EIGENVALUES OF $K_{1,k}$-FREE GRAPHS AND THE CONNECTIVITY OF THEIR INDEPENDENCE COMPLEXES

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Abstract. Let G be a graph on n vertices, with maximal degree d, and not containing $K_{1,k}$ as an induced subgraph. We prove:
(1) $\lambda(G) \leq (2 - \frac{1}{2k-2} + o(1))d$
(2) $\eta(\mathcal{I}(G)) \geq \frac{n(k-1)}{d(2k-3)+k-1}$.

Here $\lambda(G)$ is the maximal eigenvalue of the Laplacian of G, $\mathcal{I}(G)$ is the independence complex of G, and $\eta(\mathcal{C})$ denotes the topological connectivity of a complex $\mathcal{C}$ plus 2.

These results provide improved bounds for the existence of independent transversals in $K_{1,k}$-free graphs.

1. The maximum Laplace eigenvalue of $K_{1,k}$-free graphs

Let $G = (V, E)$ be a connected graph on the set of vertices $V = \{1, 2, \ldots, n\}$ with maximum degree d. The Laplacian of G is the n by n matrix $L = (L_{ij})$ where $L_{ii} = d_i$ is the degree of the vertex i, $L_{ij} = -1$ if $ij \in E$ and $L_{ij} = 0$ if $i \neq j, ij \notin E$. Let $\lambda = \lambda(G)$ denote the largest eigenvalue of L. It is easy to prove and well known that $\lambda(G) \leq 2d$, and equality holds iff G is d-regular and bipartite. If G contains no induced copy of $K_{1,k}$ this estimate can be improved, as stated in the next theorem.

Theorem 1.1. *Let $G = (V, E)$ be a (simple) graph with maximum degree d containing no induced copy of $K_{1,k}$. Let $t(d, k)$ denote the minimum possible number of edges of a graph on d vertices with no independent set of size k. Then $\lambda(G) \leq 2d - \frac{t(d,k)}{d-1}$.*

Note that by Turán's Theorem $t(d, k) = (1 + o(1))\frac{d^2}{2k-2}$, where the $o(1)$-term tends to zero as d tends to infinity, and thus for large d the above theorem provides an upper bound of $[2 - \frac{1}{2k-2} + o(1)]d$ for $\lambda(G)$. Note also that this is not very far from being tight. Indeed, consider a graph H' obtained from a $(k-1)$ regular bipartite graph H by replacing each vertex u of H by a clique V_u of size s and by replacing each edge uv of H by a complete bipartite graph connecting each vertex of V_u with each vertex of V_v. This graph is $d = ks - 1$ regular and contains no induced $K_{1,k}$. The vector assigning value 1 to each vertex of H' that belongs to V_u for some u in the first color class of H, and value -1 to each vertex of H' that belongs to V_v for some v in the second color class of H is an eigenvector of the Laplacian of H corresponding to the eigenvalue $ks - 1 + (k-1)s - (s-1) = (2k-2)s > [2 - \frac{2}{k}]d$.

Proof of Theorem 1.1: Let $G = (V, E)$ be a graph with maximum degree d and no induced copy of $K_{1,k}$, and let λ be the largest eigenvalue of the Laplacian L of G. Put $V = \{1, 2, \ldots, n\}$ and let $(x_1, x_2, \ldots, x_n)$ be an eigenvector for the eigenvalue λ, where $\sum_{i=1}^n x_i^2 = 1$. Therefore $Lx = \lambda x$ and $x^t Lx = \lambda \|x\|_2^2$. It is easy to check that $x^t Lx = \sum_{ij \in E}(x_i - x_j)^2$. Writing d_i for the degree of vertex number i, we have $\sum_{ij \in E}(x_i^2 + x_j^2) = \sum_{i=1}^n d_i x_i^2$. Combining these, we get:

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$$2d-\lambda = (2d-\lambda)\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n}(2d-2d_i)x_i^2 + \sum_{i=1}^{n}(2d_i-\lambda)x_i^2 = \sum_{i=1}^{n}(2d-2d_i)x_i^2 + \sum_{ij\in E}[2x_i^2+2x_j^2-(x_i-x_j)^2].$$

Therefore

$$2d-\lambda = \sum_{i=1}^{n}(2d-2d_i)x_i^2 + \sum_{ij\in E}(x_i+x_j)^2. \tag{1}$$

