MAXIMIZING THE NUMBER OF NONNEGATIVE SUBSETS[*]

NOGA ALON[†], HAROUT AYDINIAN[‡], AND HAO HUANG[§]

Abstract. Given a set of n real numbers, if the sum of the elements of every subset of size larger than k is negative, what is the maximum number of subsets of nonnegative sum? In this note we show that the answer is $\binom{n-1}{k-1} + \binom{n-1}{k-2} + \cdots + \binom{n-1}{0} + 1$, settling a problem of Tsukerman. We provide two proofs; the first establishes and applies a weighted version of Hall's theorem, and the second is based on an extension of the nonuniform Erdős–Ko–Rado theorem.

1. Introduction. Let $\{x_1, \ldots, x_n\}$ be a sequence of n real numbers whose sum is negative. It is natural to ask the following question: What is the maximum possible number of subsets of nonnegative sum it can have? One can set $x_1 = n - 2$ and $x_2 = \cdots = x_n = -1$. This gives $\sum_{i=1}^{n} x_i = -1 < 0$ and 2^{n-1} nonnegative subsets, since all the proper subsets containing x_1, together with the empty set, have a nonnegative sum. It is also not hard to see that this is best possible, since for every subset A either A or its complement $\{x_1, \ldots, x_n\} \backslash A$ must have a negative sum. Now a new question arises: Suppose it is known that every subset of size larger than k has a negative sum; what is the maximum number of nonnegative subsets? This question was raised recently by Tsukerman [6]. The previous problem is the special case when $k = n - 1$. A similar construction, $x_1 = k - 1$, $x_2 = \cdots = x_n = -1$, yields a lower bound $\binom{n-1}{k-1} + \cdots + \binom{n-1}{0} + 1$. In this note we prove that this is also tight.

THEOREM 1.1. *Suppose that every subset of $\{x_1, \ldots, x_n\}$ of size larger than k has a negative sum. Then there are at most $\binom{n-1}{k-1} + \cdots + \binom{n-1}{0} + 1$ subsets with nonnegative sums.*

One can further ask whether the extremal configuration $x_1 = k - 1$, $x_2 = \cdots = x_n = -1$ is unique, in the sense that the family $\mathcal{F} = \{U : \sum_{i \in U} x_i \geq 0\}$ is unique up to isomorphism. Note that when $k = n - 1$, an alternative construction $x_1 = -n$, $x_2 = \cdots = x_n = 1$ also gives 2^{n-1} nonnegative subsets, while the family $\mathcal{F}$ that it defines is nonisomorphic to the previous one. More generally, for $k = n - 1$ any sequence $X = \{x_1, \ldots, x_n\}$ of n integers whose sum is -1 contains exactly 2^{n-1} nonnegative subsets, as for any subset A of X exactly one of the two sets A and $X - A$ has a nonnegative sum. However, for every $k < n - 1$, we can prove the

uniqueness by the following result, in which the number of nonnegative elements in the set is also taken into account.

THEOREM 1.2. *Let $1 \leq t \leq k < n - 1$ be integers, and let X be a set of real numbers $\{x_1, \ldots, x_n\}$, in which there are exactly t nonnegative numbers. Suppose that the sum of elements of every subset of size greater than k is negative; then the number of nonnegative subsets is at most $2^{t-1}(\binom{n-t}{k-t} + \cdots + \binom{n-t}{0} + 1)$. This is tight for all admissible values of t, k, and n.*

