs to check only $n=1,\ldots,5$). For $k=4$, however, we have $D(4,T_4)=2^6-4!=40>2^{t_3(4)}=32$.

4.2 Orientations with no directed triangles: Preliminary lemmas

Let H be a graph, and let $H+x$ denote the graph obtained from H by adding a new vertex x and connecting it to all vertices of H. For a C_3-free orientation $\vec{H}$ of H, let $ext(\vec{H})$ denote the number of C_3-free orientations of $H+x$ that are extensions of $\vec{H}$. Let $ext(H)$ denote the maximum possible value of $ext(\vec{H})$ taken over all C_3-free orientations of H. The following lemma determines $ext(H)$ for several specific graphs, and gives a general upper bound for $ext(H)$ in terms of a spanning subgraph of H.

Lemma 4.3

1. *$ext(K_k)=k+1$.*

2. *$ext(K_4^-)=6$ where K_4^- is the graph obtained from K_4 by deleting an edge.*

3. *For all $k \geq 3$, $ext(P_k)=ext(P_{k-1})+ext(P_{k-2})$ where P_k is the path with k vertices. In particular, for all $k \geq 1$, $ext(P_k)=z_k$ where z_k is the $k+2$ element of the Fibonacci sequence.*

4. *$ext(Q)=14$ where Q is the unique tree with five vertices which is not a star and not a path.*

5. *If S_k is the star with k vertices then $ext(S_k)=2^{k-1}+1$.*

6. *If $H_1,\ldots,H_k$ are the components of a spanning subgraph of H then $ext(H) \leq \prod_{i=1}^{k} ext(H_i)$.*

Proof:

1. $ext(K_k)=k+1$ follows immediately from the fact that every C_3-free orientation of K_k must be an acyclic orientation. There are $k+1$ positions to place x in any given order.

2. Assume the vertices of K_4^- are $\{a,b,c,d\}$ where b and c are not connected. Notice that every C_3-free orientation of K_4^- must be an acyclic orientation. Any extension of an acyclic orientation of K_4^- must also be an acyclic orientation of K_5^-. Consider a topological order of $\{a,b,c,d\}$ associated with an acyclic orientation of K_4^-. If b and c are not next to each other in this topological order then there are exactly five ways to extend the orientation. If b and c are next to each other (that is, they form an antichain), then there are six ways to extend the orientation.

3. Consider an orientation of $P_k = \{v_1, \dots, v_k\}$, $k \geq 3$. Assume, w.l.o.g. that the last edge is oriented (v_{k-1}, v_k). In this case, for any extension of the P_{k-1} subpath $\{v_1, \dots, v_{k-1}\}$ the orientation (x, v_k) does not introduce directed triangles. Thus, there are at most $ext(P_{k-1})$ such extensions. For any extension that orients (v_k, x), we must orient (v_{k-1}, x) and hence there are at most $ext(P_{k-2})$ such orientations. Altogether, there are at most $ext(P_{k-1}) + ext(P_{k-2})$ such extensions. This proves $ext(P_k) \leq ext(P_{k-1}) + ext(P_{k-2})$. It is easily seen that the bound is achieved by any orientation of P_k that has no directed subpath of length two.

4. Assume that the vertices of Q are $\{a, b, c, d, e\}$ where the edges are ab, bc, cd, ce. Consider first an orientation where the two edges cd and ce are oriented differently (that is, one of the edges enters c and the other emanates from c). Any orientation of xc forces either an orientation of xd or of xe and leaves three possible orientations for xb and xa together. Thus, there are at most $3 \cdot 4 = 12$ possible extensions. Consider next an orientation where the two edges cd and ce are oriented the same (that is, both of them enter c or both emanate from c). W.l.o.g. both enter c. As before, any orientation of xd and xe but the one in which both edges enter x, forces an orientation of xc and hence leaves three possible orientations for xb and xa together. The orientation of xd and xe in which both edges enter x, allows any extension of the subpath $\{a, b, c\}$ and since $ext(P_3) = 5$ this gives, altogether, $3 \cdot 3 + 5 = 14$ possible extensions.

5. The cases $k = 1, 2$ are trivial. For $k \geq 3$, S_k has a unique root denoted r. Consider first an orientation where the root r is either a source or a sink. Assume w.l.o.g, that r is a sink. If we orient (x, r) we can orient all other edges between x and the leaves arbitrarily. This gives 2^{k-1} extensions. If we orient (r, x) then we must make x a sink and hence we only have one legal extension. Altogether, we have $2^{k-1} + 1$ extensions for an orientation of S_k in which the root is a sink (or a source). It is easily seen that all other orientations of S_k have less extensions.

