Partitioning the hypercube into smaller hypercubes

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Abstract

Denote by Q_d the d-dimensional hypercube. We estimate the number of ways the vertex set of Q_d can be partitioned into vertex disjoint smaller cubes. Among other results, we prove that the asymptotic order of this function is larger than the number of perfect matchings of Q_d by an exponential factor in the number of vertices, and not by a larger factor. We also describe and address several new (and old) related questions.

1 The problem and main results

Denote by $Q_d := \{0,1\}^d$ the d-dimensional hypercube, and let $f(d)$ be the number of partitions (tilings) of the vertex set of Q_d in which each of the classes spans a subhypercube. Let $f_S(d)$ denote the number of such partitions where each of the classes has dimension in $S \subset \{0,1,\ldots,d\}$. In particular, $f_{\leq 2}(d)$ counts the partitions where the parts are singletons, edges or spanning a 2-cube.
For a graph G, denote by $m(G)$ the number of perfect matchings in G, and by $m'(G)$ the number of matchings, and write $m(d) := m(Q_d) = f_1(d)$ and $m'(d) = m'(Q_d) = f_{\leq 1}(d)$.

An easy observation is that the number of (perfect) matchings of Q_d is a lower bound on $f(d)$. Determining or estimating the number of (perfect) matchings of

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graphs is a classical problem, see, e.g., [2, 7, 12, 17]. Although it has been studied for Q_d, besides determining it for small d, only general results are known about its asymptotics, as we discuss later. Below we collect many problems that might be of interest; and we will address several of them.

Problem 1.1. *Determine or estimate the functions*
(i) $m(d)$, *(ii)* $m'(d)$, *(iii)* $f(d)$, *(iv)* $f_{\leq 2}(d)$.

Problem (i) is natural and was studied in the literature, (ii) is a natural extension. Problem (iii) was raised by Gadouleau at the 29th British Combinatorial Conference in 2022, motivated by the paper of Bridoux, Gadouleau, and Theyssier [6]. Problem (iv) is a variant we suggest here, and seems to be closely related to (iii).

As pointed out in [6], the first few terms of $f(d)$ are given in the Online Encyclopedia of Integer Sequences (OEIS) as A018926. Starting from $d = 0$ these are: $1, 2, 8, 154, 89512$, 71319425714.

Graham and Harary [14] studied the number of perfect matchings of the hypercube. They calculated
$m(1) = 1$, $m(2) = 2$, $m(3) = 9$, $m(4) = 272$, $m(5) = 589185$.

Using dynamic programming, Östergård and Pettersson [21] determined $m(6)$ (which was also known earlier) and $m(7)$:
$m(6) = 16332454526976$, $m(7) = 391689748492473664721077609089$.

Determining precisely, or even providing satisfactory estimates for the functions $m(d)$, $m'(d)$ and f_S for some interesting families S seems to be difficult, hence it would also be interesting to compare the orders of magnitude of some of these functions. Set

$$n := 2^{d-1} \qquad \text{and} \qquad N := (d/e)^n.$$

Note that n is the number of vertices in each vertex class of Q_d, and as mentioned below, N is a rough estimate for the number of perfect matchings in it.

The following hierarchy follows from the definitions:

$$m(d) \ \leq \ m'(d) \ \leq \ f_{\leq 2}(d) \ \leq \ f(d). \tag{1}$$

We prove that the ratio $f(d)/m(d)$ is exponential in n, and wonder in which of the inequalities in (1) there is an exponential ratio.

Problem 1.2. *What is the order of magnitude of the following ratios? In particular, which one of them is an exponential function of* n*?*
(i) $m'(d)/m(d)$, *(ii)* $f_{\leq 2}(d)/m'(d)$, *(iii)* $f(d)/f_{\leq 2}(d)$, *(iv)* $f(d)/m(d)$.

Our results and conjectures are summarized below.

