

# Discrete Mathematics: Methods and Challenges

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## Abstract

Combinatorics is a fundamental mathematical discipline as well as an essential component of many mathematical areas, and its study has experienced an impressive growth in recent years. One of the main reasons for this growth is the tight connection between Discrete Mathematics and Theoretical Computer Science, and the rapid development of the latter. While in the past many of the basic combinatorial results were obtained mainly by ingenuity and detailed reasoning, the modern theory has grown out of this early stage, and often relies on deep, well developed tools. This is a survey of two of the main general techniques that played a crucial role in the development of modern combinatorics; algebraic methods and probabilistic methods. Both will be illustrated by examples, focusing on the basic ideas and the connection to other areas.

## 1 Introduction

The originators of the basic concepts of Discrete Mathematics, the mathematics of finite structures, were the Hindus, who knew the formulas for the number of permutations of a set of  $n$  elements, and for the number of subsets of cardinality  $k$  in a set of  $n$  elements, already in the sixth century. The beginning of Combinatorics as we know it today started with the work of Pascal and De Moivre in the 17th century, and continued in the 18th century with the seminal ideas of Euler in Graph Theory, with his work on partitions and their enumeration, and with his interest in latin squares. These old results are among the roots of the study of formal methods of enumeration, the development of configurations and designs, and the extensive work on Graph Theory in the last two centuries. The tight connection between Discrete Mathematics and Theoretical Computer Science, and the rapid development of the latter in recent years, led to an increased interest in combinatorial techniques and to an impressive development of the subject. It also stimulated the study of algorithmic combinatorics and combinatorial optimization.

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While many of the basic combinatorial results were obtained mainly by ingenuity and detailed reasoning, without relying on many deep, well developed tools, the modern theory has already grown out of this early stage. There are already well developed enumeration methods, some of which are based on deep algebraic tools. The probabilistic method initiated by Erdős (and to some extent, by Shannon) became one of the most powerful tools in the modern theory, and its study has been fruitful to Combinatorics, as well as to Probability Theory. Algebraic and topological techniques play a crucial role in the modern theory, and Polyhedral Combinatorics, Linear Programming and constructions of designs have been developed extensively. Most of the new significant results obtained in the area are inevitably based on the knowledge of these well developed concepts and techniques, and while there is, of course, still a lot of room for pure ingenuity in Discrete Mathematics, much of the progress is obtained by relying on the fast growing accumulated body of knowledge.

Concepts and questions of Discrete Mathematics appear naturally in many branches of mathematics, and the area has found applications in other disciplines as well. These include applications in Information Theory and Electrical Engineering, in Statistical Physics, in Chemistry and Molecular Biology, and, of course, in Computer Science. Combinatorial topics such as Ramsey Theory, Combinatorial Set Theory, Matroid Theory, Extremal Graph Theory, Combinatorial Geometry and Discrepancy Theory are related to a large part of the mathematical and scientific world, and these topics have already found numerous applications in other fields. A detailed account of the topics, methods and applications of Combinatorics can be found in [35].

This paper is mostly a survey of two of the main general techniques that played a crucial role in the development of modern combinatorics; algebraic methods and probabilistic methods. Both will be illustrated by examples, focusing on the basic ideas and the connection to other areas. The choice of topics and examples described here is inevitably biased, and is not meant to be comprehensive. Yet, it hopefully provides some of the flavor of the techniques, problems and results in the area in a way which may be appealing to researchers, even if their main interest is not Discrete Mathematics.

## **2 Dimension, geometry and information theory**

Various algebraic techniques have been used successfully in tackling problems in Discrete Mathematics over the years. These include tools from Representation Theory applied extensively in enumeration problems, spectral techniques used in the study of highly regular structures, and applications of properties of polynomials and tools from algebraic geometry in the theory of Error Correcting Codes and in the study of problems in Combinatorial Geometry. These techniques have numerous interesting applications. Yet, the most fruitful algebraic technique applied in combinatorics, which is possibly also the simplest one, is the so-called dimension argument. In its simplest form, the method

can be described as follows. In order to bound the cardinality of a discrete structure  $A$ , one maps its elements to vectors in a linear space, and shows that the set  $A$  is mapped to a linearly independent set. It then follows that the cardinality of  $A$  is bounded by the dimension of the corresponding linear space. This method is often particularly useful in the solution of extremal problems in which the extremal configuration is not unique. The method is effective in such cases because bases in a vector space can be very different from each other and yet all of them have the same cardinality. Many applications of this basic idea can be found in [13], [14], [37].

## 2.1 Combinatorial Geometry

An early application of the dimension argument appears in [49]. A set of points  $A \subset R^n$  is a two-distance set if at most 2 distinct positive distances are determined by the points of  $A$ . Let  $f(n, 2)$  denote the maximum possible size of a two-distance set in  $R^n$ . The set of all 0/1 vectors in  $R^{n+1}$  with exactly two 1's shows that  $f(n, 2) \geq n(n+1)/2$ , and the authors of [49] proved that  $f(n, 2) \leq (n+1)(n+4)/2$ . The upper bound is proved by associating each point of a two-distance set  $A$  with a polynomial in  $n$  variables, and by showing that these polynomials are linearly independent and all lie in a space of dimension  $(n+1)(n+4)/2$ . This has been improved by Blokhuis to  $(n+1)(n+2)/2$ , by showing that one can add  $n+1$  additional polynomials that lie in this space to those obtained from the two-distance set, keeping the augmented set linearly independent. See [14] and its references for more details. The precise value of  $f(n, 2)$  is not known.