6. If $H_1, \dots, H_k$ are the components of a spanning subgraph of H then trivially $ext(H) \leq \prod_{i=1}^{k} ext(H_i)$. □

The final part of lemma 4.3, applied to spanning subgraphs whose components are any mixture of paths, cliques, stars, K_4^- or Q, enables us, using any of the first five parts of Lemma 4.3, to obtain upper bounds on $ext(H)$. Hence, our aim is to translate conditions that guarantee that H has a spanning subgraph whose components are paths, cliques, stars, K_4^- or Q into conditions that force upper bounds upon $ext(H)$. Put $|H| = m$. Probably the most famous of these conditions is the following theorem of Dirac (see, e.g. [4])

Lemma 4.4 (Dirac) *If $\delta(H) \geq \lfloor m/2 \rfloor$ then H has a Hamiltonian path.* □

Since the Fibonacci sequence has the property that for $1 \le s \le k$, $z_k \le (z_s)^{k/s}$ we have, using Lemma 4.3 and Lemma 4.4:

Corollary 4.5 *If $\delta(H) \ge \lfloor m/2 \rfloor$ then $ext(H) \le z_m$. In particular, if $m \ge s \ge 1$, $ext(H) \le (z_s)^{m/s}$.* □

If H has minimum degree that is higher than $m/2$, we can decompose the vertices of H into small and dense parts. The following well known theorem of Hajnal and Szemerédi [5] specifies conditions that guarantee the existence of a vertex decomposition into small cliques.

Lemma 4.6 (Hajnal and Szemerédi) *Let k be a positive integer. If $\delta(H) \ge m(1 - 1/k)$ then H has a spanning subgraph consisting of $\lfloor m/k \rfloor$ components, each isomorphic to K_k.* □

Lemma 4.3 and Lemma 4.6 together give:

Corollary 4.7 *Let $k \ge 2$ and assume $\delta(H) = \beta m$ where $(k-2)/(k-1) < \beta \le (k-1)/k$. Then*

$$ext(H) \le (k+1)^{m/k} \left(\frac{k}{(k+1)^{1-1/k}} \right)^{(k-1)m - m\beta k}.$$

Proof: Let $x = (k-1)m - k\delta(H)$. Notice that $(x+m) = 0 \bmod k$ and $x + m\beta = (k-1)(x+m)/k$. Consider the graph H' obtained from H by adding x new vertices, and connecting them to all the original vertices of H. The new vertices are not connected to each other. Notice that $\delta(H') = \min\{\delta(H) + x, m\} = \delta(H) + x = m\beta + x = (k-1)(x+m)/k$. By Lemma 4.6, H' has a set of $(x+m)/k$ vertex disjoint copies of K_k. Deleting the newly added vertices we get that H contains x vertex-disjoint copies of K_{k-1} and $(m - (k-1)x)/k$ additional vertex-disjoint copies of K_k. Thus, using Lemma 4.3 we get

$$ext(H) \le k^x (k+1)^{(m-(k-1)x)/k} \le (k+1)^{m/k} \left(\frac{k}{(k+1)^{1-1/k}} \right)^x$$

$$= (k+1)^{m/k} \left(\frac{k}{(k+1)^{1-1/k}} \right)^{(k-1)m - m\beta k}.$$

□

The following vertex-decomposition result has been recently proved by Kawarabayashi [6]

Lemma 4.8 (Kawarabayashi) *If m is a multiple of 4, and $\delta(H) \ge 5m/8$ then H has a spanning subgraph consisting of $m/4$ components, each isomorphic to K_4^-.* □

Lemma 4.3 and Lemma 4.8 together give:

Corollary 4.9 *If $m \ge 24$, $\delta(H) > m/2$ and $\beta = \min\{5/8\ ,\ \delta(H)/m\}$ then*

$$ext(H) \le 3 \cdot 5^{5m/3 - 8m\beta/3} 6^{-m+2m\beta}.$$

Proof: We may assume that $\delta(H) \le \lceil 5m/8 \rceil$ since $ext(H)$ is a monotone decreasing parameter with respect to edge addition. Let x be the minimum integer such that $(x+m) = 0 \bmod 4$ and $x + m\beta \ge 5(x+m)/8$. Notice that

$$x \in [5m/3 - 8m\beta/3, 5m/3 - 8m\beta/3 + 4].$$

Consider the graph H' obtained from H by adding x new vertices, and connecting them to all the original vertices of H. The new vertices are not connected to each other. Notice that $\delta(H') = \min\{\delta(H) + x, m\} = \delta(H) + x$ (since $m \ge 24$, $\beta > 0.5$ and either $\delta(H) = \beta m$ or else $\beta = 5/8$ and $\delta(H) \le \lceil 5m/8 \rceil$). Now,

$$\delta(H') = \delta(H) + x \ge m\beta + x \ge 5(x+m)/8 = 5|H'|/8.$$

By Lemma 4.8, H' has a K_4^- factor. Let S be a K_4^- copy in this factor. If S contains zero new vertices then S is already inside H. If S contains one new vertex then the subgraph of S inside H contains a P_3. If S contains two new vertices then they must be the two degree-two vertices of the K_4^-, and the subgraph of S inside H is a K_2. For $t = 0, 1, 2$ let t_i denote the number of copies in the K_4^--factor with t new vertices. Then, $t_1 + 2t_2 = x$ and $t_0 + t_1 + t_2 = (m+x)/4$. Furthermore, by Lemma 4.3,

$$ext(H) \le ext(K_4^-)^{t_0} ext(P_3)^{t_1} ext(K_2)^{t_2} = 6^{t_0} 5^{t_1} 3^{t_2}.$$

The last inequality subject to the above constraints is maximized when $t_1 = x$, $t_2 = 0$ and $t_0 = (m+x)/4 - x$. Thus,

$$ext(H) \le 6^{(m+x)/4 - x} 5^x \le \frac{5^4}{6^3} \cdot 5^{5m/3 - 8m\beta/3} 6^{-m+2m\beta} < 3 \cdot 5^{5m/3 - 8m\beta/3} 6^{-m+2m\beta}.$$

$\square$

4.3 Orientations with no directed triangles: The proof

In this section we prove:

Theorem 4.10 *For all $n \ge 600000$, $D(n, C_3) = 2^{\lfloor n^2/4 \rfloor}$.*

The constant 600000 can be improved considerably to less than 10000 at the price of additional case analysis. We prefer the "cleaner" proof. The proof of Theorem 4.10 is based upon the following lemma:

Lemma 4.11 *If $n \ge 320$, and G is a graph with n vertices, then at least one of the following must hold:*

1. $D(G) \le 2^{\lfloor n^2/4 \rfloor}$.

2. *There exists a vertex x of minimum degree such that if H is the subgraph of G induced by the neighbors of x then $ext(H) \leq 0.94 \cdot 2^{\lfloor n/2 \rfloor}$. Thus, $D(G) \leq 0.94 \cdot 2^{\lfloor n/2 \rfloor} D(G-x)$.*

3. *$\delta(G) = \lfloor n/2 \rfloor$, there exist two vertices x and y such that $D(G) \leq 2^{\lfloor n/2 \rfloor} D(G-x)$ and $D(G-x) \leq 0.94 \cdot 2^{\lfloor (n-1)/2 \rfloor} D(G - \{x,y\})$.*

The rest of this section is dedicated to the proof of Lemma 4.11 but we first show how Lemma 4.11 yields Theorem 4.10.

Proof of Theorem 4.10 given Lemma 4.11: Let $n \geq 600000$ and let G be a graph with n vertices. We show that $D(G) \leq 2^{\lfloor n^2/4 \rfloor}$. Consider $D(320)$. Trivially, $D(320) \leq 2^{160 \cdot 319}$. Now, let $v_1, \ldots, v_{n-318}$ be a sequence of vertices of G that satisfies the following: If G_i is the subgraph of G obtained by deleting $v_1, \ldots, v_{i-1}$ for $i = 1, \ldots, n-317$ ($G_1 = G$), then for all $i = 1, \ldots, n-319$, either $D(G_i) \leq 2^{\lfloor (n-i+1)^2/4 \rfloor}$ or else $D(G_i) \leq 0.94 \cdot 2^{\lfloor (n-i+1)/2 \rfloor} D(G_{i+1})$ or else $D(G_i) \leq 0.94 \cdot 2^{\lfloor (n-i+1)/2 \rfloor + \lfloor (n-i)/2 \rfloor} D(G_{i+2})$. According to Lemma 4.11, such a sequence exists.