Let $\mathbf{T}$ be the set of triangles in G. For each triangle T in G on the vertices i,j,q define

$$S(T) = (x_i+x_j)^2 + (x_j+x_q)^2 + (x_q+x_i)^2.$$

Clearly

$$S(T) = x_i^2 + x_j^2 + x_q^2 + (x_i+x_j+x_q)^2 \geq x_i^2 + x_j^2 + x_q^2.$$

Fix a vertex i of G, and let $N = N(i)$ be the set of its d_i neighbors. Since G contains no induced copy of $K_{1,k}$ the induced subgraph of G on N contains no independent set of size k and thus spans at least $t(d_i,k)$ edges. It follows that i is contained in at least $t(d_i,k)$ triangles of G. We thus conclude that

$$\sum_{T\in\mathbf{T}} S(T) \geq \sum_{i=1}^{n} t(d_i,k)x_i^2. \tag{2}$$

On the other hand, since G has maximum degree d, every edge is contained in at most $(d-1)$ triangles, and therefore

$$\sum_{T\in\mathbf{T}} S(T) \leq (d-1)\sum_{ij\in E}(x_i+x_j)^2. \tag{3}$$

By (2) and (3)

$$(d-1)\sum_{ij\in E}(x_i+x_j)^2 \geq \sum_{i=1}^{n} t(d_i,k)x_i^2,$$

and therefore, by (1),

$$2d-\lambda \geq \sum_{i=1}^{n}[2(d-d_i)+\frac{t(d_i,k)}{d-1}]x_i^2 \geq \sum_{i=1}^{n}\frac{t(d,k)}{d-1}x_i^2 = \frac{t(d,k)}{d-1},$$

here we used the fact that $2(d-d_i)(d-1) \geq t(d,k) - t(d_i,k)$ for all $d_i \leq d$. One way to verify that this inequality holds (with room to spare) is as follows. Let $T(d,k)$ be the complement of the Turan graph, having d vertices and $t(d,k)$ edges (so, $T(d,k)$ is the union of $k-1$ disjoint cliques). Observe that $T(d_i,k)$ is an induced subgraph of $T(d,k)$, which has $t(d,k)$ edges. Therefore one can get $T(d,k)$ from $T(d_i,k)$ by adding vertices one by one, where each new vertex is adjacent to a subset of the existing ones, and hence the number of edges added per added vertex never exceeds $d-1 \leq 2(d-1)$. This completes the proof. □

2. The connectivity of the independence complex of a $K_{1,k}$-free graph.

A simplicial complex $\mathcal{C}$ is called (homotopically) *k-connected* if for every $-1 \leq j \leq k$, every continuous function $f : S^j \to ||\mathcal{C}||$ can be extended to a continuous function $\tilde{f} : B^{j+1} \to ||\mathcal{C}||$ (here $||\mathcal{C}||$ is the underlying space of the geometric realization of $\mathcal{C}$). Intuitively, this means that there is no hole of dimension $k+1$ or less, where a "hole of dimension d" is an image of S^{d-1} that is not filled (it is the missing filling that is of dimension d). The *connectivity* $\eta(\mathcal{C})$ of $\mathcal{C}$ is the largest k for which $\mathcal{C}$ is k-connected, plus 2 (this differs from the ordinary definition of connectivity, in which the 2 is not added. The addition of 2 simplifies the statements of the theorems). Another version of connectivity is the homological connectivity: $\eta_H(\mathcal{C})$ is the maximal k such that $H_i(\mathcal{C}) = 0$ for all $i \leq k-2$. It is known that $\eta_H(\mathcal{C}) \geq \eta(\mathcal{C})$ (see, e.g., [12]), and that they are equal if $\eta(\mathcal{C}) \geq 3$ (the latter follows by a theorem of Hurewicz, [12]).

The complex of independent sets of vertices in a graph G is denoted by $\mathcal{I}(G)$. In [2] the following lower bound on $\eta_H(\mathcal{I}(G))$ was proved:

Theorem 2.1. $\eta_H(\mathcal{I}(G)) \geq \frac{|V(G)|}{\lambda(G)}$.

This yields, among other things, the following:

Corollary 2.2. *For any graph G with maximum degree d, $\eta_H(\mathcal{I}(G)) \geq \frac{n}{2d}$.*

The corollary was proved in [11] by simpler methods, and for homotopic η it was proved in [4].

Combining Theorem 1.1 with Theorem 2.1 yields lower bounds on $\eta_H(\mathcal{I}(G))$ for any $K_{1,k}$-free graph G. These bounds can be improved:

Theorem 2.3. *If $k > 2$ and G is a $K_{1,k}$-free graph on n vertices with maximum degree d, then $\eta_H(\mathcal{I}(G)) \geq \frac{n(k-1)}{d(2k-3)+k-1}$.*

This was proved in [7] for $k = 3$, namely for claw free graphs. The present proof is shorter.

