Probabilistic Methods in Coloring and Decomposition Problems

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Abstract

Numerous problems in Graph Theory and Combinatorics can be formulated in terms of the existence of certain colorings of graphs or hypergraphs. Many of these problems can be solved or partially solved by applying probabilistic arguments. In this paper we discuss several examples that illustrate the methods used. This is mainly a survey paper, but it contains some new results as well.

1 Probability and Coloring; Older Examples

Probabilistic methods have been useful in combinatorics for almost fifty years. Many examples dealing with various branches of Combinatorics can be found in [20] and in [31]. Coloring is one of the most popular areas in Combinatorics and in Graph Theory and has been the source of many intriguing problems for decades. It is therfore not surprising that there are various rather old known applications of probabilistic techniques in different coloring problems. In this section we discuss briefly some representing examples.

Recall that the *Ramsey number* $r(k,l)$ is the smallest integer r such that in any 2-coloring of the edges of the complete graph K_r on r vertices there is always either a red K_k or a blue K_l. The fact that $r(k,l)$ is finite for every two integers k and l is the content of Ramsey Theorem [29]. One of the first applications of the Probabilistic Method in Combinatorics is the lower bound of Erdös for the Ramsey numbers, obtained in 1947. (For simplicity we only state the case $k=l$.)

Theorem 1.1 (Erdös [17]) *If* $\binom{n}{k}2^{1-\binom{k}{2}}<1$ *then* $r(k,k)>n$. *Therefore,*

$$r(k,k)>(1+o(1))\frac{k}{e\sqrt{2}}2^{k/2}.$$

The proof is very simple. Simply color the edges of K_n randomly by two colors by choosing the color of each edge randomly and independently with equal probability to be either red or blue. The expected number of monochromatic copies of K_k is clearly $\binom{n}{k}2^{1-\binom{k}{2}}<1$ and hence there is a coloring with neither a red K_k nor a blue K_k, showing that indeed $r(k,k)>n$. A simple (though a little tedious) derivation of the asymptotics using Stirling's Formula shows that the largest n for which the last inequality holds is indeed $(1+o(1))\frac{k}{e\sqrt{2}}2^{k/2}$, completing the proof. $\Box$

Although this is a very simple example, it demonstrates the power of the basic Probabilistic Method. The bound it supplies is still essentially the best known bound for $r(k,k)$ (it has only been improved by a factor of 2 by a more delicate probabilistic argument). There is no known explicit coloring that supplies a bound which is exponential in k.

Turning to a more complicated example, we mention another result of Erdös, which is one of the most pleasing applications of the Probabilistic Method. The *girth* of a graph is the length of the shortest cycle in it.

Theorem 1.2 (Erdös [18]) *For every two integers $k \geq 2$ and $l \geq 3$ there exists a graph G with chromatic number $\chi(G) > k$ and with girth at least l.*

Here is a sketch of the proof of this theorem. Fix a real number $\epsilon < 1/l$, define $p = n^{\epsilon-1}$ and let H be a random graph on n labelled vertices chosen by picking each pair of vertices as an edge randomly and independently with probability p. It is not too difficult to show that the expected number of cycles of length smaller than l in H is $o(n)$ and that the probability that H contains an independent set of size at least $\lceil \frac{3}{p} ln\ n \rceil$ is $o(1)$. Therefore, with positive probability (if n is sufficiently large) H contains no such independent set and has less than $n/2$ of these short cycles. Let G be the graph obtained from such an H by omitting a vertex from each of these short cycles. Then the girth of G is at least l and since it contains no independent set of size $\lceil \frac{3}{p} ln\ n \rceil$ and its number of vertices is at least $n/2$ its chromatic number is at least $n/(2\frac{3}{p} ln\ n) \geq n^{\epsilon}/(6 ln\ n)$. To complete the proof we can now simply choose a sufficiently large n. $\square$

Another beautiful coloring result proved by probabilistic means is due to Erdös and Lovász, and deals with hypergraph coloring. A hypergraph is *k-uniform* if each of its edges contains precisely k vertices. It is *k-regular* if each of its vertices is contained in precisely k edges. A hypergraph is *2-colorable* if there is a two-coloring of the set of its vertices so that none of its edges is monochromatic. Erdös and Lovász proved the following result.

Theorem 1.3 *For each $k \geq 9$, every k-regular, k-uniform hypergraph is two colorable.*

The proof is a simple consequence of the Lovász Local Lemma, proved in [19] (see also, e.g., [31]), which supplies a way of showing that certain events hold with positive probability, although this probability may be extremely small. The exact statement (for the symmetric case) is the following.