For every fixed k and n with $k < n - 1$, the expression in Theorem 1.2 is strictly decreasing in t. Indeed, if $1 \leq t < t + 1 \leq k \leq n$, then, using Pascal's identity,

$$2^{t-1}\left(\binom{n-t}{k-t} + \cdots + \binom{n-t}{0} + 1\right) - 2^t\left(\binom{n-t-1}{k-t-1} + \cdots + \binom{n-t-1}{0} + 1\right)$$

$$= 2^{t-1}\left[\sum_{i=0}^{k-t}\binom{n-t}{i} - \sum_{i=1}^{k-t}\binom{n-t-1}{i-1} - \sum_{i=0}^{k-t-1}\binom{n-t-1}{i} - 1\right]$$

$$= 2^{t-1}\left[\binom{n-t}{k-t} - \binom{n-t-1}{k-t-1} + \binom{n-t}{0} - \binom{n-t-1}{0} - 1\right]$$

$$= 2^{t-1}\left[\binom{n-t-1}{k-t} - 1\right].$$

The last quantity is strictly positive for all $t < k < n - 1$ (and is zero if $k = n - 1$).

Therefore, the above theorem implies Theorem 1.1 as a corollary and shows that it is tight for $k < n - 1$ only when there is exactly one nonnegative number. The bound in Theorem 1.2 is also tight by taking $x_1 = k - t$, $x_2 = \cdots = x_t = 0$, $x_{t+1} = \cdots = x_n = -1$. In this example, the sum of any $k + 1$ elements is negative, and a subset is nonnegative if and only if it is either of the form $\{x_1\} \cup S \cup T$, where S is an arbitrary subset of $\{x_2, \ldots, x_t\}$ and T is a subset of $\{x_{t+1}, \ldots, x_n\}$ having size at most $k - t$, or when it is a subset of $\{x_2, \ldots, x_t\}$.

The rest of this short paper is organized as follows. In section 2 we prove a Hall-type theorem and deduce from it the existence of perfect matchings in certain bipartite graphs. This enables us to obtain Theorem 1.2 as a corollary. Section 3 includes a strengthening of the nonuniform version of the Erdős–Ko–Rado theorem, which leads to an alternative proof of Theorem 1.1. In the last section, we discuss some further research directions.

2. The main result. The following lemma is useful in proving the existence of a perfect matching when combined with the fundamental theorem of Hall [3]. It can also be reformulated and viewed as a strengthening of Hall's theorem itself.

LEMMA 2.1. *In a bipartite graph G with two parts A and B, suppose there exist partitions $A = A_1 \cup \cdots \cup A_k$ and $B = B_1 \cup \cdots \cup B_l$ such that, for every $i \in [k], j \in [l]$, in the induced bipartite graph $G[A_i, B_j]$ all the vertices in A_i have equal degrees and all the vertices in B_j have equal degrees too. Define an auxiliary bipartite graph H on the same vertex set, and replace every nonempty $G[A_i, B_j]$ by a complete bipartite graph. Then G contains a perfect matching if and only if H contains a perfect matching.*

Proof. The "only if" part is obvious since G is a subgraph of H. In order to prove the "if" part, note first that if H contains a perfect matching, then $\sum_{i=1}^{k}|A_i| = \sum_{j=1}^{l}|B_j|$. We will verify that the graph G satisfies the conditions in Hall's theorem: for any subset $X \subset A$, its neighborhood has size $|N_G(X)| \geq |X|$. Set $Y = N_G(X)$ and

$$X_i = X \cap A_i, \quad Y_j = Y \cap B_j,$$

and define two sequences of numbers $\{x_i\}$, $\{y_j\}$ so that

$$|X_i| = x_i|A_i|, \qquad |Y_j| = y_j|B_j|.$$

Consider the pairs (i, j) such that $G[A_i, B_j]$ is nonempty. In this induced bipartite subgraph suppose that every vertex in A_i has degree d_1 and that every vertex in B_j has degree d_2. Double-counting the number of edges gives $d_1 \cdot |A_i| = d_2 \cdot |B_j|$. On the other hand, we also have $d_1 \cdot |X_i| \leq d_2 \cdot |Y_j|$, since every vertex in X_i has exactly d_1 neighbors in Y_j and every vertex in Y_j has at most d_2 neighbors in X_i. Combining these two inequalities, we have $y_j \geq x_i$ for every pair (i, j) such that $G[A_i, B_j]$ is nonempty. We claim that these inequalities imply that $|Y| \geq |X|$; i.e.,