The proof requires some preliminaries. For a graph G and a vertex $v \in V(G)$ we denote by $G - v$ the graph obtained from G by the removal of v and the edges adjacent to it, and by $G \wr v$ the graph $G - v - N(v)$, where $N(v)$ denotes the set of neighbors of v. Then $\mathcal{I}(G \wr v) = link_{\mathcal{I}(G)}(v)$ (namely, the complex consisting of those sets whose union with v belongs to $\mathcal{I}(G)$). A standard application of the exactness of the Mayer-Vietoris sequence yields:

Lemma 2.4. *For any complex $\mathcal{C}$ and vertex $v \in V(\mathcal{C})$*

$$\eta_H(\mathcal{C}) \geq \min(\eta_H(\mathcal{C} - v), \eta_H(link_{\mathcal{C}}(v)) + 1).$$

Here is a short explanation why this is true. Given two complexes $\mathcal{A}$ and $\mathcal{B}$ (each considered as a set of edges, that in this context are usually called 'simplices'), the Mayer-Vietoris sequence is a naturally defined sequence of homomorphisms

$$\ldots \to H_{n+1}(\mathcal{A} \cup \mathcal{B}) \to H_n(\mathcal{A} \cap \mathcal{B}) \to H_n(\mathcal{A}) \oplus H_n(\mathcal{B}) \to H_n(\mathcal{A} \cup \mathcal{B}) \to H_{n-1}(\mathcal{A} \cap \mathcal{B}) \to \ldots$$

The Mayer-Vietoris theorem (to be found in any standard textbook on algebraic topology, like [12, 13]) is that this sequence is exact. This means that if two terms (groups) that are two apart are null, then so is the term (group) between them. In particular, if $H_n(\mathcal{A}) \oplus H_n(\mathcal{B}) = 0$ and $H_{n-1}(\mathcal{A} \cap \mathcal{B}) = 0$ then $H_n(\mathcal{A} \cup \mathcal{B}) = 0$. In terms of connectivity, this means:

$$\eta_H(\mathcal{A} \cup \mathcal{B}) \geq \min(\eta_H(\mathcal{A}), \eta_H(\mathcal{B}), \eta_H(\mathcal{A} \cap \mathcal{B}) + 1). \tag{4}$$

Let $\mathcal{C} = \mathcal{I}(G)$, $\mathcal{A} = \mathcal{C} - v$ and $\mathcal{B} = link_{\mathcal{C}}(v) * \{v\}$ (here "*" denotes the join operation, so $\mathcal{B} = link_{\mathcal{C}}(v) \cup \{I + v \mid I \in link_{\mathcal{C}}(v)\}$). Then $\mathcal{A} \cap \mathcal{B} = link_{\mathcal{C}}(v)$. Clearly, $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$. Since $\mathcal{B}$ is contractible to v, we have $\eta_H(\mathcal{B}) = \infty$, and hence the lemma follows from (4).

By the lemma, for any graph G and vertex $v \in V(G)$ the following is true: :

$$\eta_H(\mathcal{I}(G)) \geq \min(\eta_H(\mathcal{I}(G - v)), \eta_H(\mathcal{I}(G \wr v)) + 1). \tag{5}$$

A lower bound on η_H obtained from this inequality can be formulated in terms of a game between two players, CON and NON, on the graph G. CON wants to show high connectivity, NON wants to thwart this attempt. At each step, CON chooses a vertex v of the graph remaining at this stage, the starting point being the graph G. NON can then either remove the offered vertex from the graph (we call such a step "deletion"), or remove it and its neighbors (we call such a step "explosion"). The payoff of a game to CON is the number of explosions, or ∞ if there appears at some stage an isolated vertex. We define $\Psi(G)$ to be the maximum, over all strategies of CON, of the minimal payoff. The bound on η is then stated as:

Theorem 2.5. $\eta_H(\mathcal{I}(G)) \geq \Psi(G)$.

Remark 2.6. A similar result, using another segment of the Mayer-Vietoris sequence, was proved by Meshulam. For an edge $e = uv$ denote by $G \wr e$ the graph $(G \wr u) \wr v$.