Borsuk [21] asked if any compact set of at least 2 points in  $R^d$  can be partitioned into at most  $d+1$  subsets of smaller diameter. Let  $m(d)$  be the smallest integer  $m$  so that any such set can be partitioned into at most  $m$  subsets of smaller diameter. Borsuk's question is whether  $m(d) = d+1$  (the  $d+1$  points of a simplex show that  $m(d)$  is at least  $d+1$ .) Kahn and Kalai [42] gave an example showing that this is not the case for all sufficiently large  $d$ , by applying a theorem of Frankl and Wilson [33]. Improved versions of their construction have been obtained by Nilli in 1994, by Raigorodski in 1997, by Hinrichs in 2001 and by Hinrichs and Richter in 2002. The last two results are based on some properties of the Leech Lattice and give a construction showing that already in dimension  $d = 298$ , more than  $d+1$  subsets may be needed. All the constructions and the proofs of their properties are based on the dimension argument. Here is a brief sketch of one of them.

Let  $n = 4p$ , where  $p$  is an odd prime, and let  $\mathcal{F}$  be the set of all vectors  $\mathbf{x} = (x_1, \dots, x_n) \in \{-1, 1\}^n$ , where  $x_1 = 1$  and the number of negative coordinates of  $\mathbf{x}$  is even. One first proves the following.

$$\text{If } \mathcal{G} \subset \mathcal{F} \text{ contains no two orthogonal vectors then } |\mathcal{G}| \leq \sum_{i=0}^{p-1} \binom{n-1}{i}. \quad (1)$$

This is done by associating each member of  $\mathcal{G}$  with a multilinear polynomial of degree at most  $p - 1$  in  $n - 1$  variables, so that all the obtained polynomials are linearly independent. Having established (1), define  $S = \{\mathbf{x} * \mathbf{x} : \mathbf{x} \in \mathcal{F}\}$ , where  $\mathcal{F}$  is as above, and  $\mathbf{x} * \mathbf{x}$  is the tensor product of  $\mathbf{x}$  with itself, i.e., the vector of length  $n^2$ ,  $(x_{ij} : 1 \leq i, j \leq n)$ , where  $x_{ij} = x_i x_j$ . The norm of each vector in  $S$  is  $n$  and the scalar product between any two members of  $S$  is non-negative. Moreover, by (1) any set of more than  $\sum_{i=0}^{p-1} \binom{n-1}{i}$  members of  $S$  contains an orthogonal pair, i.e., two points the distance between which is the diameter of  $S$ . It follows that  $S$  cannot be partitioned into less than  $2^{n-2} / \sum_{i=0}^{p-1} \binom{n-1}{i}$  subsets of smaller diameter. This shows that  $m(d) \geq c_1^{\sqrt{d}}$  for some  $c_1 > 1$ . An upper bound of  $m(d) \leq c_2^d$  where  $c_2 = \sqrt{3/2} + o(1)$  is known, but determining the correct order of magnitude of  $m(d)$  is an open question. The following conjecture seems plausible.

**Conjecture 2.1** *There is a constant  $c > 1$  such that  $m(d) > c^d$  for all  $d \geq 1$ .*

An *equilateral set* (or a simplex) in a metric space, is a set  $A$ , so that the distance between any pair of distinct members of  $A$  is  $b$ , where  $b \neq 0$  is a constant. Trivially, the maximum cardinality of such a set in  $R^n$  with respect to the (usual)  $l_2$ -norm is  $n+1$ . Somewhat surprisingly, the situation is far more complicated for the  $l_1$  norms. The  $l_1$ -distance between two points  $\vec{a} = (a_1, \dots, a_n)$  and  $\vec{b} = (b_1, \dots, b_n)$  in  $R^n$  is  $\|\vec{a} - \vec{b}\|_1 = (\sum_{k=1}^n |a_k - b_k|)$ . Let  $e(l_1^n)$  denote the maximum possible cardinality of an equilateral set in  $l_1^n$ . The set of standard basis vectors and their negatives shows that  $e(l_1^n) \geq 2n$ . Kusner [39] conjectured that this is tight, i.e., that  $e(l_1^n) = 2n$  for all  $n$ . For  $n \leq 4$  this is proved in [44]. For general  $n$ , the best known upper bound is  $e(l_1^n) \leq c_1 n \log n$  for some absolute positive constant  $c_1$ . This is proved in [9] by an appropriate dimension argument. Each vector in an equilateral set of  $m$  vectors in  $R^n$  is mapped to a vector in  $l_2^t$  for an appropriate  $t = t(m, n)$ , by applying a probabilistic technique involving randomized rounding. It is then shown, using a simple argument based on the eigenvalues of the Gram matrix of these new vectors, that they span a space of dimension at least  $c_2 m$ , implying that  $c_2 m \leq t(m, n)$  and supplying the desired result. The precise details require some work, and can be found in [9].