Classical results easily imply Proposition 1.3, which determines the main term of the asymptotics of $m(d)$, partially solving Problem 1.1 (i).[1] Proposition 1.4 partially solves Problem 1.1 (ii) and Problem 1.2 (i), showing that $m'(d)/m(d)$ is subexponential. For both statements we state general results about regular bipartite graphs, which imply the required bounds for the hypercube. Proposition 1.5 shows that $f_{\le 2}(d)/m'(d)$ and $f_{\le 2}(d)/m(d)$ are (at least) exponential in n, addressing Problem 1.2 (ii) and (iv). In Proposition 1.6 we show that the ratio $f(d)/m(d)$ is at most exponential in n, and a modification of its proof implies that allowing to use 2-dimensional subcubes in the partitions has a large impact on their numbers, see Proposition 1.8.

Our final result, Proposition 1.11, shows that there are many *irreducible tight* partitions. The precise definition of this notion appears before the statement of the result.

We proceed with the formal statements of all the results.

Proposition 1.3. *(i) Let G be an a-regular bipartite graph with vertex classes of size b, where $a \ge 2$. Then*

$$e^{b/2a} \cdot \left(\frac{a}{e}\right)^b \le m(G) \le [\sqrt{2\pi a} \cdot e^{1/(12a)}]^{b/a} \cdot \left(\frac{a}{e}\right)^b = 2^{(1+o(1))\cdot \log a \cdot (b/2a)} \cdot \left(\frac{a}{e}\right)^b, \quad (2)$$

when both $a, b \to \infty$.

(ii) In particular for Q_d we have

$$e^{n/2d} \cdot N \le m(d) \le 2^{(1+o(1))\cdot \log d \cdot (n/2d)} \cdot N. \quad (3)$$

The following result is addressing Problem 1.1 (ii) and Problem 1.2 (i). The lower bound in Proposition 1.4 (i) also follows from known results in [7], and the upper bound from the ones in [17]. In particular, these results imply that $m'(G)/m(G) = e^{(2+o(1))n/\sqrt{d}}$. Here we describe a short and simple proof, which does not provide the optimal constant in the exponent obtained in the upper bound.

Proposition 1.4. *(i) Let G be an a-regular bipartite graph with vertex classes of size b, where $a \ge 1$ and $b \to \infty$. Then*

$$m'(G) = m(G) \cdot 2^{\Theta(b/\sqrt{a})} = \left(\frac{a}{e}\right)^b \cdot 2^{\Theta(b/\sqrt{a})}. \quad (4)$$

(ii) In particular for Q_d we have

$$m'(d) = m(d) \cdot 2^{\Theta(n/\sqrt{d})} = N \cdot 2^{\Theta(n/\sqrt{d})}. \quad (5)$$

[1]The proof of the upper bound, and that of a slightly weaker lower bound on $m(d)$ as stated in Proposition 1.3, are posted in several class notes at various course websites, see also [19]. Here we provide a short proof, for completeness.

Next, we show that the number of cube partitions of Q_d is exponentially larger than the number of matchings.

Proposition 1.5. *There exists a constant $c > 1$ such that for all $d \geq 3$*

$$c^n \cdot m'(d) \;\leq\; f_{\leq 2}(d) \;\leq\; f(d). \tag{6}$$

The following result is a simple upper bound on $f(d)$. Note that a sketch of the proof appeared already in [23].

Proposition 1.6.

$$f(d) \leq (d+1)^n \;\leq e^{n+n/d} \cdot N.$$

The following Propositions show the effect of allowing 1- and 2-dimensional (and in general small dimensional) subcubes in a partition.

Proposition 1.7. *For every fixed $r \geq 2$*

$$N^{r/2^{r-1}-o(1)} \leq f_{0,r}(d) \leq f_{0,r,r+1,r+2,\ldots}(d) \leq N^{r/2^{r-1}+o(1)}.$$

Proposition 1.8.

$$f_{0,1,3,4,\ldots}(d) \leq \exp(20n/d^{1/4}) \cdot N \leq \exp(20n/d^{1/4}) \cdot f_1(d) = \exp(20n/d^{1/4}) \cdot m(d).$$

It is natural to ask what happens if only 2-dimensional subcubes are allowed to be in a partition. We believe that the same bound as in Proposition 1.7, with $r = 2$ holds.