Lemma 1.4 *Let $A_1, \ldots, A_n$ be events in an arbitrary probability space. Suppose that the probability of each of the n events is at most p, and suppose that each event A_i is mutually independent of all but at most b of the other events A_j. If $ep(b+1) < 1$ then with positive probability none of the events A_i holds.*

Here is the proof of Theorem 1.3 based on this lemma. Let (V, E) be a k-uniform, k-regular hypergraph, and let $f : V \mapsto \{0, 1\}$ be a random 2-coloring obtained by choosing, for each $v \in V$ randomly and independently, $f(v) \in \{0, 1\}$ according to a uniform distribution. For each $e \in E$ let

A_e denote the event that f restricted to e is a constant, i.e., that e is monochromatic. It is obvious that $Prob(A_e) = 2^{-(k-1)}$ for every e, and that each event A_e is mutually independent of all the events A_g but those for which $g \cap e \neq \emptyset$. Since there are at most $k(k-1)$ edges g that intersect e we can substitute $b = k(k-1)$ and $p = 2^{-(k-1)}$ in Lemma 1.4 and conclude that for $k \geq 9$ with positive probability none of the events A_e holds, completing the proof. $\Box$

We note that a different, algebraic proof of the statement of Theorem 1.3 (that works for all $k \geq 8$) is given in [12].

The final result we mention in this section is due to Beck. Its proof relies on an elegant but somewhat complicated probabilistic recoloring, whose details we omit.

Theorem 1.5 (Beck [15]) *There exists a constant $c > 0$ such that every k-uniform hypergraph with at most $ck^{1/3}2^k$ edges is 2-colorable.*

We note that there are k uniform hypergraphs with $O(k^2 2^k)$ edges which are not 2-colorable and that the problem of determining more precisely the asymptotic behaviour of the minimum possible number of edges in a k-uniform hypergraph which is not 2-colorable is still open.

2 Acyclic Coloring

A vertex coloring of a graph G is called *acyclic* if no two adjacent vertices of G have the same color and there is no 2-colored cycle of G. The *acyclic chromatic number* of G, denoted by $A(G)$, is the minimum number of colors in an acyclic coloring of G. For a positive integer d, let $A(d)$ denote the maximum possible value of $A(G)$, as G ranges over all graphs with maximum degree d.

It is easy to show that $A(d) \leq d^2 + 1$. To see this observe that if G has maximum degree d then the vertices of G can be colored sequentially, using $d^2 + 1$ colors, where each vertex v in its turn is colored by a color that differs from those of the already colored vertices of distance at most 2 from v. This gives an acyclic coloring of G and shows that $A(d) \leq d^2 + 1$. In 1976 Erdös (cf. [4]) conjectured that $A(d) = o(d^2)$ as d tends to infinity. This conjecture is proved in [13], where the following stronger result is established.

Theorem 2.1 *There exists an absolute constant $c \leq 50$ such that for every d, $A(d) \leq cd^{4/3}$.*

The estimate given in this theorem is not far from the truth, as shown in the following additional result proved in [13].

Theorem 2.2 *There exists a positive absolute constant b such that for every d,*

$$A(d) \geq bd^{4/3}/(\log d)^{1/3}.$$

Another result proved in [13] is that the *edges* of every graph G with maximum degree d can be colored with $O(d)$ colors so that no two adjacent edges have the same color and there is no two-colored cycle. This result is obtained by applying a more general but technical result (whose exact statement is omitted) about acyclic vertex coloring to the line graph of G.

The proofs of all the results mentioned above are probabilistic. Theorem 2.2 is proved by considering a random graph G on n labelled vertices chosen by picking every pair of vertices to be an edge, randomly and independently, with probability $p = 3(\frac{\log n}{n})^{1/4}$. It is easy to see that with high probability the maximum degree of a vertex of G is at most $6n^{3/4}(\log n)^{1/4}$. It is slightly more difficult to show that with high probability in any vertex coloring of G with at most $n/2$ colors there is a two colored cycle of length 4. Thus, with positive probability, $A(G) > n/2 > \Omega(d^{4/3}/(\log d)^{1/3})$, implying the assertion of Theorem 2.2. The details appear in [13].

The proof of Theorem 2.1 is more complicated. It is based on the general (non-symmetric) Lovász Local Lemma, which is the following generalization of Lemma 1.4 (see, e.g., [31] for the proof).

Lemma 2.3 *Let $A_1, \ldots, A_n$ be events in an arbitrary probability space. Let the graph $H = (V, E)$ on the nodes $\{1, \ldots, n\}$ be a* dependency graph *for the events A_i, that is, assume that for each i, A_i is mutually independent of the family of events $\{A_j : \{i, j\} \notin E\}$. If there are reals $0 \leq y_i < 1$ such that for all i*

$$Pr(A_i) \leq y_i \prod_{\{i,j\} \in E} (1 - y_j)$$

then the probability that no A_i holds is at least $\prod_{i=1}^{n}(1 - y_i) > 0$.