Assume first that for some i, $1 \leq i \leq n-319$, $D(G_i) \leq 2^{\lfloor (n-i+1)^2/4 \rfloor}$. Let i be minimal with this property. In this case we have

$$D(G) \leq \prod_{j=1}^{i-1} 2^{\lfloor (n-j+1)/2 \rfloor} D(G_i) = 2^{\lfloor n^2/4 \rfloor - \lfloor (n-i+1)^2/4 \rfloor} D(G_i) \leq$$

$$2^{\lfloor n^2/4 \rfloor - \lfloor (n-i+1)^2/4 \rfloor} 2^{\lfloor (n-i+1)^2/4 \rfloor} \leq 2^{\lfloor n^2/4 \rfloor}$$

as required.

Assume next that for each $i = 1, \ldots, n-319$, either the second or third condition in Lemma 4.11 holds for G_i. In either case we have

$$D(G_i) \leq 0.94 \cdot 2^{\lfloor (n-i+1)/2 \rfloor + \lfloor (n-i)/2 \rfloor} D(G_{i+2}).$$

Therefore:

$$D(G) \leq (0.94)^{\lfloor (n-319)/2 \rfloor} \prod_{i=1}^{n-319} 2^{\lfloor (n-i+1)/2 \rfloor} D(320) = (0.94)^{\lfloor (n-319)/2 \rfloor} 2^{\lfloor n^2/4 \rfloor - \lfloor 319^2/4 \rfloor} D(320) \leq$$

$$2^{\lfloor n^2/4 \rfloor} 2^{25600} (0.94)^{\lfloor (n-319)/2 \rfloor} \leq 2^{\lfloor n^2/4 \rfloor}$$

as required. □

The proof of Lemma 4.11 is divided into several parts according to the structure of G. From here onwards we assume, whenever necessary, that G has $n \geq 320$ vertices.

Lemma 4.12 *If G is a complete bipartite graph then $D(G) \leq 2^{\lfloor n^2/4 \rfloor}$. If $n \geq 7$ and G has a vertex v of degree $n-1$ and $G \setminus \{v\}$ is a complete bipartite graph on $n-1$ vertices then $D(G) \leq 2^{\lfloor n^2/4 \rfloor}$. In particular, for both of these graphs, the first condition in Lemma 4.11 holds.*

Proof: The first claim is trivial as the number of edges of a complete bipartite graph is at most $\lfloor n^2/4 \rfloor$. To see the second claim, assume that the vertex classes of $G \setminus \{v\}$ are $x_1, \ldots, x_k$ and $y_1, \ldots, y_l$ where $k + l = n - 1$ and $1 \leq l \leq k$. Consider an orientation $\vec{S}$ of the star S induced by $\{v, y_1, \ldots, y_l\}$. We count the number of C_3-free orientations of $x_i \cup S$ that extend $\vec{S}$. There are two cases. If v is either a sink or a source in $\vec{S}$ then there are exactly $2^l + 1$ extensions. If v is neither a source nor a sink, then assume l_1 edges enter v and $l_2 = l - l_1$ edges emanate from v. If we orient (x_i, v) then there are 2^{l_1} possible extensions. If we orient (v, x_i) then there are 2^{l_2} possible extensions. In any case, there are at most $2^{l_1} + 2^{l_2} \leq 2^{l-1} + 2$ extensions. As there are $2^l - 2$ orientations of S in which v is neither a source nor a sink we have, for all $n \geq 7$ and $l + k = n - 1$, $1 \leq l \leq k$

$$D(G) \leq 2(2^l + 1)^k + (2^l - 2)(2^{l-1} + 2)^k \leq 2^{\lfloor n^2/4 \rfloor}.$$

□

In the following lemmas x denotes a vertex of minimum degree of G, and H denotes the subgraph of G induced by the neighbors of x. Hence $|H| = \delta(G)$. Notice also that if $|H| = \lfloor n/2 \rfloor + t$ then the minimality of x shows that $\delta(H) \geq 2t - 1$.