## 2.2 Capacities and graph powers

Let  $G = (V, E)$  be a simple, undirected graph. The power  $G^n$  of  $G$  is the graph whose set of vertices is  $V^n$  in which two distinct vertices  $(u_1, u_2, \dots, u_n)$  and  $(v_1, v_2, \dots, v_n)$  are adjacent iff for all  $i$  between 1 and  $n$  either  $u_i = v_i$  or  $u_i v_i \in E$ . The *Shannon capacity*  $c(G)$  of  $G$  is the limit  $\lim_{n \rightarrow \infty} (\alpha(G^n))^{1/n}$ , where  $\alpha(G^n)$  is the maximum size of an independent set of vertices in  $G^n$ . This limit exists, by super-multiplicativity, and it is always at least  $\alpha(G)$ .

The study of this parameter was introduced by Shannon in [61], motivated by a question in Information Theory. Indeed, if  $V$  is the set of all possible letters a channel can transmit in one

use, and two letters are adjacent if they may be confused, then  $\alpha(G^n)$  is the maximum number of messages that can be transmitted in  $n$  uses of the channel with no danger of confusion. Thus  $c(G)$  represents the number of distinct messages *per use* the channel can communicate with no error while used many times.

Calculation of  $c(G)$  seems to be very hard. For example  $c(C_5) = \sqrt{5}$  was only shown in 1979 by Lovász [50], and  $c(C_7)$  remains unknown. Certain polynomially computable upper bounds on  $c(G)$  are known including Lovász's theta function  $\theta(G)$ , and other upper bounds are due to Haemers and to Schrijver.

Another upper bound, based on the dimension argument and related to the bound of Haemers [40], is described in [3], where it is applied to solve a problem of Shannon on the capacity of the disjoint union of two graphs. The (*disjoint*) union of two graphs  $G$  and  $H$ , denoted  $G + H$ , is the graph whose vertex set is the disjoint union of the vertex sets of  $G$  and of  $H$  and whose edge set is the (disjoint) union of the edge sets of  $G$  and  $H$ . If  $G$  and  $H$  are graphs of two channels, then their union represents the *sum* of the channels corresponding to the situation where either one of the two channels may be used, a new choice being made for each transmitted letter. Shannon proved that for every  $G$  and  $H$ ,  $c(G + H) \geq c(G) + c(H)$  and that equality holds in many cases. He conjectured that in fact equality always holds. In [3] it is shown that this is false in the following strong sense.

**Theorem 2.2** *For every  $k$  there is a graph  $G$  so that the Shannon capacity of the graph and that of its complement  $\bar{G}$  satisfy  $c(G) \leq k, c(\bar{G}) \leq k$ , whereas  $c(G + \bar{G}) \geq k^{(1+o(1))\frac{\log k}{8 \log \log k}}$  and the  $o(1)$ -term tends to zero as  $k$  tends to infinity.*

Therefore, the capacity of the disjoint union of two graphs can be much bigger than the capacity of each of the two graphs. Strangely enough, it is not even known if the maximum possible capacity of a disjoint union of two graphs  $G$  and  $H$ , each of capacity at most  $k$ , is bounded by any function of  $k$ . It seems very likely that this is the case.

### 3 Polynomials, addition and graph coloring

The study of algebraic varieties, that is, sets of common roots of systems of polynomials, is the main topic of algebraic geometry. The most elementary property of a univariate nonzero polynomial over a field is the fact that it does not have more roots than its degree. This elementary property is surprisingly effective in Combinatorics: it plays a major role in the theory of error correcting codes, and has many applications in the study of finite geometries – see, e.g., [14]. A similar property holds for polynomials of several variables, and can also be used to supply results in Discrete Mathematics.

In this section we describe a general result of this type, which is called in [4] *Combinatorial Nullstellensatz*, and briefly sketch some of its applications in Additive Number Theory and in Graph Theory.

### 3.1 Combinatorial Nullstellensatz

Hilbert's Nullstellensatz (see, e.g., [65]) is the fundamental theorem that asserts that if  $F$  is an algebraically closed field, and  $f, g_1, \dots, g_m$  are polynomials in the ring of polynomials  $F[x_1, \dots, x_n]$ , where  $f$  vanishes over all common zeros of  $g_1, \dots, g_m$ , then there is an integer  $k$  and polynomials  $h_1, \dots, h_m$  in  $F[x_1, \dots, x_n]$  so that

$$f^k = \sum_{i=1}^m h_i g_i.$$

In the special case  $m = n$ , where each  $g_i$  is a univariate polynomial of the form  $\prod_{s \in S_i} (x_i - s)$  for some  $S_i \subset F$ , a stronger conclusion holds. It can be shown that if  $F$  is an arbitrary field,  $f, g_i, S_i$  are as above, and  $f$  vanishes over all the common zeros of  $g_1, \dots, g_n$  (that is;  $f(s_1, \dots, s_n) = 0$  for all  $s_i \in S_i$ ), then there are polynomials  $h_1, \dots, h_n \in F[x_1, \dots, x_n]$  satisfying  $\deg(h_i) \leq \deg(f) - \deg(g_i)$  so that

$$f = \sum_{i=1}^n h_i g_i.$$

As a consequence of the above one can prove the following,

**Theorem 3.1** *Let  $F$  be an arbitrary field, and let  $f = f(x_1, \dots, x_n)$  be a polynomial in  $F[x_1, \dots, x_n]$ . Suppose the degree  $\deg(f)$  of  $f$  is  $\sum_{i=1}^n t_i$ , where each  $t_i$  is a nonnegative integer, and suppose the coefficient of  $\prod_{i=1}^n x_i^{t_i}$  in  $f$  is nonzero. If  $S_1, \dots, S_n$  are subsets of  $F$  with  $|S_i| > t_i$ , then there are  $s_1 \in S_1, s_2 \in S_2, \dots, s_n \in S_n$  so that*

$$f(s_1, \dots, s_n) \neq 0.$$

The detailed proof, as well as many applications, can be found in [4]. A quick application, first proved in [5], is the assertion that for any prime  $p$ , any loopless graph  $G = (V, E)$  with average degree bigger than  $2p - 2$  and maximum degree at most  $2p - 1$  contains a  $p$ -regular subgraph.