(1)
$$\sum_{j=1}^{l} |B_j| y_j \geq \sum_{i=1}^{k} |A_i| x_i.$$

To prove the claim it suffices to find $d_{i,j} \geq 0$ defined on every pair (i, j) with nonempty $G[A_i, B_j]$ such that

$$\sum_{i,j} d_{i,j}(y_j - x_i) = \sum_{j=1}^{l} |B_j| y_j - \sum_{i=1}^{k} |A_i| x_i.$$

In other words, the conditions for Hall's theorem would be satisfied if the following system has a solution:

(2) $$\sum_{i} d_{i,j} = |B_j|, \qquad \sum_{j} d_{i,j} = |A_i|, \qquad d_{i,j} \geq 0, \qquad d_{i,j} = 0 \quad \text{if } G[A_i, B_j] = \emptyset.$$

The standard way to prove that there is a solution is by considering an appropriate flow problem. Construct a network with a source s, a sink t, and vertices $a_1, \ldots, a_k$ and $b_1, \ldots, b_l$. The source s is connected to every a_i with capacity $|A_i|$, and every b_j is connected to the sink t with capacity $|B_j|$. For every pair (i, j) there is an edge from a_i to b_j. Its capacity is $+\infty$ if $G[A_i, B_j]$ is nonempty and 0 otherwise. Then (2) is feasible if and only if there exists a flow of value $\sum_i |A_i| = \sum_j |B_j|$. Now we consider an arbitrary cut in this network: $(s \cup \{a_i\}_{i \in U_1} \cup \{b_j\}_{j \in U_2}, t \cup \{a_i\}_{i \in [k] \setminus U_1} \cup \{b_j\}_{j \in [l] \setminus U_2})$. Its capacity is finite only when for every $i \in U_1$, $j \in [l] \setminus U_2$, $G[A_i, B_j]$ is empty. Therefore in the auxiliary graph H, if we take $Z = \cup_{i \in U_1} A_i$, then the degree condition $|N_H(Z)| \geq |Z|$ implies that $\sum_{j \in U_2} |B_j| \geq \sum_{i \in U_1} |A_i|$, and thus the capacity of this cut is equal to

$$\sum_{i \in [k] \setminus U_1} |A_i| + \sum_{j \in U_2} |B_j| \geq \sum_{i \in [k] \setminus U_1} |A_i| + \sum_{i \in U_1} |A_i| = \sum_{i=1}^{k} |A_i|.$$

Therefore the minimum cut in this network has capacity at least $\sum_{i=1}^{k} |A_i|$, and there is a cut of exactly this capacity, namely the cut consisting of all edges emanating from the source s. By the max-flow min-cut theorem, we obtain a maximum flow of the same size, and this provides us with a solution $d_{i,j}$ to (2), which verifies the Hall's condition (1) for the graph G.   □

Remark. Lemma 2.1 can also be reformulated in the following way: given G with the properties stated, define the reduced auxiliary graph H' on the vertex set

$A' \cup B'$, where $A' = [k]$, $B' = [l]$, such that $i \in A'$ is adjacent to $j \in B'$ if $G[A_i, B_j]$ is nonempty. If, for every subset $X \subset A'$, $\sum_{j \in N_{H'}(X)} |B_j| \geq \sum_{i \in X} |A_i|$, then G has a perfect matching. For the case of partitioning A and B into singletons, this is exactly Hall's theorem. Moreover, if we let $k = 1$ and $l = 1$ in Lemma 2.1, it immediately implies the well-known result that every d-regular bipartite graph has a perfect matching.