Lemma 4.13 *If $\delta(G) < \lfloor n/2 \rfloor$ then the second condition in Lemma 4.11 holds.*

Proof:

$$ext(H) \leq 2^{|H|} \leq 2^{\lfloor n/2 \rfloor - 1} = 0.5 \cdot 2^{\lfloor n/2 \rfloor}.$$

□

Lemma 4.14 *If $\delta(G) = \lfloor n/2 \rfloor$ and G is not complete bipartite then either the second or third condition in Lemma 4.11 holds.*

Proof: If H has at least one edge then, by Lemma 4.3,

$$ext(H) \leq ext(K_2) ext(K_1)^{|H|-2} = 3 \cdot 2^{\lfloor n/2 \rfloor - 2} = 0.75 \cdot 2^{\lfloor n/2 \rfloor}.$$

Otherwise, we may assume that every vertex of minimum degree has an isolated neighborhood. Let y be a neighbor of x. Then, $deg(x) = \lfloor n/2 \rfloor \leq deg(y) \leq n - |H| = n - \lfloor n/2 \rfloor$. Hence, if n is even then $deg(y) = n/2$ so its neighborhood is also isolated. This forces G to be a complete bipartite graph, contradicting the assumption in the statement of the lemma. If n is odd then either $deg(y) = (n-1)/2$ or else $deg(y) = (n+1)/2$. In the first case, y has minimum degree $(n-3)/2$ in $G - x$ so by Lemma 4.13 applied to $G - x$ and y we have

$$D(G) \leq 2^{\lfloor n/2 \rfloor} D(G - x) \leq 0.5 \cdot 2^{\lfloor n/2 \rfloor} \cdot 2^{\lfloor (n-1)/2 \rfloor} D(G - \{x, y\}).$$

In the second case, we can assume that every neighbor of x has degree $(n+1)/2$. This forces all the neighbors of x to have the same neighborhood (namely $V(G) \setminus H$). Since G is not complete bipartite, this common neighborhood is not isolated, and, furthermore, each neighbor of x has minimum degree $(n-1)/2$ in $G-x$. As in the first part of this lemma, if H' is the neighborhood of a neighbor y of x in $G-x$ then

$$ext(H') \leq ext(K_2)ext(K_1)^{|H'|-2} = 3 \cdot 2^{\lfloor (n-1)/2 \rfloor -2} \leq 0.75 \cdot 2^{\lfloor (n-1)/2 \rfloor}.$$

Thus,

$$D(G) \leq 2^{\lfloor n/2 \rfloor} D(G-x) \leq 0.75 \cdot 2^{\lfloor n/2 \rfloor} \cdot 2^{\lfloor (n-1)/2 \rfloor} D(G - \{x, y\}).$$

$\square$

Lemma 4.15 *If $\delta(G) = \lfloor n/2 \rfloor + 1$ and G is none of the graphs from lemma 4.12 then the second condition in Lemma 4.11 holds.*

Proof: Assume first that H contains a P_5. Since $ext(P_5) = z_5 = 13$ we have

$$ext(H) \leq 13 \cdot ext(K_1)^{|H|-5} = 13 \cdot 2^{\lfloor n/2 \rfloor -4} = \frac{13}{16} 2^{\lfloor n/2 \rfloor}.$$

Next, assume H has no P_5 but has a P_4. Every vertex of H has at least $\lfloor n/2 \rfloor + 1$ neighbors in G, so we have $\delta(H) \geq 1$. Since $|H| > 4$, there exists a vertex v not on the P_4. Let vu be an edge. If u is not a vertex of the P_4 then by Lemma 4.3

$$ext(H) \leq ext(P_4)ext(P_2)2^{|H|-6} = 8 \cdot 3 \cdot 2^{\lfloor n/2 \rfloor -5} = 0.75 \cdot 2^{\lfloor n/2 \rfloor}.$$

Otherwise, u must be an inner vertex of the P_4, and hence H contains the unique tree on five vertices that is neither a path nor a star, which we denoted Q in Lemma 4.3. Recall that $ext(Q) = 14$. Hence,

$$ext(H) \leq ext(Q)2^{|H|-5} = 14 \cdot 2^{\lfloor n/2 \rfloor -4} = 0.875 \cdot 2^{\lfloor n/2 \rfloor}.$$

Next, assume that H has no P_3 (we leave the case where H has no P_4 and has a P_3 to the end). In this case, since $\delta(H) \geq 1$ we must have $\delta(H) = 1$ and H must be a perfect matching. By Lemma 4.3 we have, for all $n \geq 14$,

$$ext(H) = ext(K_2)^{|H|/2} = 3^{(\lfloor n/2 \rfloor +1)/2} \leq 0.75 \cdot 2^{\lfloor n/2 \rfloor}.$$