To prove it, let  $(a_{v,e})_{v \in V, e \in E}$  denote the incidence matrix of  $G$  defined by  $a_{v,e} = 1$  if  $v \in e$  and  $a_{v,e} = 0$  otherwise. Associate each edge  $e$  of  $G$  with a variable  $x_e$  and consider the polynomial

$$f = \prod_{v \in V} [1 - (\sum_{e \in E} a_{v,e} x_e)^{p-1}] - \prod_{e \in E} (1 - x_e),$$

over  $GF(p)$ . Applying Theorem 3.1 with  $t_i = 1$  and  $S_i = \{0, 1\}$  for all  $i$ , we conclude that there are values  $x_e \in \{0, 1\}$  such that  $f(x_e : e \in E) \neq 0$ . It is now easy to check that in the subgraph

consisting of all edges  $e \in E$  for which  $x_e = 1$  all degrees are divisible by  $p$ , and since the maximum degree is smaller than  $2p$  all positive degrees are precisely  $p$ , as needed.

Pyber applied the above result to solve a problem of Erdős and Sauer and prove that any simple graph on  $n$  vertices with at least  $200n \log n$  edges contains a 3-regular subgraph. Pyber, Rödl and Szemerédi proved that this is not very far from being best possible, by showing, using probabilistic arguments, that there are simple graphs on  $n$  vertices with at least  $cn \log \log n$  edges that contain no 3-regular subgraphs. See [58] for some further related results.

### 3.2 Additive Number Theory

The Cauchy-Davenport Theorem, which has numerous applications in Additive Number Theory, is the statement that if  $p$  is a prime, and  $A, B$  are two nonempty subsets of  $Z_p$ , then

$$|A + B| \geq \min\{p, |A| + |B| - 1\}.$$

Cauchy proved this theorem in 1813, and applied it to give a new proof to a lemma of Lagrange in his well known 1770 paper that shows that every positive integer is a sum of four squares. Davenport formulated the theorem as a discrete analogue of a conjecture of Khintchine about the Schnirelman density of the sum of two sequences of integers. There are numerous extensions of this result, see, e.g., [56]. A simple algebraic proof of this result is given in [7], and its main advantage is that it extends easily and gives several related results. This proof can be described as a simple application of Theorem 3.1. If  $|A| + |B| > p$ , then the result is trivial, as the sets  $A$  and  $g - B$  intersect, for each  $g \in Z_p$ . Otherwise, assuming the result is false and  $|A + B| \leq |A| + |B| - 2$ , let  $C$  be a subset of  $Z_p$  satisfying  $A + B \subset C$  and  $|C| = |A| + |B| - 2$ . Define  $f = f(x, y) = \prod_{c \in C} (x + y - c)$  and apply Theorem 3.1 with  $t_1 = |A| - 1, t_2 = |B| - 1, S_1 = A, S_2 = B$  to get a contradiction.

Using similar (though somewhat more complicated) arguments, the following related result is proved in [7].

**Proposition 3.2** *Let  $p$  be a prime, and let  $A_0, A_1, \dots, A_k$  be nonempty subsets of the cyclic group  $Z_p$ . If  $|A_i| \neq |A_j|$  for all  $0 \leq i < j \leq k$  and  $\sum_{i=0}^k |A_i| \leq p + \binom{k+2}{2} - 1$  then*

$$|\{a_0 + a_1 + \dots + a_k : a_i \in A_i, a_i \neq a_j \text{ for all } i \neq j\}| \geq \sum_{i=0}^k |A_i| - \binom{k+2}{2} + 1.$$

The very special case of this proposition in which  $k = 1, A_0 = A$  and  $A_1 = A - \{a\}$  for an arbitrary element  $a \in A$  implies that if  $A \subset Z_p$  and  $2|A| - 1 \leq p + 2$  then the number of sums  $a_1 + a_2$  with  $a_1, a_2 \in A$  and  $a_1 \neq a_2$  is at least  $2|A| - 3$ . This supplies a short proof of a result of Dias Da Silva and Hamidoune [23], which settles a conjecture of Erdős and Heilbronn (cf., e.g., [27]).

Snevily [62] conjectured that for any two sets  $A$  and  $B$  of equal cardinality in any abelian group of odd order, there is a renumbering  $a_i, b_i$  of the elements of  $A$  and  $B$  so that all sums  $a_i + b_i$  are pairwise distinct.

For the cyclic group  $Z_p$  of prime order, this follows easily from Theorem 3.1 by considering the polynomial  $f = \prod_{i < j} (x_i - x_j) \prod_{i < j} (a_i + x_i - a_j - x_j)$  with  $S_1 = \dots = S_k = B$ .

More generally, Dasgupta et al. [24], proved the conjecture for any cyclic group of odd order, by applying the polynomial method for polynomials over  $Q[\omega]$ , where  $\omega$  is an appropriate root of unity, and by considering  $G$  as a subgroup of the multiplicative group of this field. Further related results appear in [63].