COROLLARY 2.2. *For $m \geq r + 1$, let G be the bipartite graph with two parts A and B such that both parts consist of subsets of $[m]$ of size between 1 and r. $S \in A$ is adjacent to $T \in B$ if and only if $S \cap T = \emptyset$ and $|S| + |T| \geq r + 1$. Then G has a perfect matching.*

Proof. For $1 \leq i \leq r$, let $A_i = B_i = \binom{[m]}{i}$, i.e., all the i-subsets of $[m]$. Let us consider the bipartite graph $G[A_i, B_j]$ induced by $A_i \cup B_j$. Note that when $i + j \leq r$ or $i + j > m$, $G[A_i, B_j]$ is empty, while when $r + 1 \leq i + j \leq \min\{2r, m\}$, every vertex in A_i has degree $\binom{m-i}{j}$ and every vertex in B_j has degree $\binom{m-j}{i}$. Therefore, by Lemma 2.1, it suffices to check that the reduced auxiliary graph H' satisfies the conditions in the above remark. We discuss the following two cases.

First, suppose $m \geq 2r$. Note that in the reduced graph H', $A' = B' = [r]$, every vertex i in A' is adjacent to the vertices $\{r + 1 - i, \ldots, r\}$ in B'. The only inequalities we need to verify are: for every $1 \leq t \leq r$, $\sum_{j=r+1-t}^{r} |B_j| \geq \sum_{i=1}^{t} |A_i|$. Note that

$$\sum_{j=r+1-t}^{r} |B_j| = \sum_{i=1}^{t} \binom{m}{r-t+i} \geq \sum_{i=1}^{t} \binom{m}{i}.$$

The last inequality holds because the function $\binom{m}{k}$ is increasing in k when $k \leq m/2$.

Now we consider the case $r + 1 \leq m \leq 2r - 1$. In this case every vertex i in A' is adjacent to vertices from $r + 1 - i$ to $\min\{r, m - i\}$. More precisely, if $1 \leq i \leq m - r$, then i is adjacent to $\{r + 1 - i, \ldots, r\}$ in B', and if $m - r + 1 \leq i \leq r$, then i is adjacent to $\{r + 1 - i \ldots, m - i\}$ in B'. It suffices to verify the conditions for $X = \{1, \ldots, t\}$ when $t \leq r$, and for $X = \{s, \ldots, t\}$ when $m - r \leq s \leq t \leq r$. In the first case $N_{H'}(X) = \{r + 1 - t, \ldots, r\}$, and the desired inequality holds since

$$\sum_{j=r+1-t}^{r} \binom{m}{j} = \sum_{i=1}^{r} \binom{m}{i} - \sum_{i=1}^{r-t} \binom{m}{i} \geq \sum_{i=1}^{r} \binom{m}{i} - \sum_{i=t+1}^{r} \binom{m}{i} = \sum_{i=1}^{t} \binom{m}{i}.$$

For the second case, $N_{H'}(X) = \{r + 1 - t, \ldots, m - s\}$, and since $m \geq r + 1$,

$$\sum_{j=r+1-t}^{m-s} \binom{m}{j} = \sum_{i=s}^{m-r+t-1} \binom{m}{i} \geq \sum_{i=s}^{t} \binom{m}{i}.$$

This concludes the proof of the corollary. □

We are now ready to deduce Theorem 1.2 from Corollary 2.2.

Proof of Theorem 1.2. Without loss of generality, we may assume that $x_1 \geq x_2 \geq \cdots \geq x_n$ and $x_1 + \cdots + x_{k+1} < 0$. Suppose that there are $t \leq k$ nonnegative numbers, i.e., $x_1 \geq \cdots \geq x_t \geq 0$ and $x_{t+1}, \ldots, x_n < 0$. If $t = 1$, then every nonempty subset of nonnegative sum must contain x_1, which gives at most $\binom{n-1}{k-1} + \cdots + \binom{n-1}{0} + 1$ nonnegative subsets in total, as needed.