Finally, assume that H has no P_4 but has a P_3. Assume first that H is not a star. Denote the vertices of a P_3 in H by (a, b, c) where b is the middle vertex. If (a, c) is also an edge then the fact that H has no P_4 and the fact that $|H| > 3$ show that there exists an edge both of whose endpoints are not in $\{a, b, c\}$. If (a, c) is not an edge, then the fact that H is not a star implies that there is some edge not incident with b. The fact that H has no P_4 shows that such an edge has both of its endpoints outside $\{a, b, c\}$. We have shown the existence of vertex disjoint P_3 and K_2. By Lemma 4.3,

$$ext(H) \leq ext(P_3)ext(K_2)2^{|H|-5} = 5 \cdot 3 \cdot 2^{\lfloor n/2 \rfloor -4} = \frac{15}{16} 2^{\lfloor n/2 \rfloor}.$$

Finally, assume that the neighborhood of every minimum degree vertex induces a star. It is not difficult to check that this forces G to be the unique graph with n vertices, n odd, having a vertex of degree $n-1$ and the remaining $n-1$ vertices induce a complete bipartite graph with $(n-1)/2$ vertices in each partite class. Thus, G is one of the graphs from Lemma 4.12, contradicting the assumption in the current lemma. $\square$

Lemma 4.16 *If $\delta(G) = \lfloor n/2 \rfloor + t$ and $1 < t < (\lfloor n/2 \rfloor + 2)/3$ then the second condition in Lemma 4.11 holds.*

Proof: Recall that $\delta(H) \geq 2t-1$. Let P be a longest path in H. Notice that $|P| \geq \delta(H)+1 \geq 2t \geq 4$. We consider two cases: $t \geq 3$ and $t = 2$. Assume first that $t \geq 3$. If $|P| \geq 4t-1$ then

$$ext(H) \leq z_{4t-1}2^{|H|-4t+1} = z_{4t-1}2^{\lfloor n/2 \rfloor -3t+1} \leq z_{11}^{(4t-1)/11}2^{\lfloor n/2 \rfloor -3t+1} = \left(\frac{233^{4/11}}{8}\right)^t \cdot 2^{\lfloor n/2 \rfloor} \cdot \frac{2}{233^{1/11}} <$$

$$(0.91)^3 \cdot 2^{\lfloor n/2 \rfloor} \cdot 1.22 < 0.92 \cdot 2^{\lfloor n/2 \rfloor}.$$

If P has at most $4t-2$ vertices then, as in Dirac's theorem, there is also a cycle C with the same set of vertices as that of P, and hence, by the maximality of P, the subgraph of H induced by this set of vertices is a connected component of H. Since $|H| > 4t-2$ this is not the whole graph, and hence we can find another path in another component whose number of vertices is at least $\delta(H)+1 \geq 2t$. We therefore have two vertex-disjoint paths of length (at least) $2t$ each and thus

$$ext(H) \leq z_{2t}^2 2^{|H|-4t} = z_{2t}^2 2^{\lfloor n/2 \rfloor -3t} \leq z_6^{2t/3}2^{\lfloor n/2 \rfloor -3t} = 21^{2t/3}2^{\lfloor n/2 \rfloor -3t} \leq (0.96)^t 2^{\lfloor n/2 \rfloor} \leq 0.89 \cdot 2^{\lfloor n/2 \rfloor}.$$

Consider next the case $t = 2$. If $|P| \geq 8$ we have

$$ext(H) \leq z_8 2^{\lfloor n/2 \rfloor -6} = \frac{55}{64} \cdot 2^{\lfloor n/2 \rfloor}.$$

If $|P| \leq 6$ we have, as in the case for $t \geq 3$, that there is also a cycle C with the same set of vertices as that of P, and hence, by the maximality of P, the subgraph of H induced by this set of vertices is a connected component of H. Since $|H| > 6$ we can find another path in another component whose number of vertices is at least $\delta(H)+1 \geq 4$. We have found two vertex-disjoint paths of length at least four each. If one of them has length at least five then

$$ext(H) \leq z_4 z_5 2^{\lfloor n/2 \rfloor -7} = \frac{104}{128}2^{\lfloor n/2 \rfloor}.$$

If both have length four then the maximality of P implies that its vertices induce a K_4. Thus,

$$ext(H) \leq ext(K_4) \cdot z_4 2^{\lfloor n/2 \rfloor -6} = \frac{40}{64}2^{\lfloor n/2 \rfloor}.$$