To prove Theorem 2.1 we must show, given a graph $G = (V, E)$ with maximum degree d, that $A(G) \leq 50d^{4/3}$. Put $x = \lfloor 50d^{4/3} \rfloor$, and let $f : V \mapsto \{1, 2, \ldots x\}$ be a random vertex-coloring of G where for each $v \in V$ independently, the color $f(v) \in \{1, \ldots, x\}$ is chosen randomly according to a uniform distribution. To complete the proof it suffices to show that with positive (though, maybe, very small) probability, f is an acyclic coloring of G. This is done by applying the last lemma to a properly defined set of events. The actual choice of the appropriate events is somewhat tricky. Let us describe these events, omitting the computation required in the full proof. A pair of non-adjacent vertices u, v of G is called a *special pair* if u and v have more than $d^{2/3}$ common neighbours. The events considered are of the following four types.

1. **Type I**: For each pair of adjacent vertices u and v of G, let $A_{u,v}$ be the event that $f(u) = f(v)$.

2. **Type II**: For each induced path of length 4 $v_0v_1v_2v_3v_4$ in G, let $B_{v_0v_1v_2v_3v_4}$ be the event that $f(v_0) = f(v_2) = f(v_4)$ and $f(v_1) = f(v_3)$.

3. **Type III**: For each induced $4-$cycle $v_1v_2v_3v_4$ in G, in which neither $\{v_1, v_3\}$ nor $\{v_2, v_4\}$ is a special pair, let $C_{v_1v_2v_3v_4}$ be the event that $f(v_1) = f(v_3)$ and $f(v_2) = f(v_4)$.

4. **Type IV**: For each special pair u, v in G let $D_{u,v}$ be the event that $f(u) = f(v)$.

It is not too difficult to check that if none of the events of the four types above holds then f is an acyclic coloring of G. It can also be shown, by choosing appropriately the required real numbers appearing in Lemma 2.3, that indeed with positive probability none of these events holds. By making the required computation, omitted here, the proof of Theorem 2.1 can be completed. □

3 The Chromatic Index of Hypergraphs

Proving an old conjecture of Erdös and Hanani concerning the existence of "almost designs", Rödl developed a probabilistic technique which turned out to be very successful in tackling various difficult coloring problems. His basic idea is, very roughly, that when we try to prove the existence of a large matching in a hypergraph with certain regularity properties it is helpful to first choose randomly a *small* number of edges, delete all the edges that intersect them (including the chosen ones that intersect other chosen ones), and repeat, while maintaining the regularity properties of the hypergraph, until a large matching is obtained. The full proof is rather complicated, and can be found in [30]. Rödl's result has been generalized by Frankl and Rödl [21], and by Pippenger and Spencer [28]. The main result in [28] deals with the chromatic index of uniform hypergraphs.

Recall that the *chromatic index* of a hypergraph $H = (V, E)$, denoted by $\chi'(H)$, is the minimum number of colors in an edge-coloring of H, so that no two intersecting edges have the same color, i.e., each color class forms a matching. For two vertices u and v of H, let us denote by $d(u)$ the number of edges of H containing u, and by $d(u, v)$ the number of edges of H that contain both u and v.

Theorem 3.1 ([28]) *For every $k \geq 2$ and every $\epsilon > 0$ there exists a $\delta > 0$ such that if H is a k-uniform hypergraph on a set V of vertices satisfying, for some integer D,* $(1-\delta)D < d(v) \leq D$ *for all $v \in V$ and $d(u, v) < \delta D$ for all distinct $u, v \in V$, then $\chi'(H) < (1+\epsilon)D$.*

One immediate application of this theorem is a tight asymptotic estimate for the chromatic index of a Steiner Triple System on n points. A *Steiner Triple System* on n points is a 3-uniform hypergraph on n vertices such that each pair of vertices is contained in precisely one edge. Clearly each such hypergraph is $(n-1)/2$-regular and it trivially satisfies the assumptions of the last theorem for any $\delta > 0$, provided n is sufficiently large. Therefore, the above theorem implies that as n tends to infinity, the chromatic index of any Steiner Triple System on n vertices is $(1 + o(1))n/2$. In particular, each such system contains a matching of size $(1 + o(1))n/3$.

Kahn [23] has recently generalized Theorem 3.1 significantly, and proved the following result.

Theorem 3.2 *For every $k \geq 2$ and every $\epsilon > 0$ there exists a $\delta > 0$ such that the following statement is true:*
If H is a hypergraph on a set V of vertices in which every edge has at most k vertices, and if $d(v) \leq D$ for all $v \in V$ and $d(u, v) < \delta D$ for all distinct $u, v \in V$, and if C is a set of colors and for each edge e of H we have a subset $C(e) \subset C$ satisfying $|C(e)| > (1+\epsilon)D$, then there is a coloring $f : E(H) \mapsto C$ of the edges of H so that for each edge e, $f(e) \in C(e)$ and such that each color class is a matching.

This Theorem supplies a proof that the well known conjecture of Erdös, Faber and Lovász is approximately correct (see [22]). It also shows that for any $\epsilon > 0$, if D is sufficiently large then for every D-regular simple graph $G = (V, E)$ and any family of sets $\{C(e)\}_{e \in E}$ satisfying $|C(e)| \geq (1 + \epsilon)D$, there is a proper edge coloring f of G satisfyiung $f(e) \in C(e)$ for all $e \in E$.