Additional applications of Theorem 3.1 in additive number theory can be found in [4].

### 3.3 Graph Coloring

Theorem 3.1 has various applications in the study of Graph Coloring, which is the most popular area in Graph Theory. We sketch below the basic approach, following [12]. See also [52], [53] for a related method.

A *vertex coloring* of a graph  $G$  is an assignment of a color to each vertex of  $G$ . The coloring is *proper* if adjacent vertices get distinct colors. The *chromatic number*  $\chi(G)$  of  $G$  is the minimum number of colors used in a proper vertex coloring of  $G$ . An *edge coloring* of  $G$  is, similarly, an assignment of a color to each edge of  $G$ . It is *proper* if adjacent edges receive distinct colors. The minimum number of colors in a proper edge coloring of  $G$  is the *chromatic index*  $\chi'(G)$  of  $G$ . This is equal to the chromatic number of the line graph of  $G$ .

A graph  $G = (V, E)$  is *k-choosable* if for every assignment of sets of integers  $S(v) \subset Z$ , each of size  $k$ , to the vertices  $v \in V$ , there is a proper vertex coloring  $c : V \mapsto Z$  so that  $c(v) \in S(v)$  for all  $v \in V$ . The *choice number* of  $G$ , denoted  $ch(G)$ , is the minimum integer  $k$  so that  $G$  is  $k$ -choosable. Obviously, this number is at least the chromatic number  $\chi(G)$  of  $G$ . The choice number of the line graph of  $G$ , denoted by  $ch'(G)$ , is usually called the *list chromatic index* of  $G$ , and it is clearly at least the chromatic index  $\chi'(G)$  of  $G$ .

The study of choice numbers was introduced, independently, by Vizing [67] and by Erdős, Rubin and Taylor [29]. There are many graphs  $G$  for which the choice number  $ch(G)$  is strictly larger than the chromatic number  $\chi(G)$  (a complete bipartite graph with 3 vertices in each color class is one such example). In view of this, the following conjecture, suggested independently by various researchers including Vizing, Albertson, Collins, Tucker and Gupta, which apparently appeared first in print in [17], is somewhat surprising.

**Conjecture 3.3 (The list coloring conjecture)** *For every graph  $G$ ,  $ch'(G) = \chi'(G)$ .*



This conjecture asserts that for *line graphs* there is no gap at all between the choice number and the chromatic number. Many of the most interesting results in the area are proofs of special cases of this conjecture, which is still wide open.

The *graph polynomial*  $f_G = f_G(x_1, x_2, \dots, x_n)$  of a graph  $G = (V, E)$  on a set  $V = \{1, \dots, n\}$  of  $n$  vertices is defined by  $f_G(x_1, x_2, \dots, x_n) = \prod\{(x_i - x_j) : i < j, ij \in E\}$ . This polynomial has been studied by various researchers, starting already with Petersen [57] in 1891.

Note that if  $S_1, \dots, S_n$  are sets of integers, then there is a proper coloring assigning to each vertex  $i$  a color from its list  $S_i$ , if and only if there are  $s_i \in S_i$  such that  $f_G(s_1, \dots, s_n) \neq 0$ . This condition is precisely the one appearing in the conclusion of Theorem 3.1, and it is therefore natural to expect that this theorem can be useful in tackling coloring problems. By applying it to line graphs of planar cubic graphs, and by interpreting the appropriate coefficient of the corresponding polynomial combinatorially, it can be shown, using a known result of Vigneron [66] and the Four Color Theorem, that the list chromatic index of every 2-connected cubic planar graph is 3. This is a strengthening of the Four Color Theorem, which is well known to be equivalent to the fact that the chromatic index of any such graph is 3. An extension of this result appears in [25].

Additional results on graph coloring and choice numbers using the above algebraic approach are described in the survey [2]. These include the fact that the choice number of every planar bipartite graph is at most 3, thus solving a conjecture raised in [29], and the assertion, proved in [32], that if  $G$  is a graph on  $3n$  vertices, whose set of edges is the disjoint union of a Hamilton cycle and  $n$  pairwise vertex-disjoint triangles, then the choice number and the chromatic number of  $G$  are both 3.

## 4 The probabilistic method

The discovery that deterministic statements can be proved by probabilistic reasoning, led already in the middle of the previous century to several striking results in Analysis, Number Theory, Combinatorics and Information Theory. It soon became clear that the method, which is now called *the probabilistic method*, is a very powerful tool for proving results in Discrete Mathematics. The early results combined combinatorial arguments with fairly elementary probabilistic techniques, whereas the development of the method in recent years required the application of more sophisticated tools from Probability Theory. In this section we illustrate the method and describe several recent results. More material can be found in the recent books [11], [16], [41] and [55].

## 4.1 Thresholds for random properties

The systematic study of Random Graphs was initiated by Erdős and Rényi whose first main paper on the subject is [28]. Formally,  $G(n, p)$  denotes the probability space whose points are graphs on a fixed set of  $n$  labelled vertices, where each pair of vertices forms an edge, randomly and independently, with probability  $p$ . The term “the random graph  $G(n, p)$ ” means, in this context, a random point chosen in this probability space. Each graph property  $A$  (that is, a family of graphs closed under graph isomorphism) is an event in this probability space, and one may study its probability  $Pr[A]$ , that is, the probability that the random graph  $G(n, p)$  lies in this family. In particular, we say that  $A$  holds *almost surely* if the probability that  $G(n, p)$  satisfies  $A$  tends to 1 as  $n$  tends to infinity. There are numerous papers dealing with random graphs, and the two recent books [16], [41] provide excellent extensive accounts of the known results in the subject.