Suppose $t \geq 2$. We first partition all the subsets of $\{1, \ldots, t\}$ into 2^{t-1} pairs (A_i, B_i), with the property that $A_i \cup B_i = [t]$, $A_i \cap B_i = \emptyset$, and $1 \in A_i$. This can be done by pairing every subset with its complement. For every i consider the bipartite

graph G_i with vertex set $V_{i,1} \cup V_{i,2}$ such that $V_{i,1} = \{A_i \cup S : S \subset \{t+1, \ldots, n\}, |S| \leq k - t\}$ and $V_{i,2} = \{B_i \cup S : S \subset \{t+1, \ldots, n\}, |S| \leq k - t\}$. Note that if a nonempty subset with index set U has a nonnegative sum, then $|U \cap \{t+1, \ldots, n\}| \leq k - t$; otherwise $U \cup \{1, \ldots, t\}$ gives a nonnegative subset with more than k elements. Therefore every nonnegative subset is a vertex of one of the graphs G_i. Moreover, we can define the edges of G_i in such a way that $A_i \cup S$ is adjacent to $B_i \cup T$ if and only if $S, T \subset \{t+1, \ldots, n\}$, $S \cap T = \emptyset$, and $|S| + |T| \geq k - t + 1$. Note that, by this definition, two adjacent vertices cannot both correspond to nonnegative subsets; otherwise $S \cup T \cup \{1, \ldots, t\}$ gives a nonnegative subset of size larger than k. Applying Corollary 2.2 with $m = n - t$, $r = k - t$, we conclude that there is a matching saturating all the vertices in G_i except A_i and B_i. Therefore the number of nonnegative subsets in G_i is at most $\binom{n-t}{k-t} + \cdots + \binom{n-t}{0} + 1$. Note that this number remains the same for different choices of (A_i, B_i), so the total number of nonnegative subsets is at most $2^{t-1}(\binom{n-t}{k-t} + \cdots + \binom{n-t}{0} + 1)$.   □

3. A strengthening of the nonuniform Erdős–Ko–Rado (EKR) theorem. A conjecture of Manickam, Miklós, and Singhi (see [4], [5]) asserts that for any integers n, k satisfying $n \geq 4k$ every set of n real numbers with a nonnegative sum has at least $\binom{n-1}{k-1}$ k-element subsets whose sum is also nonnegative. The study of this problem (see, e.g., [1] and the references therein) reveals a tight connection between questions about nonnegative sums and problems in extremal finite set theory. A connection of the same flavor exists for the problem studied in this note, as explained in what follows.

The EKR theorem [2] has the following nonuniform version: for integers $1 \leq k \leq n$ the maximum size of an intersecting family of subsets of sizes up to k is equal to $\binom{n-1}{k-1} + \binom{n-1}{k-2} + \cdots + \binom{n-1}{0}$. The extremal example is the family of all the subsets of size at most k containing a fixed element. This result is a direct corollary of the uniform EKR theorem, together with the obvious fact that each such family cannot contain both a set and its complement. In this section we show that the following strengthening is also true. It also provides an alternative proof of Theorem 1.1.

THEOREM 3.1. *Let $1 \leq k \leq n - 1$, and let $\mathcal{F} \subset 2^{[n]}$ be a family consisting of subsets of size at most k, where $\emptyset \notin \mathcal{F}$. Suppose that for every two subsets $A, B \in \mathcal{F}$, if $A \cap B = \emptyset$, then $|A| + |B| \leq k$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1} + \binom{n-1}{k-2} + \cdots + \binom{n-1}{0}$.*

Proof. Define $\binom{[n]}{\leq k} = \{A \subset [n] : |A| \leq k\}$. Let us first observe that if $\mathcal{F}$ is an upset in $\binom{[n]}{\leq k}$ (that is, $A \in \mathcal{F}$ implies that $\{B \in \binom{[n]}{\leq k} : B \supseteq A\} \subset \mathcal{F}$), then $\mathcal{F}$ is an intersecting family, and hence the bound for $|\mathcal{F}|$ holds. Suppose there exist $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$; thus $|A| + |B| \leq k$. Since $k \leq n - 1$ and $\mathcal{F}$ is an upset, there exists a $C \in \mathcal{F}$ such that $A \subset C$, $C \cap B = \emptyset$, and $|C| + |B| > k$, which is a contradiction.