Problem 1.9. *Determine the asymptotic behaviour of the number $f_2(d)$.*

Additionally, we left open another question addressing Problem 1.2 (iii).

Problem 1.10. *Is it true that $f(d)/f_{\leq 2}(d)$ is subexponential in n?*

A subcube partition is an *irreducible*, if no subcube is spanned by a subfamily of the partition, and it is *tight* if the partition is ‘proper’, in the sense that every coordinate is used, i.e., every coordinate is fixed in at least one subcube of the partition. For example, the partition of the 2-cube into the two subcubes $\{0*\}$, in which the first coordinate is fixed to 0, and $\{1*\}$, in which the first coordinate is fixed to 1, is not a tight partition, as the second coordinate is not used.

Irreducible subcube partitions appear in a work of Kullmann and Zhao [18] and variants are described in several other papers, see [10] for relevant references. Note that it is not immediately clear that there exists an irreducible tight partition for every given (large) d.

Peitl and Szeider [22] enumerated all tight irreducible subcube partitions for $d = 3, 4$, and asked whether there are infinitely many such partitions. Filmus, Hirsch, Kurz, Ihringer, Riazanov, Smal, and Vinyals [10] answered this question in the affirmative, giving many explicit constructions of tight irreducible subcube partitions. Here we prove the existence of many more tight irreducible partitions.

Proposition 1.11. *The number of irreducible tight partitions of the d-dimensional cube is at least c^n for some absolute constant $c > 1$, where, as before, $n = 2^{d-1}$.*

We shall frequently use the Stirling formula:

$$\sqrt{2\pi m} \cdot \left(\frac{m}{e}\right)^m \;\le\; m! \;\le\; e^{1/(12m)} \cdot \sqrt{2\pi m} \cdot \left(\frac{m}{e}\right)^m .$$

Furthermore, we use the binary entropy function $h(x) := -x \log_2 x - (1-x)\log_2(1-x)$ to estimate the binomial coefficients for $0 < x < 1/2$:

$$\sum_{k \le xn} \binom{n}{k} \;=\; \Theta(1) \cdot \binom{n}{xn} = 2^{(1+o(1))h(x)\cdot n}. \tag{7}$$

All logarithms throughout the paper have base 2, unless otherwise indicated. To simplify the presentation, we omit all floor and ceiling signs whenever these are not crucial. We also assume, whenever this is needed, that d (and hence also n) is sufficiently large.

In Section 2 we prove Propositions 1.3, 1.4, and 1.5. In Section 3 we prove Propositions 1.6, 1.7, 1.8, and 1.11.

Remark: We mention very briefly two motivations for the study of $f(d)$. First, $f(d)$ is the number of so-called bijective commutative Boolean networks of dimension d. See [6] for the definition of this notion and the fact that there is a bijection between partitions of the d-cube into subcubes and such networks.

Secondly, $f(d)$ is the number of instances of SAT on d Boolean variables such that any truth assignment fails to satisfy exactly one clause of the instance. Indeed, to any clause (e.g. $x_1 \vee \neg x_3$) we can associate the subcube of assignments that fail to satisfy it ($x_1 = 0, x_3 = 1$ in the example above). Then those subcubes partition the d-cube if and only if any truth assignment belongs to exactly one of them, i.e., it fails to satisfy exactly one clause. This is equivalent to the description from OEIS, which reads: "the number of ways to make a tautology from disjoint terms with d Boolean variables".

2 Matchings and 2-dimensional subcubes

First, we list some of the classical results that we shall use in our proofs.

Bregman-Minc inequality: The celebrated Bregman-Minc inequality, conjectured by Minc [20] and proved by Bregman [5] (see also [25], [3], [24] for short proofs) implies that the maximum possible number of perfect matchings in an a-regular bipartite graph with b vertices in each vertex class is at most $(a!)^{b/a}$. Equality is achieved when b is divisible by a, by a vertex disjoint collection of complete bipartite graphs. When the degrees of the vertices on one side are $a_1, \ldots, a_b$, with average degree a, the following upper bound holds: $\Pi_i (a_i!)^{1/a_i} \leq (a!)^{b/a}$.