One of the important discoveries of Erdős and Rényi was the discovery of *threshold functions*. A function  $r(n)$  is called a threshold function for a graph property  $A$ , if when  $p(n)/r(n)$  tends to 0, then  $G(n, p(n))$  does not satisfy  $A$  almost surely, whereas when  $p(n)/r(n)$  tends to infinity, then  $G(n, p(n))$  satisfies  $A$  almost surely. Thus, for example, they identified the threshold function for the property of being connected very precisely: if  $p(n) = \frac{\ln n}{n} + \frac{c}{n}$ , then, as  $n$  tends to infinity, the probability that  $G(n, p(n))$  is connected tends to  $e^{-e^{-c}}$ .

A graph property is *monotone* if it is closed under the addition of edges. Note that many interesting graph properties, like hamiltonicity, non-planarity, connectivity or containing at least 10 vertex disjoint triangles are monotone.

Bollobás and Thomason [18] proved that *any* monotone graph property has a threshold function. Their proof applies to any monotone family of subsets of a finite set, and holds even without the assumption that the property  $A$  is closed under graph isomorphism.

Friedgut and Kalai [30] showed that the symmetry of graph properties can be applied to obtain a sharper result. They proved that for any monotone graph property  $A$ , if  $G(n, p)$  satisfies  $A$  with probability at least  $\epsilon$ , then  $G(n, q)$  satisfies  $A$  with probability at least  $1 - \epsilon$ , for  $q = p + O(\log(1/2\epsilon)/\log n)$ .

The proof follows by combining two results. The first is a simple but fundamental lemma of Margulis [51] and Russo [60], which is useful in Percolation Theory. This lemma can be used to express the derivative with respect to  $p$  of the probability that  $G(n, p)$  satisfies  $A$  as a sum of contributions associated with the single potential edges. The second result is a theorem of [19], which is proved using Harmonic Analysis, that asserts that at least one such contribution is always large. The symmetry implies that all contributions are the same and the result follows. See also [64] for some related results. These results hold for every transitive group of symmetries. In [20]

it is shown that one can, in fact, prove that the threshold for graph properties is even sharper, by taking into account the precise group of symmetries induced on the edges of the complete graph by permuting the vertices. It turns out that for every monotone graph property and for every fixed  $\epsilon > 0$ , the width of the interval in which the probability the property holds increases from  $\epsilon$  to  $1 - \epsilon$  is at most  $c_\delta / (\log n)^{2-\delta}$  for all  $\delta > 0$ . The power 2 here is tight, as shown by the property of containing a clique of size, say,  $\lfloor 2 \log_2 n \rfloor$ .

It is natural to call the threshold for a monotone graph property *sharp* if for every fixed positive  $\epsilon$ , the width  $w$  of the interval in which the probability that the property holds increases from  $\epsilon$  to  $1 - \epsilon$  satisfies  $w = o(p)$ , where  $p$  is any point inside this interval. In [31] Friedgut obtained a beautiful characterization of all monotone graph properties for which the threshold is sharp. Roughly speaking, a property does not have a sharp threshold if and only if it can be approximated well in the relevant range of the probability  $p$  by a property that is determined by constant size witnesses. Thus, for example, the property of containing 5 vertex disjoint triangles does not have a sharp threshold, whereas the property of having chromatic number bigger than 10 does. A similar result holds for hypergraphs as well. The proofs combine probabilistic and combinatorial arguments with techniques from Harmonic analysis.

## 4.2 Ramsey Numbers

Let  $H_1, H_2, \dots, H_k$  be a sequence of  $k$  finite, undirected, simple graphs. The (multicolored) *Ramsey number*  $r(H_1, H_2, \dots, H_k)$  is the minimum integer  $r$  such that in every edge coloring of the complete graph on  $r$  vertices by  $k$  colors, there is a monochromatic copy of  $H_i$  in color  $i$  for some  $1 \leq i \leq k$ . By a (special case of) a well known theorem of Ramsey (c.f., e.g., [38]), this number is finite for every sequence of graphs  $H_i$ .

The determination or estimation of these numbers is usually a very difficult problem. When all graphs  $H_i$  are complete graphs with more than two vertices, the only values that are known precisely are those of  $r(K_3, K_m)$  for  $m \leq 9$ ,  $r(K_4, K_4)$ ,  $r(K_4, K_5)$  and  $r(K_3, K_3, K_3)$ . Even the determination of the asymptotic behaviour of Ramsey numbers up to a constant factor is a hard problem, and despite a lot of efforts by various researchers (see, e.g., [38], [22] and their references), there are only a few infinite families of graphs for which this behaviour is known.

In one of the first applications of the probabilistic method in Combinatorics, Erdős [26] proved that if  $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$  then  $R(k, k) > n$ , that is, there exists a 2-coloring of the edges of the complete graph on  $n$  vertices containing no monochromatic clique of size  $k$ . The proof is extremely simple; the probability that a random two-edge coloring of  $K_n$  contains a monochromatic  $K_k$  is at most  $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$ , and hence there is a coloring with the required property.