4 Star Arboricity and Radio Networks

A *star forest* is a forest whose connected components are stars. The *star arboricity* of a graph G, denoted by $st(G)$, is the minimum number of star forests whose union covers all edges of G. For an integer $d \geq 1$ define $st(d) = Max\{st(G)\}$, where the maximum is taken over all simple graphs with maximum degree d.

The star arboricity of graphs was introduced by Akiyama and Kano [2] and has been studied in various papers. In particular it is shown in [3] that for every complete multipartite graph with equal color classes and with maximum degree d, $st(G) \leq \lceil d/2 \rceil + 2$. The asymptotic behaviour of $st(d)$ is determined in [14], improving a previous estimate from [5]. This is stated in the following theorem.

Theorem 4.1 *There exist two positive constants c_1 and c_2 such that for every $d \geq 1$,*

$$\frac{d}{2} + c_1 \log d \leq st(d) \leq \frac{d}{2} + c_2 \log d.$$

The somewhat complicated proof is probabilistic, and is based on the Local Lemma (Lemma 1.4).

The study of star arboricity is naturally suggested by the analysis of certain communication networks. A *radio network* is a synchronous network of processors that communicate by transmitting messages to their neighbors. A processor P can receive at most one message in one step. Let us mention here two possible models.

Type I : P receives a message from its neighbor Q in a given step if P is silent, Q transmits and P chooses to receive from Q in this step.

Type II: P receives a message from its neighbor Q if P is silent, and Q is the *only* neighbor of P that transmits in this step.

Suppose, now, that the model is the Type I model and the network is represented by an undirected graph $G = (V, E)$ whose vertices are the processors and two are adjacent if they can transmit to each other. Suppose, further, that we need to transmit once along every edge (in one of the two possible directions), say, in order to check that there is indeed a connection between each adjacent pair. It is easy to see that the minimum number of steps in which we can finish all the required transmissions is precisely $st(G)$, since the set of edges corresponding to the transmissions performed in a single step forms a star forest. Theorem 4.1 thus supplies an upper bound which is sometimes almost tight for the minimum number of required steps.

What about the Type II networks? This model is much more popular, and has been considered in many papers (see, e.g., [7] and its many references). For simplicity let us only consider the following case. Let G be a bipartite graph with classes of vertices A and B representing the processors. Each edge ab, with $a \in A$ and $b \in B$ represents a transmission that a has to transmit to b. What is the minimum number of steps in which all these transmissions can be performed? It is not too difficult to see that this is precisely the minimum number of colors in an edge coloring of G in which each color class is an *induced* star forest with the centers of each star lying in A. The following result is a special case of a theorem proved in [7], [8] that bounds this number for graphs with a given maximum degree.

Theorem 4.2 *There exist two positive constants c and b such that*
(i) The edges of any bipartite graph with maximum degree d can be covered by $cd \log d$ induced star forests whose centers lie in the first color class.
(ii) For every d there is a bipartite graph with maximum degree d whose edges cannot be covered by less than $bd \log d$ induced star forests as above.

The proof of both parts are probabilistic, and rely, among other combinatorial arguments, on the Local Lemma and on the FKG Inequality.

5 Linear Arboricity

A *linear forest* is a forest in which every connected component is a path. The *linear arboricity* $la(G)$ of a graph G is the minimum number of linear forests in G, whose union is the set of all edges of G. The following conjecture, known as the *linear arboricity conjecture*, was raised in [1]:

Conjecture 5.1 (The Linear Arboricity Conjecture) *The linear arboricity of every d-regular graph is $\lceil (d+1)/2 \rceil$.*

Notice that since every d-regular graph G on n vertices has $nd/2$ edges, and every linear forest in it has at most $n-1$ edges, the inequality

$$la(G) \geq \frac{nd}{2(n-1)} > \frac{d}{2}$$

is immediate. Since $la(G)$ is an integer this gives $la(G) \geq \lceil (d+1)/2 \rceil$. The difficulty in Conjecture 5.1 lies in proving the converse inequality: $la(G) \leq \lceil (d+1)/2 \rceil$. Note also that since every graph G with maximum degree Δ is a subgraph of a Δ-regular graph (which may have more vertices, as well as more edges than G), the linear arboricity conjecture is equivalent to the statement that the linear arboricity of every graph G with maximum degree Δ is at most $\lceil (\Delta+1)/2 \rceil$.

Although this conjecture received a considerable amount of attention, the best general result concerning it, proved without any probabilistic arguments, is that $la(G) \leq \lceil 3\Delta/5 \rceil$ for even Δ and that $la(G) \leq \lceil (3\Delta+2)/5 \rceil$ for odd Δ. In this section we sketch a proof of the fact that for every $\epsilon > 0$ there is a $\Delta_0 = \Delta_0(\epsilon)$ such that for every $\Delta \geq \Delta_0$ the linear arboricity of every graph with maximum degree Δ is less than $\left(\frac{1}{2}+\epsilon\right)\Delta$. This result (with a somewhat more complicated proof) appears in [10] and its proof relies heavily on probabilistic arguments. The proof we sketch here is different and supplies a much better estimate for the error term.