Theorem 2.7. [11] *For any edge e:*

$$\eta_H(\mathcal{I}(G)) \geq \min(\eta_H(\mathcal{I}(G-e)), \eta_H(\mathcal{I}(G \wr e)) + 1).$$

This means that in the above game one can also offer NON edges, alongside vertices. There is no example known in which it is provably not enough to offer CON edges to obtain the best bound. Thomassé and Rao [14] gave an example in which it is not enough to use vertex offers.

Proof of Theorem 2.3. By (the trivial part of) Brooks' theorem, for every induced subgraph K of G we have $\alpha(K) \geq \frac{|V(K)|}{d+1}$. Let u be a vertex of degree d, and let N be its neighborhood. Choose inductively vertices $v_1, \ldots, v_d$ in N, so that v_i has maximal degree in $G_i = G[N - \{v_j \mid j < i\}]$. Since G is $K_{1,k}$-free, $\alpha(G[N]) < k$, and hence by the observation above

$$deg_{G_i}(v_i) \geq \frac{d-i+1}{k-1} - 1. \tag{6}$$

Play now the game by offering NON one by one the vertices $z_i = v_{d-i+1}$. Not wishing to isolate u, NON will explode one of them, say z_p. Writing $i = d-p+1$, by (6) $deg_{G_i}(z_p) \geq \frac{d-i+1}{k-1} - 1 = \frac{p}{k-1} - 1$. The number of vertices removed before the explosion is $p-1$, and the number of vertices removed by the explosion is at most $d+1$, but of those $deg_{G_i}(v_i)$ were removed before the explosion. Thus the total number of vertices removed is at most

$$d + 1 + p - 1 - (\frac{d-i+1}{k-1} - 1)$$

Since $p \leq d$, this is not more than $\frac{2k-3}{k-1}d + 1$. We have thus forced NON to perform one explosion, and paid the price of removal of at most $\frac{2k-3}{k-1}d+1$ vertices. Repeating this procedure until the graph is exhausted (or an isolated vertex appears) shows that $\Psi(G) \geq \frac{n(k-1)}{d(2k-3)+k-1}$. □

Conjecture 2.8. *If $G = (V, E)$ is $K_{1,k}$-free and has maximum degree d then $\eta_H(\mathcal{I}(G)) \geq \frac{|V|}{d+k-1}$.*

If true, then this conjecture is sharp when d is divisible by $k-1$, as shown by taking G to be a Turán graph, the complement of the disjoint union of $\frac{d}{k-1} + 1$ cliques of size $k-1$. Here $\eta_H(\mathcal{I}(G)) = 1$, namely $\mathcal{I}(G)$ is disconnected.

Theorem 2.9. *Conjecture 2.8 is true for line graphs of simple graphs, with $k = 3$, namely if G is the line graph of a simple graph H then*

$$\eta_H(\mathcal{I}(G)) \geq \frac{|V(G)|}{\Delta(G) + 2}.$$

Proof. Let A be the $E(H) \times V(H)$ incidence matrix of H, where $V(G) = E(H) = \{e_1 \ldots, e_n\}$. Then AA^T is the adjacency matrix of G, plus $2I$. Hence $L(G) = D - AA^T + 2I$, where D is the diagonal matrix with $D_{ii} = deg_G(e_i)$. Since AA^T is positive semi definite, this implies that $\lambda_1(L(G)) \leq \Delta(G) + 2$, which by Theorem 2.1 proves the desired result. □

Remark 2.10. The entire discussion can be carried out also in homotopic terms. For this purpose we have to prove Lemma 2.4 also for homotopic η, namely:

Lemma 2.11. *For any complex $\mathcal{C}$ and vertex $v \in V(\mathcal{C})$*

$$\eta(\mathcal{C}) \geq \min(\eta(\mathcal{C} - v), \eta(link_{\mathcal{C}}(v)) + 1).$$