A particularly interesting example of an infinite family for which the asymptotic behaviour of the Ramsey number is known, is the following result of Kim [43] together with that of Ajtai, Komlós and Szemerédi [1].

**Theorem 4.1** ([43], [1]) *There are two absolute positive constants  $c_1, c_2$  such that*

$$c_1 m^2 / \log m \leq r(K_3, K_m) \leq c_2 m^2 / \log m$$

for all  $m > 1$ .

The upper bound, proved in [1], is probabilistic, and applies a certain random greedy algorithm. The lower bound is proved by a “semi-random” construction and proceeds in stages. The detailed analysis is subtle, and is based on certain large deviation inequalities.

Even less is known about the asymptotic behaviour of multicolored Ramsey numbers, that is, Ramsey numbers with at least 3 colors. The asymptotic behaviour of  $r(K_3, K_3, K_m)$ , for example, has been very poorly understood until recently, and Erdős and Sós conjectured in 1979 (c.f., e.g., [22]) that

$$\lim_{m \rightarrow \infty} \frac{r(K_3, K_3, K_m)}{r(K_3, K_m)} = \infty.$$

This has been proved recently, in a strong sense, in [10], where it is shown that in fact  $r(K_3, K_3, K_m)$  is equal, up to logarithmic factors, to  $m^3$ . A more complicated, related result proved in [10], that supplies the asymptotic behaviour of infinitely many families of Ramsey numbers up to a constant factor is the following.

**Theorem 4.2** *For every  $t > 1$  and  $s \geq (t - 1)! + 1$  there are two positive constants  $c_1, c_2$  such that for every  $m > 1$*

$$c_1 \frac{m^t}{\log^t m} \leq r(K_{t,s}, K_{t,s}, K_{t,s}, K_m) \leq c_2 \frac{m^t}{\log^t m},$$

where  $K_{t,s}$  is the complete bipartite graph with  $t$  vertices in one color class and  $s$  vertices in the other.

The proof combines spectral techniques, character sum estimates, and probabilistic arguments.

### 4.3 Turán type results

For a graph  $H$  and an integer  $n$ , the Turán number  $ex(n, H)$  is the maximum possible number of edges in a simple graph on  $n$  vertices that contains no copy of  $H$ . The asymptotic behavior of these numbers for graphs of chromatic number at least 3 is well known, see, e.g., [15]. For bipartite graphs  $H$ , however, much less is known, and there are relatively few nontrivial bipartite graphs  $H$  for which the order of magnitude of  $ex(n, H)$  is known.

A result of Füredi [34] implies that for every fixed bipartite graph  $H$  in which the degrees of all vertices in one color class are at most  $r$ , there is some  $c = c(H) > 0$  such that  $ex(n, H) \leq cn^{2-1/r}$ . As observed in [6], this result can be derived from a simple and yet surprisingly powerful probabilistic lemma, variants of which have been proved and applied by various researchers starting with Rödl and including Kostochka, Gowers and Sudakov (see [46], [36], [47].) The lemma asserts, roughly, that every graph with sufficiently many edges contains a large subset  $A$  in which every  $a$  vertices have many common neighbors. The proof uses a process that may be called a *dependent random choice* for finding the set  $A$ ;  $A$  is simply the set of all common neighbors of an appropriately chosen random set  $R$ . Intuitively, it is clear that if some  $a$  vertices have only a few common neighbors, it is unlikely all the members of  $R$  will be chosen among these neighbors. Hence, we do not expect  $A$  to contain any such subset of  $a$  vertices. This simple idea can be extended. In particular, it can be used to bound the Turán numbers of degenerate bipartite graphs.

A graph is *r-degenerate* if every subgraph of it contains a vertex of degree at most  $r$ . An old conjecture of Erdős asserts that for every fixed  $r$ -degenerate bipartite graph  $H$ ,  $ex(n, H) \leq O(n^{2-1/r})$ , and the above technique suffices to show that there is an absolute constant  $c > 0$ , such that for every such  $H$ ,  $ex(n, H) \leq n^{2-c/r}$ .

Further questions and results about Turán numbers can be found in [6], [15] and their references.

## 5 Algorithms and explicit constructions

The rapid development of Theoretical Computer Science and its tight connection to Discrete Mathematics motivated the study of the algorithmic aspects of algebraic and probabilistic techniques. Can a combinatorial structure, or a substructure of a given one, whose existence is proved by algebraic or probabilistic means, be constructed *explicitly* (that is, by an efficient deterministic algorithm)? Can the algorithmic problems corresponding to existence proofs be solved by efficient procedures? The study of these questions often requires tools from other branches of mathematics.

As described in subsection 3.3, if  $G$  is a graph on  $3n$  vertices, whose set of edges is the disjoint union of a Hamilton cycle and  $n$  pairwise vertex-disjoint triangles, then the chromatic number of  $G$  is 3. Can we solve the corresponding algorithmic problem efficiently? That is, is there a polynomial time, deterministic or randomized algorithm, that given an input graph as above, colors it properly with 3 colors? Similarly, as mentioned in subsection 3.3, the list chromatic index of any planar cubic 2-connected graph is 3. Can we color properly the edges of any given planar cubic 2-connected graph using given lists of three colors per edge, in polynomial time?