It is convenient to deduce the result for undirected graphs from its directed version. A *d-regular digraph* is a directed graph in which the indegree and the outdegree of every vertex is precisely d. A linear directed forest is a directed graph in which every connected component is a directed path. The *di-linear arboricity* $dla(G)$ of a directed graph G is the minimum number of linear directed forests in G whose union covers all edges of G. The directed version of the Linear Arboricity Conjecture, first stated in [26] is:

Conjecture 5.2 *For every d-regular digraph D,*

$$dla(D) = d + 1.$$

Note that since the edges of any (connected) undirected $2d$-regular graph G can be oriented along an Euler cycle, so that the resulting oriented digraph is d-regular, the validity of Conjecture 5.2 for d implies that of Conjecture 5.1 for $2d$.

It is easy to prove that any graph with n vertices and maximum degree d contains an independent set of size at least $n/(d+1)$. The following proposition shows that at the price of decreasing the size of such a set by a constant factor we can guarantee that it has a certain structure.

Proposition 5.3 *Let $H = (V,E)$ be a graph with maximum degree d, and let $V = V_1 \cup V_2 \cup \cdots \cup V_r$ be a partition of V into r pairwise disjoint sets. Suppose each set V_i is of cardinality $|V_i| \geq 2ed$,* *where $e = 2.71828...$ is the basis of the natural logarithm. Then there is an independent set of vertices $W \subseteq V$, that contains a vertex from each V_i.*

Here is a sketch of the proof. Clearly we may assume that each set V_i is of cardinality precisely $g = \lceil 2ed \rceil$ (otherwise, simply replace each V_i by a subset of cardinality g of it, and replace H by its induced subgraph on the union of these r new sets). Let us pick from each set V_i randomly and independently a single vertex according to a uniform distribution. Let W be the random set of the vertices picked. To complete the proof it suffices to show that with positive probability W is an independent set of vertices in H. This can be deduced from Lemma 1.4. For each edge f of H, let A_f be the event that W contains both ends of f. Lemma 1.4 easily implies that with positive probability none of these events holds, implying that W is independent. $\square$

It is worth noting that a much stronger assertion than that proved in the last proposition is also true. This is stated in the following theorem, whose proof, that combines probabilistic arguments with some additional combinatorial ideas, appears in [11].

Theorem 5.4 *There is an absolute constant c with the following property: For any two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same set of vertices, where G_1 has maximum degree at most d and G_2 is a vertex disjoint union of cliques of size cd each, the chromatic number of the graph $G = (V, E_1 \cup E_2)$ is precisely cd.*

Returning to Linear Arboricity we next observe that Proposition 5.3 suffices to proves Conjecture 5.2 for digraphs with no short directed cycle. The *directed girth* of a digraph is the minimum length of a directed cycle in it.

Theorem 5.5 *Let $G = (U, F)$ be a d-regular digraph with directed girth $g \geq 4ed$. Then*

$$dla(G) = d + 1 \ .$$

Proof. As is well known, Hall's Theorem implies that F can be partitioned into d pairwise disjoint 1-regular spanning subgraphs $F_1 \ldots F_d$ of G. Each F_i is a union of vertex disjoint directed cycles $C_{i1}, C_{i2} \ldots C_{ir_i}$. Let V_1, $V_2 \ldots V_r$ be the sets of *edges* of all the cycles $\{C_{ij} : 1 \leq i \leq d, 1 \leq j \leq r_i\}$. Clearly $V_1, V_2 \ldots V_r$ is a partition of the set F of all edges of G, and by the girth condition, $|V_i| \geq g \geq 4ed$ for all $1 \leq i \leq r$. Let H be the line graph of G, i.e., the graph whose set of vertices is the set F of edges of G in which two edges are adjacent iff they share a common vertex in G. Clearly H is $2d-2$ regular. As the cardinality of each V_i is at least $4ed \geq 2e(2d-2)$, there is, by Proposition 5.3, an independent set of H containing a member from each V_i. But this means that there is a matching M in G, containing at least one edge from each cycle C_{ij} of the 1-factors $F_1 \ldots F_d$. Therefore $M, F_1 \setminus M, F_2 \setminus M \ldots F_d \setminus M$ are $d+1$-directed linear forests in G (one of which is a matching) that cover all its edges. Hence

$$dla(G) \leq d+1 \ .$$

As G has $|U| \cdot d$ edges and each directed linear forest can have at most $|U|-1$ edges,

$$dla(G) \geq |U|d/(|U|-1) > d.$$

Thus $dla(G) = d+1$, completing the proof. $\Box$

The last theorem shows that the assertion of Conjecture 5.2 holds for digraphs with sufficiently large (directed) girth. In order to deal with digraphs with small girth, we show that most of the edges of each regular digraph can be decomposed to a relatively small number of almost regular digraphs with high girth. To do this, we need the following statement.