Finally, assume $|P| = 7$. If there is an edge connecting two vertices outside P then

$$ext(H) \leq ext(K_2)ext(P_7)2^{\lfloor n/2 \rfloor - 7} = \frac{102}{128} 2^{\lfloor n/2 \rfloor}.$$

Otherwise, by the maximality of $P = (a_1, \ldots, a_7)$ and the fact that $\delta(H) \geq 2t - 1 = 3$, every vertex outside of P is adjacent to at least three vertices among $\{a_2, a_3, a_4, a_5, a_6\}$. If some vertex $b \notin P$ is adjacent to a_3 then we have two paths of length four, namely, $P^1 = (a_1, a_2, a_3, b)$ and $P^2 = (a_4, a_5, a_6, a_7)$. Let $c \notin P$ and $c \neq b$. Since c has three neighbors on these paths and none of them are from $\{a_1, a_7, b\}$ we have that H contains a P_4 and a Q. Thus,

$$ext(H) \leq ext(Q) z_4 2^{\lfloor n/2 \rfloor - 7} = \frac{112}{128} 2^{\lfloor n/2 \rfloor}.$$

A similar argument holds if some vertex $b \notin P$ is adjacent to a_5. We may therefore assume that the neighborhood of all vertices outside P is precisely $\{a_2, a_4, a_6\}$. Let b_1, b_2, b_3 be such vertices. Then, $\{b_1, a_7, a_6, a_5, a_4\}$ induces a subgraph that contains a Q and $\{a_1, a_2, a_3, b_2, b_3\}$ induce a subgraph that contains the star S_5 where a_2 is the root. We therefore have

$$ext(H) \leq ext(Q)ext(S_5)2^{\lfloor n/2 \rfloor - 8} = 14 \cdot 17 \cdot 2^{\lfloor n/2 \rfloor - 8} = \frac{238}{256} 2^{\lfloor n/2 \rfloor}.$$

$\square$

Lemma 4.17 *If $\delta(G) = \lfloor n/2 \rfloor + t$ and $0.205n \geq t \geq (\lfloor n/2 \rfloor + 2)/3$ then the second condition in Lemma 4.11 holds.*

Proof: In this case we have $\delta(H) \geq 2t - 1 > \lfloor |H|/2 \rfloor - 1$. Hence, H is Hamiltonian. By Corollary 4.5 we have, since $n \geq 320$ and $|H| > n/2 \gg 21$,

$$ext(H) \leq (z_{21})^{|H|/21} = 28657^{(\lfloor n/2 \rfloor + t)/21} \leq 28657^{n(0.705/21)} \leq 1.41136^n \leq 0.94 \cdot 2^{\lfloor n/2 \rfloor}.$$

$\square$

Lemma 4.18 *If $\delta(G) = \lfloor n/2 \rfloor + t$ and $0.205n < t < 0.259n$ then the second condition in Lemma 4.11 holds.*

Proof: Put $\beta = \max\{5/8, \delta(H)/|H|\}$. Notice that $\beta > 0.5$. By Corollary 4.9 we have

$$ext(H) \leq 3 \cdot 5^{5|H|/3 - 8|H|\beta/3} 6^{-|H| + 2|H|\beta}.$$

Put $t = \alpha n$. Notice that $|H| \leq (0.5 + \alpha)n$ and either $\beta = 5/8$ or else $\beta \geq 0.99(2\alpha/(0.5 + \alpha))$. If $\beta = 5/8$ we have

$$ext(H) \leq 3 \cdot 6^{|H|/4} \leq 3 \cdot 6^{0.759n/4} \leq 3 \cdot (1.405)^n \leq 0.94 \cdot 2^{\lfloor n/2 \rfloor}.$$

If $\beta \geq 0.99(2\alpha/(0.5+\alpha))$ we have

$$ext(H) \leq 3 \cdot 6^{n(0.5+\alpha)(\frac{\ln 5}{\ln 6}(\frac{5}{3}-\frac{8\beta}{3})+2\beta-1)} \leq 3 \cdot 6^{n(0.5+\alpha)(0.4971-0.395\beta)} \leq$$

$$3 \cdot 6^{n(0.5+\alpha)(0.4971-\frac{2\alpha}{0.5+\alpha}0.391)} \leq 3 \cdot 6^{n(0.24855-\alpha 0.2849)} \leq 3 \cdot 6^{0.19015n} < 0.75 \cdot 2^{\lfloor n/2 \rfloor}.$$