These problems, as well as the algorithmic versions of additional applications of Theorem 3.1, are open. Of course, an algorithmic version of the theorem itself would provide efficient procedures for

solving all these questions. The input for such an algorithm is a polynomial in  $n$  variables over a field described, say, by a polynomial size arithmetic circuit. Suppose that this polynomial satisfies the assumptions of Theorem 3.1, and that the fact it satisfies it can be checked efficiently. The algorithm should then find, efficiently, a point  $(s_1, s_2, \dots, s_n)$  satisfying the conclusion of Theorem 3.1.

Unfortunately, it seems unlikely that such a general result can exist, as it would imply that there are no one-way permutations. Indeed, let  $F : \{0, 1\}^n \mapsto \{0, 1\}^n$  be a  $1-1$  function, and suppose that for any  $x = (x_1, \dots, x_n) \in \{0, 1\}^n$ , the value of  $F(x)$  can be computed efficiently. Every Boolean function can be expressed as a multilinear polynomial over  $GF(2)$ , and hence, when we wish to find an  $x$  such that  $F(x) = y = (y_1, \dots, y_n)$ , we can write it as a system of multilinear polynomials over  $GF(2)$ :  $F_i(x) = y_i$  for all  $1 \leq i \leq n$ . Equivalently, this can be written as  $\prod_{i=1}^n (F_i(x) + y_i + 1) \neq 0$ . This last equation has a unique solution, implying that its left hand side, written as a multilinear polynomial, is of full degree  $n$  (since otherwise it is easy to check that it attains the value 1 an even number of times). It follows that the assumptions of Theorem 3.1 with  $f = \prod_{i=1}^n (F_i(x) + y_i + 1)$ ,  $t_i = 1$  and  $S_i = GF(2)$  hold. Thus, the existence of an efficient algorithm as above would enable us to invert  $F$  efficiently, implying that there cannot be any one-way permutations. As this seems unlikely, it may be more productive (and yet challenging) to try and develop efficient procedures for solving the particular algorithmic problems corresponding to the results obtained by the theorem.

Probabilistic proofs also suggest the study of the corresponding algorithmic problems. This is related to the study of randomized algorithms, a topic which has been developed tremendously during the last decade. See, e.g., [54] and its many references. In particular, it is interesting to find explicit constructions of combinatorial structures whose existence is proved by probabilistic arguments. "Explicit" here means that there is an efficient algorithm that constructs the desired structure in time polynomial in its size. Constructions of this type, besides being interesting in their own, have applications in other areas. Thus, for example, explicit constructions of error correcting codes that are as good as the random ones are of interest in information theory, and explicit constructions of certain Ramsey type colorings may have applications in derandomization – the process of converting randomized algorithms into deterministic ones.

It turns out, however, that the problem of finding a good explicit construction is often very difficult. Even the simple proof of Erdős, described in subsection 4.2, that there are two-edge colorings of the complete graph on  $\lfloor 2^{m/2} \rfloor$  vertices containing no monochromatic clique of size  $m$ , leads to an open problem which seems very difficult. Can we construct, explicitly, such a coloring of a complete graph on  $n \geq (1 + \epsilon)^m$  vertices, in time which is polynomial in  $n$ , where  $\epsilon > 0$  is any positive absolute constant ?

This problem is still open, despite a lot of efforts. The best known explicit construction is due to Frankl and Wilson [33], who gave an explicit two-edge coloring of the complete graph on

$m^{(1+o(1))\frac{\log m}{4\log\log m}}$  vertices with no monochromatic clique on  $m$  vertices.

The construction of explicit two-edge colorings of large complete graphs  $K_n$  with no red  $K_s$  and no blue  $K_m$  for fixed  $s$  and large  $m$  also appears to be very difficult. Using probabilistic arguments it can be shown that there are such colorings for  $n$  which is  $c\left(\frac{m}{\log m}\right)^{(s+1)/2}$  for some absolute constant  $c > 0$ . The best known explicit construction, however, given in [8], works only for  $m^{\delta\sqrt{\log s/\log\log s}}$ , for some absolute constant  $\delta > 0$ . The description of the construction is not complicated but the proof of its properties relies on tools from various mathematical areas. These include some ideas from algebraic geometry obtained in [45], the well known bound of Weil on character sums, spectral techniques and their connection to the pseudo-random properties of graphs, the known bounds of [48] for the problem of Zarankiewicz and the well known Erdős-Rado bound for the existence of  $\Delta$ -systems.

The above example is typical, and illustrates the fact that tools from various mathematical disciplines often appear in the design of explicit constructions of combinatorial structures. Other examples that demonstrate this fact are the construction of Algebraic Geometry codes, and the construction of sparse pseudo-random graphs called expanders.

## 6 Some future challenges

Several specific open problems in Discrete Mathematics are mentioned throughout this article. These, and many additional ones, provide interesting challenges for future research in the area. We conclude with some brief comments on two more general future challenges.

It seems safe to predict that in the future there will be additional incorporation of methods from other mathematical areas in Combinatorics. However, such methods often provide non-constructive proof techniques, and the conversion of these to algorithmic ones may well be one of the main future challenges of the area. Another interesting recent development is the increased appearance of Computer aided proofs in Combinatorics, starting with the proof of the Four Color Theorem, and including automatic methods for the discovery and proof of hypergeometric identities – see [59]. A successful incorporation of such proofs in the area, without losing its special beauty and appeal, is another challenge. These challenges, the fundamental nature of the area, its tight connection to other disciplines, and the many fascinating specific open problems studied in it, ensure that Discrete Mathematics will keep playing an essential role in the general development of science in the future as well.

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