Lemma 5.6 *Let $G=(V,E)$ be a d-regular directed graph, where $d \geq 100$, and let p be an integer satisfying $10\sqrt{d} \leq p \leq 20\sqrt{d}$. Then, there is a p-coloring of the vertices of G by the colors $0,1,\ldots,p-1$ with the following property; for each vertex $v \in V$ and each color i, the numbers $N^+(v,i) = |\{u \in V; (v,u) \in E$ and u is colored $i\}|$ and $N^-(v,i) = |\{u \in V : (u,v) \in E$ and u is colored $i\}|$ satisfy:*

$$
\begin{aligned}
|N^+(v,i) - \tfrac{d}{p}| &\leq 3\sqrt{d/p}\sqrt{\log d} \ , \\
|N^-(v,i) - \tfrac{d}{p}| &\leq 3\sqrt{d/p}\sqrt{\log d} \ .
\end{aligned}
\tag{5.1}
$$

The proof is again probabilistic. Let $f : V \to \{0,1,\ldots,p-1\}$ be a random vertex coloring of V by p colors, where for each $v \in V$, $f(v) \in \{0,1,\ldots,p-1\}$ is chosen according to a uniform distribution. Combining the standard estimates for Binomial distributions with Lemma 1.4 one can show that with positive probability f is a coloring satisfying the assertion of the lemma. We omit the details.

We are now ready to deal with general regular digraphs. Let $G = (V, E)$ be an arbitrary d-regular digraph. Throughout the argument we assume, whenever it is needed, that d is sufficiently large. Let p be a prime satisfying $10d^{1/2} \leq p \leq 20d^{1/2}$ (it is well known that for every n there is a prime between n and $2n$). By Lemma 5.6 there is a vertex coloring $f : V \to \{0, 1 \ldots p-1\}$ satisfying (5.1). For each i, $0 \leq i < p$, let $G_i = (V, E_i)$ be the spanning subdigraph of G defined by $E_i = \{(u, v) \in E : f(v) \equiv (f(u) + i) \ mod \ p\}$. By inequality (5.1) the maximum indegree Δ_i^- and the maximum outdegree Δ_i^+ in each G_i is at most $\frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\log d}$. Moreover, for each $i > 0$, the length of every directed cycle in G_i is divisible by p. Thus, the directed girth g_i of G_i is at least p. Since each G_i can be completed, by adding vertices and edges, to a Δ_i-regular digraph with the same girth g_i and with $\Delta_i = \max\ (\Delta_i^+, \Delta_i^-)$, and since $g_i > 4e\Delta_i$ (for all sufficiently large d), we conclude, by Theorem 5.5, that $dla(G_i) \leq \Delta_i + 1 \leq \frac{d}{p} + 3\sqrt{\frac{d}{p}}\sqrt{\log d} + 1$ for all $1 \leq i < p$. For G_0, we only apply the trivial inequality

$$dla(G_0) \leq 2\Delta_0 \leq 2\frac{d}{p} + 6\sqrt{\frac{d}{p}}\sqrt{\log d}$$

obtained by, e.g., embedding G_0 as a subgraph of a Δ_0-regular graph, splitting the edges of this graph into Δ_0 1-regular spanning subgraphs, and breaking each of these 1-regular spanning subgraphs into two linear directed forests. The last two inequalities, together with the fact that $10\sqrt{d} \leq p \leq 20\sqrt{d}$ imply

$$dla(G) \leq d + \frac{d}{p} + 3\sqrt{pd}\sqrt{\log d} + 3\sqrt{\frac{d}{p}}\sqrt{\log d} + p - 1 \leq d + c \cdot d^{3/4}(\log d)^{1/2}$$

We have thus proved;

Theorem 5.7 *There is an absolute constant $c > 0$ such that for every d-regular digraph G*

$$dla(G) \leq d + cd^{3/4}(\log d)^{1/2} \ .$$

□

We note that by using recursion to cover G_0 instead of the naive method above (and by changing the parameters), we can improve the error term to $c'd^{2/3}(\log d)^{1/3}$. Since the edges of any undirected $d = 2f$-regular graph can be oriented so that the resulting digraph is f-regular, and since any $(2f-1)$-regular undirected graph is a subgraph of a $2f$-regular graph the last theorem implies;

Theorem 5.8 *There is an absolute constant $c > 0$ such that for every undirected d-regular graph G*

$$la(G) \leq \frac{d}{2} + cd^{3/4}(\log d)^{1/2}.$$

□

6 The Algorithmic Aspect

In a typical application of the probabilistic method we try to prove the existence of a combinatorial structure (or a substructure of a given structure) with certain prescribed properties. To do so, we show that a randomly chosen element from an appropriately defined sample space satisfies all the required properties with positive probability. In most applications, this probability is not only positive, but is actually high and frequently tends to 1 as the parameters of the problem tend to infinity. In such cases, the proof usually supplies an efficient randomized algorithm for producing a structure of the desired type, and in many cases this algorithm can be derandomized and converted into an efficient deterministic one. By efficient we mean here, as usual, an algorithm whose running time -(or expected running time, in case we consider randomized algorithms)- is polynomial in the length of the input.