$\square$

Lemma 4.19 *If $0.8n > \delta(G) \geq 0.758n$ then the second condition in Lemma 4.11 holds.*

Proof: Put $\delta(G) = \alpha n$. We may assume $\delta(H) \leq 0.75|H|$ since $ext(H)$ is monotone decreasing with respect to edge addition. Notice that if $\delta(H) = \beta|H|$ then we have $0.75 \geq \beta \geq \frac{2\alpha-1}{\alpha} > 2/3$. By Corollary 4.7 with $k=4$ we have

$$ext(H) \leq 5^{\alpha n/4}(1.197)^{3\alpha n-4\beta\alpha n} \leq (5^{\alpha/4}(1.197)^{4-5\alpha})^n \leq$$

$$(5^{0.1895}(1.197)^{0.21})^n \leq 1.4089^n < 0.94 \cdot 2^{\lfloor n/2 \rfloor}.$$

$\square$

Lemma 4.20 *If $\frac{k+1}{k+2}n > \delta(G) \geq \frac{k}{k+1}n$ For $k=4,5$. then the second condition in Lemma 4.11 holds.*

Proof: Since $|H| = \delta(G)$ we have $\delta(H) \geq \frac{k-1}{k}|H|$. By Lemma 4.6, H has $\lfloor |H|/k \rfloor$ vertex-disjoint copies of K_k In particular H has a spanning subgraphs whose components are $\lfloor |H|/k \rfloor$ copies of K_k and at most $k-1$ isolated vertices. Thus,

$$ext(H) \leq (k+1)^{\lfloor |H|/k \rfloor} 2^{k-1} \leq (k+1)^{(\frac{k+1}{k+2}n)/k} 2^{k-1} < 1.3985^n \cdot 16 < 0.94 \cdot 2^{\lfloor n/2 \rfloor}$$

where the last inequality is valid for $k=4,5$ and for all $n \geq 320$. $\square$

Lemma 4.21 *If $\delta(G) \geq \frac{6}{7}n$ then the second condition in Lemma 4.11 holds.*

Proof: As in the previous lemma we get $\delta(H) \geq \frac{5}{6}n$ and

$$ext(H) \leq 7^{n/6} 2^5 \leq 0.94 \cdot 2^{\lfloor n/2 \rfloor}$$

where the last inequality is valid for all $n \geq 320$. $\square$

5 Concluding remarks and open problems

- Another interesting problem is to determine $D(n, m, T)$, that is, the maximum possible number of T-free orientations of a graph with n vertices and m edges. By (1) we trivially have $D(n, m, T) = 2^m$ whenever $m \leq t_{k-1}(n)$, where k is the number of vertices of T. The problem becomes considerably more difficult for $m > t_{k-1}(n)$. Even for $T = C_3$ the exact values for all (n, m) pairs are unknown. Using the fact that every non-transitive tournament contains a triangle we trivially have $D(n, \binom{n}{2}, C_3) = n!$. It is also not difficult to prove the following proposition

  Proposition 5.1

  1. $D(n, \binom{n}{2} - 1, C_3) = (n-1)!(n-1)$ *for* $n \geq 2$.
  2. $D(n, \binom{n}{2} - 2, C_3) = n! - 2(n-1)! + (n-2)! + 2(n-3)!$ *for* $n \geq 4$. □

- A careful examination of the constants in the proof of Theorem 4.10 shows that the theorem holds for all $n \geq 10000$ (in fact, slightly less). It is of some interest to determine $D(n, C_3)$ for all n. Using a computer program we have $D(n, C_3) = n!$ for $n = 1, \ldots, 7$. The same program yields $D(8, C_3) = 2^{16}$. The case $n = 9$ is too large for a straightforward computer verification. We conjecture that the following holds for all $n \geq 1$

  $$D(n, C_3) = \max\{2^{\lfloor n^2/4 \rfloor}, n!\}.$$

  In particular, Theorem 4.10 is conjectured to hold for all $n \geq 8$.

- It would be interesting to generalize Theorem 1.1 to the situation of finding the number of H-free orientations, where H is any directed graph, not necessarily a tournament. In fact, it is not difficult to generalize Lemma 2.1 to apply also for $H = T(t)$, where t is any positive integer and $T(t)$ is the directed graph obtained from the k-vertex tournament T by replacing each vertex with an independent set of size t. In particular, this shows that an asymptotic version of Theorem 1.1 holds for $T(t)$.

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