To make this less of a miracle, let us present the idea of a proof, with no claim to rigor. A function between two geometric realizations of simplicial complexes is called *simplicial* if it maps simplices to simplices, and its restriction to every simplex is linear. Let $k = \min(\eta(\mathcal{C} - v), \eta(link_{\mathcal{C}}(v)) + 1)$. It is known that to prove $\eta(\mathcal{C}) \geq k$ it suffices to show that every simplicial map f from a triangulation T of $S = S^{k-2}$ to $||\mathcal{C}||$ is extendable to the ball B^{k-1}. It is possible to show that in fact it is enough to prove this for f that is injective on every simplex. We shall show that f can be continuously extended to a function from some triangulation T' of B^{k-1} whose boundary is T, to $||\mathcal{C}||$. Let v_i ($i = 1, \ldots, m$, where possibly $m = 0$) be the vertices of T mapped by f to v. Intuition tells us that the link of v_i in T is an image of S^{k-3}, and indeed this is true if T is a PL (piecewise linear) triangulation, i.e. has a subdivision isomorphic to a subdivision of the boundary of a simplex. Hence by the assumption that $\eta(link_{\mathcal{C}}(v)) \geq k - 1$ it follows that there exists a filling L_i of $link_T(v_i)$ in $link_C(v)$. The union of all these L_i, together with $S \setminus \bigcup_{i \leq m} link_S(v_i)$, is an image of S^{k-2} in $\mathcal{C} - v$. By the assumption that $\eta(\mathcal{C} - v) \geq k$ it follows that this union can be filled in $\mathcal{C} - v$. Together with $\bigcup_{i \leq m} L_i * v_i$ this forms a filling of S in $\mathcal{C}$.

3. Applications to independent transversals

Let $G = (V, E)$ be a graph with maximum degree d and let $V = V_1 \cup V_2 \cup \ldots \cup V_m$ be a partition of V into pairwise disjoint sets. An *independent transversal* in G with respect to this partition is an independent set of G containing exactly one vertex in each V_i. In [8] it was proved that if $|V_i| \geq 2d$ for all $i \leq m$ then there exists an independent transversal (in [6] it was proved before that $|V_i| \geq 25d$ suffices). In [10] the following topological version of Hall's theorem was proved:

Theorem 3.1. *If $\eta_H(\mathcal{I}(G[\bigcup_{i \in I} V_i])) \geq |I|$ for every $I \subseteq [m]$ then there exists an independent transversal.*

This improved upon the homotopic version of this theorem (namely, with η replacing η_H), that was proved in [5]. Theorem 2.1 yields that if the largest eigenvalue of the Laplacian of every induced subgraph of G is bounded by h, then sets V_i of size h suffice. Theorem 1.1 therefore implies that for graphs that contain no induced copy of $K_{1,k}$ sets V_i of size $(2 - \frac{1}{2k-2} + o(1))d$ suffice. The combination of Theorems 3.1 and 2.3 provides the better:

Theorem 3.2. *If G is $K_{1,k}$-free and $|V_i| \geq \frac{d(2k-3)+k-1}{(k-1)}$ then there exists an independent transversal.*

Remark 3.3. An anonymous referee kindly provided also a combinatorial proof of this result. The topological proof has some advantage, however: it can be applied to a situation in which any matroid $\mathcal{M}$ and a graph G share the same vertex set. In the setting of independent transversals $\mathcal{M}$ is a partition matroid. In [1] a generalization of Theorem 3.1 was proved: a sufficient condition for the existence of a base of $\mathcal{M}$ that is independent in G is that for every set X of vertices $\eta(\mathcal{I}(G[X])) \geq \rho(\mathcal{M}.X)$ where $\mathcal{M}.X$ is the contraction of $\mathcal{M}$ to X. Plugging in the bound above proves a generalization of Theorem 3.2.

Theorem 2.1 can be applied to line graphs of r-uniform linear hypergraphs (that is, hypergraphs in which no two edges share more than one common vertex). For such line graphs, the most negative eigenvalue of the adjacency matrix is at least $-r$, as the adjacency matrix can be written as $BB^T - rI$, where B is the incidence matrix of the hypergraph. This, therefore, implies, by the above reasoning, that any partition into sets V_i of size at least $d + r$, where d is the maximum degree of the line graph, admits an independent transversal. In particular this applies to any partition of the triangles of a Steiner Triple System on n vertices (and $n(n-1)/6$ triangles) into sets of size at least $3n/2 + O(1)$. It seems plausible that the constant $3/2$ here can be reduced, possibly even to $1/2$. A similar question regarding line graphs of simple graphs is worth studying as well.

We close this short paper with two questions.

(1) (Improving the estimate in Theorem 1.1) Is it true that in a $K_{1,k}$-free graph with maximum degree d the maximum Laplace eigenvalue is no larger than $(2-\frac{2}{k}+o(1))d$? As mentioned after the statement of Theorem 1.1, this estimate, if correct, is tight.

(2) Do the results that follow for the existence of independent transversals for $K_{1,k}$-free graphs hold also for graphs that contain no induced copy of $K_{k,k}$? In [3] it was shown that for such graphs $\eta(\mathcal{I}(G)) \geq \frac{|V(G)|}{2d-1}$, implying that if $V(G)$ is partitioned into sets of size $2d-1$ then there exists an independent transversal.

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