There are, however, certain examples, where one can prove the existence of the required combinatorial structure by probabilistic arguments that deal with rare events; events that hold with positive probability which is exponentially small in the size of the input. Such proofs usually yield neither randomized nor deterministic efficient procedures for the corresponding algorithmic problems.

A class of examples demonstrating this phenomenon is the class of results proved by applying the Local Lemma. Many examples are given in the previous sections of this paper. For several years there has been no known method of converting the proofs of any of these examples into an efficient algorithm. Very recently J. Beck [16] found such a method, that works for many of the examples mentioned here, with some loss in the constants. Beck demonstrated his method by considering the problem of hypergraph 2-coloring, an example which generalizes the one described in Theorem 1.3. The derivation of the following theorem from the local lemma is almost identical to that of Theorem 1.3

Theorem 6.1 *If $e(d+1) < 2^{n-1}$ then any n-uniform hypergraph in which no edge intersects more than d other edges is 2-colorable.*

Let H be such an n-uniform hypergraph with N edges, and suppose n, d are fixed. Can we find a proper two coloring of H (i.e., a vertex coloring in which no edge is monochromatic) efficiently? Beck showed that indeed we can, in case d is somewhat smaller, say $d < O(2^{n/11})$. In this case there is a randomized as well as a deterministic algorithm whose running time is polynomial in N for finding a proper two coloring. Beck's method does not seem to provide a parallel efficient algorithm (i.e., an algorithm that runs in poly-logarithmic time using a polynomial number of processors). We describe here a modified version of his algorithm which is parallelizable. For simplicity we describe the randmoized version of the algorithm and only comment briefly on the possibilities to derandomize and parallelize it. Let us denote, as usual, the binary entropy function by $H(x) = -x \log_2 x - (1-x) \log_2 (1-x)$.

Theorem 6.2 *Suppose n, d are fixed and suppose that for some $\alpha > 0$*

$$2ed^3 < 2^{n(1-H(\alpha))}$$

and

$$e(d+1) < 2^{\alpha n}.$$

Then there is a randomized algorithm that finds a proper 2-coloring of any given n-uniform hypergraph H with N edges in which no edge intersects more than d others in expected running time $N^{O(1)}$. This algorithm can be derandomized and parallelized, providing a deterministic algorithm that finds a proper coloring in time $O(\log N)$ using $N^{O(1)}$ processors.

We note that for large n, any $d \leq 2^{n/8}$ satisfies the above (by taking an appropriate $\alpha > 1/8$). We also note that by assuming that d is smaller, say that $d < 2^{n/20}$, the expecetd running time can be reduced to almost linear in N.

Here is the randmoized algorithm. In the First Pass we color all the vertices of H, randomly and independently by two colors, where each point is colored either red or blue with equal probability. Call an edge *bad* if at most αn of its points are red or at most αn of its points are blue. The probability of a fixed edge to be bad is clearly at most $2 \sum_{i \leq \alpha n} \binom{n}{i} / 2^n \leq 2 \cdot 2^{(H(\alpha)-1)n}$. Put $p = 2 \cdot 2^{(H(\alpha)-1)n}$. Let B denote the set of all bad edges.

Let G be the dependency graph for the problem, i.e., the graph whose vertices are the edges of H in which two are adjacent iff they intersect. Observe that if S is an independent set in G then the probability that $S \subset B$ is at most $p^{|S|}$, since these $|S|$ events are mutually independent. Let us call a set of vertices C of G a $1,2$-tree if the $A_i \in C$ are such that drawing an arc between $A_i, A_j \in C$ if their distance in G is either 1 or 2 the resulting graph is connected. (This is simply the set of vertices of a connected subgraph in the square of G).

Lemma 6.3 *There exists a positive constant c such that almost surely every $1,2$-tree in G all of whose vertices belong to B has size at most $c \log N$.*

Proof Call $T \subseteq G$ a $2,3$-tree if the $A_i \in T$ are such that all their mutual distances in G are at least two and so that, drawing an arc between $A_i, A_j \in T$ if their distance is either 2 or 3 the resulting graph is connected. We first bound the number of $2,3$-trees of size u in G. Consider the graph on the set of vertices of G in which two vertices are adjacent if their distance in G is either 2 or 3. Every $2,3$-tree on a set T of vertices of G must contain a tree on T in this new graph. The new graph has maximum degree smaller than $D = d^3$. It is well known (see [25]) that an infinite D-regular rooted tree contains precisely $\frac{1}{(D-1)u+1}\binom{Du}{u}$ rooted subtrees of size u, and this easily imlies that the number of trees of size u containing one specific given vertex in any graph with maximum degree at most D does not exceed this number, which is smaller than $(eD)^u$.