Next let us show that, applying so-called pushing up operations $S_i(\mathcal{F})$, we can transform $\mathcal{F}$ to an upset $\mathcal{F}^* \subset \binom{[n]}{\leq k}$ of the same size, without violating the property of $\mathcal{F}$. This, together with the observation above, will complete the proof. For $i \in [n]$ we define $S_i(\mathcal{F}) = \{S_i(A) : A \in \mathcal{F}\}$, where

$$S_i(A) = \begin{cases} A & \text{if } A \cup \{i\} \in \mathcal{F} \text{ or } |A| = k, \\ A \cup \{i\} & \text{otherwise.} \end{cases}$$

It is clear that $|S_i(\mathcal{F})| = |\mathcal{F}|$, and applying finitely many operations $S_i(\mathcal{F})$, $i \in [n]$, we come to an upset $\mathcal{F}^* \subset \binom{[n]}{\leq k}$. To see that $S_i(\mathcal{F})$ does not violate the property of $\mathcal{F}$, let $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$, where $\mathcal{F}_1 = \{A \in \mathcal{F} : i \in A\}$, $\mathcal{F}_0 = \mathcal{F} \setminus \mathcal{F}_1$. Thus $S_i(\mathcal{F}_1) = \mathcal{F}_1$. What

we have to show is that for each pair $A, B \in \mathcal{F}$ the pair $S_i(A), S_i(B)$ satisfies the condition in the theorem as well. In fact, the only doubtful case is when $A, B \in \mathcal{F}_0$, $A \cap B = \emptyset$, $|A| + |B| = k$. The subcase when $S_i(A) = A \cup \{i\}, S_i(B) = B \cup \{i\}$ is also clear. Thus, it remains to consider the situation when $S_i(A) = A$ (or $S_i(B) = B$). In this case $(A \cup \{i\}) \in \mathcal{F}$, since $|A|, |B| \leq k - 1$. Moreover, $(A \cup \{i\}) \cap B = \emptyset$ and $|A \cup \{i\}| + |B| = k + 1$, a contradiction.   □

To see that Theorem 3.1 implies Theorem 1.1, take $\mathcal{F} = \{F : \emptyset \neq F \subset \{1, \ldots, n\}, \sum_{i \in F} x_i \geq 0\}$. The family $\mathcal{F}$ satisfies the conditions in Theorem 3.1 since if $A, B \in \mathcal{F}$, then $\sum_{i \in A} x_i \geq 0$, $\sum_{i \in B} x_i \geq 0$. If, moreover, $A \cap B = \emptyset$, then $\sum_{i \in A \cup B} x_i \geq 0$, and it follows that $|A \cup B| \leq k$.

4. Concluding remarks. We have given two different proofs of the following result: for a set of n real numbers, if the sum of elements of every subset of size larger than k is negative, then the number of subsets of nonnegative sum is at most $\binom{n-1}{k-1} + \cdots + \binom{n-1}{0} + 1$. The connection between questions of this type and extremal problems for hypergraphs that appears here as well as in [1] and some of its references is interesting and deserves further study.

Another intriguing question motivated by the first proof is the problem of finding an explicit perfect matching for Corollary 2.2 without resorting to Hall's theorem. When r is small or $r = m-1$, one can construct such a perfect matching, but it seems that things get more complicated when r is closer to $m/2$.

Note added in proof. Alexey Pokrovskiy recently informed us that he has found an alternative proof of Theorem 3.1. His proof is very different from the one given here, combining the Kruskal–Katona theorem with certain EKR-type inequalities.

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