Van der Waerden inequality: In 1926 Van der Waerden [27] conjectured that the minimum possible value of the permanent of a b-by-b doubly stochastic matrix is $b!/b^b$, achieved by the matrix in which all entries are $1/b$. Proofs of this conjecture were given in the early 80s by Falikman [9] and by Egorychev [8], see also Gyires [16].

Schrijver's bound: Schrijver [26] (see also [15] for subsequent work) proved that every a-regular bipartite graph with b vertices in each class has at least

$$\left(\frac{(a-1)^{a-1}}{a^{a-2}}\right)^b$$

perfect matchings.

Proof of Proposition 1.3 (i). The Bregman-Minc inequality gives that

$$m(G) \leq (a!)^{b/a} \leq [\sqrt{2\pi a} \cdot \left(\frac{a}{e}\right)^a \cdot e^{1/(12a)}]^{b/a} = 2^{(1/2+o(1))\log a \cdot (b/a)} \cdot \left(\frac{a}{e}\right)^b.$$

The Van der Waerden inequality gives the following lower bound on $m(G)$:

$$m(G) \geq \frac{b!}{b^b} \cdot a^b \geq \sqrt{2\pi b} \cdot \left(\frac{b}{e}\right)^b \cdot b^{-b} \cdot a^b.$$

Using Schrijver's bound, and some delicate estimates on e, we obtain the following improved asymptotics for the lower bound:

$$\begin{aligned} m(G) \geq \left(\frac{(a-1)^{a-1}}{a^{a-2}}\right)^b &= \left(\frac{a}{e}\right)^b \left(\frac{e \cdot (a-1)^{a-1}}{a^{a-1}}\right)^b \\ \geq \ & \left(\frac{a}{e}\right)^b \cdot \left(\left(1 + \frac{1}{a-1} + \frac{1}{2(a-1)^2} + \frac{1}{6(a-1)^3}\right)\left(1 - \frac{1}{a}\right)\right)^{(a-1)b} \\ = \ & \left(\frac{a}{e}\right)^b \cdot \left(1 + \frac{1}{2a(a-1)} + \frac{1}{6a(a-1)^2}\right)^{(a-1)b} \geq \left(\frac{a}{e}\right)^b \cdot e^{b/2a}. \end{aligned}$$

The inequality in the second line above is equivalent to the statement that

$$e^{1/(a-1)} \geq 1 + \frac{1}{a-1} + \frac{1}{2(a-1)^2} + \frac{1}{6(a-1)^3},$$

which follows from the fact that the right-hand-side is the sum of the four first terms in the power series of the left-hand-side, in which all terms are positive. The final inequality is equivalent to the fact that

$$e^{1/(2a(a-1))} \leq 1 + \frac{1}{2a(a-1)} + \frac{1}{6a(a-1)^2}.$$

This inequality holds for every $a \geq 2$ since the power series of the left-hand-side is

$$1 + \frac{1}{2a(a-1)} + \frac{1}{8a^2(a-1)^2} + \dots$$

The first two terms here are equal to the first two terms in the expression above, and it is easy to see that for $a \geq 2$ the sum of all the remaining terms is smaller than $\frac{1}{6a(a-1)^2}$.

(ii) Follows from (i) by setting $a := d, b := n$ and $N = \left(\frac{d}{e}\right)^n$. $\square$

Proof of Proposition 1.4. (i) Note that the second equation instantly follows from (2), we just need to prove the first equation.