For any particular $2,3$-tree T we know that $\Pr[T \subseteq B] \leq p^u$. Hence the expected number of $2,3$-trees of size u $T \subseteq S$ is at most $N(eDp)^u$. As $eDp < 1$ by assumption (recall d, n and hence D, p are constants), for an appropriately chosen positive c_1, if $u = c_1 \log N$ this term is $o(1)$. Thus almost surely there is no $2,3$-tree of size bigger than $c_1 \log N$ all of whose vertices are in B. We actually want to bound the size of any $1,2$-tree C of G. A maximal $2,3$-tree T in such a C must have the property that every $A_i \in C$ is a neighbor (in G) of an $A_j \in T$. There are less than d (a constant) A_i neighbors of any given A_j so that $c_1 \log N \geq |T| \geq |C|/d$ and so

$$|C| \leq c \log N$$

completing the proof of the lemma. $\square$

Let us call the First Pass *successful* if, for the constant c appearing in Lemma 6.3, there is no $1,2$-tree of size greater than $c \log n$ all of whose vertices lie in B. By the last lemma the probability the First Pass is successful is close to 1. In case it is not, we simply repeat the entire procedure. In expected *linear* time the First Pass is successful.

We can now fix the coloring by recoloring, in the Second Pass, the vertices of H that belong to the bad edges. Let us call an edge *dangerous* if it contains at least αn vertices that belong to bad edges. (Thus, in particular, bad edges are also dangerous). Observe that if an edge is not dangerous then it will not become monochromatic after the recoloring. This is because less than αn of its points will change color, and it has at least αn points of each color before the recoloring. Thus we only have to worry about the dangerous edges. However, if we recolor all the vertices in bad edges randomly and independently than we recolor at least αn vertices in each dangerous edge, and hence the probability it becomes monochromatic does not exceed $2^{-\alpha n}$. Since each dangerous edge intersects at most d others it follows from the assumptions in Theorem 6.2 and from the Lovász Local Lemma that there exists a recoloring in which no edge is monochromatic.

The crucial point is that the recoloring of the points in the edges of each maximal $1,2$-tree C of bad edges can be done separately. This is becuase there is no dangerous edge that intersects edges from two distinct such maximal $1,2$-trees, and hence it suffices to recolor the points in the edges of each such C in a way that only makes sure that no dangerous edge intersecting an edge in C becomes dangerous. Since each of the $1,2$-trees C as above has only $O(\log N)$ vertices that have to be recolored, we can find the required recoloring *by brute force!* Examining all possible two colorings in each such C only takes time $O(2^{O(\log N)}) = N^{O(1)}$ and hence doing it for all the above C-s can be done in polynomial time.

This completes the description of the randomized algorithm with expected polynomial running time. In case $d < 2^{cn}$ for a smaller c we can make another pass similar to the first one in each $1,2$-tree seperately, get new $1,2$ trees of size $O(\log logN)$ and complete as before obtaining an expected running time which is nearly linear- $O(N(\log N)^{O(1)})$. We omit the detailed computation.

The randomized algorithm above is trivially parallelizable and can be implemented on a standard $EREW$-PRAM in time $O(\log N)$ using $N^{O(1)}$ parallel processors. (See [24] for the basic definitions of an $EREW$-PRAM and the complexity classes NC and NC^1.) Moreover, the algorithm can be derandomized maintaining the running time (with some increase in the number of processors), showing that the problem can be solved in NC^1. To see this observe that the recoloring step is deterministic even in the version described above, so the only problem is the derandomization of the First Pass. This can be done by applying the techniques from [27] or [9]. The basic idea is that for every constant c there is a constant $b = b(c)$ such that for every m there are explicit sample spaces of size at most m^b in which one can embed m random variables taking the values $0, 1$ in which every set of $c \log m$ of the variables are nearly independent. Instead of describing the details let us simply mention that the First Pass here can be performed deterministically as follows. Let $q \geq N^b$ be a prime, where b is a constant dependending on the constant c in Lemma 6.3. Let χ be the quadratic character defined on the elements of the finite field $GF(q)$, i.e., $\chi(x) = 1$ if x is a quadratic residue modulo q and $\chi(x) = -1$ otherwise. Define a family of q two-colorings of the set of vertices of H as follows. For each $i \in GF(q)$, the color of the j-th point in the i-th coloring is blue if $\chi(i - j) = 1$ and is red if $\chi(i - j) = -1$. Using the results in [9] (based on the ideas in [27] (see also [6])), it can be shown that at least one of the q colorings defined above will produce a successful First Pass. All these (deterministically defined) colorings can be checked in parallel, completing the proof of Theorem 6.2. The full details will appear somewhere else. $\square$

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