To prove the lower bound, set $t = b/\sqrt{a}$. For every perfect matching M of G, let F be a random subset of t edges of M, chosen uniformly among all subsets of cardinality t of M. Note that $M - F$ is a matching (of size $b - t$). This provides $m(G) \cdot \binom{b}{t}$ matchings, but the same matching may be obtained multiple times. More precisely, the number of times such a matching $M - F$ appears is exactly the number of perfect matchings in the induced subgraph of G on the set of vertices $V(F)$ covered by the edges of F. The expected number of edges in this induced subgraph is exactly

$$t + (b \cdot a - b)\frac{\binom{t}{2}}{\binom{b}{2}} < t + b.$$

Indeed, the subgraph contains exactly t edges of M, and each edge that does not belong to M lies in this induced subgraph with probability $\binom{t}{2}/\binom{b}{2} < t^2/b^2 = 1/a$. The above estimate thus follows from the linearity of expectation. By Markov's Inequality it follows that with probability at least, say, $1/3a$, the number of edges in this induced subgraph is smaller than $b + 2t$. This gives that with probability at least

$1/3a$ the induced subgraph of G on $V(F)$ has average degree smaller than $(b+2t)/t = \sqrt{a}+2$. By Minc Conjecture and the fact that as shown in [1], Corollary 2.3 (see also [3], page 66) the upper bound provided by the Bregman-Minc inequality for the permanent of a $\{0,1\}$-matrix with a given number of 1 entries is obtained when all the rows have the same sum, it follows that the number of perfect matchings spanned by $V(F)$ is at most

$$[(\sqrt{a}+2)!]^{t/(\sqrt{a}+2)} \leq [(1+o(1))\sqrt{a}/e]^t.$$

Therefore, in each of these cases the matching $M-F$ obtained is counted at most $[(1+o(1))\sqrt{a}/e]^t$ times. It follows that the number of matchings of size $b-t$ in G is at least

$$\frac{1}{3a} \cdot m(G) \cdot \frac{\binom{b}{t}}{[(1+o(1))\sqrt{a}/e]^t} = m(G) \cdot [(1+o(1))e^2]^{b/\sqrt{a}},$$

implying the desired lower bound.

For the proof of the upper bound for $m'(G)$ denote by $M(t)$ the set of matchings in G with $b-t$ edges, where $0 \leq t \leq b$. For a given t, there are $\binom{b}{t}^2$ ways to choose the set of uncovered vertices, and given this choice, by the Bregman-Minc inequality, there are at most $(a!)^{(b-t)/a}$ ways to place a perfect matching on the rest of the vertices. Therefore

$$|M(t)| \leq \binom{b}{t}^2 \cdot (a!)^{(b-t)/a}.$$

To bound the sum of these terms for all t define $x = (a!)^{1/a}$ and observe that this sum satisfies

$$\sum_t \binom{b}{t}^2 x^{b-t} \leq \left(\sum_t \binom{b}{t}(\sqrt{x})^{b-t}\right)^2 = (1+\sqrt{x})^{2b} = (\sqrt{x})^{2b}\left(1+\frac{1}{\sqrt{x}}\right)^{2b}.$$

Using Stirling's formula it is easy to check that

$$(\sqrt{x})^{2b} = (a!)^{b/a} \leq \left(\frac{a}{e}\right)^b e^{O(b\log a/a)} \qquad \text{and} \qquad \left(1+\frac{1}{\sqrt{x}}\right)^{2b} \leq e^{O(b/\sqrt{x})} = e^{O(b/\sqrt{a})}.$$

This provides the required upper bound $\left(\frac{a}{e}\right)^b \cdot 2^{O(b/\sqrt{a})}$.
(ii) Follows from (i) by setting $a := d,\ b := n$ and $N = \left(\frac{d}{e}\right)^n$. □

Next we describe the proof of Proposition 1.5, starting with an outline of the proof. We have to prove that there is a constant $c > 1$ such that $f(d) > c^n \cdot m'(d)$. To do so we fix a small positive constant α and choose randomly and uniformly $(1+o(1))\alpha n$ 2-dimensional subcubes of Q_d. Whenever two such chosen cubes have common vertices, we remove one of them, noting that typically the number of subcubes removed is only $O(\alpha^2)n$. This gives a collection B of some $\beta n = (\alpha - O(\alpha^2))n$ 2-dimensional subcubes, and by lower bounding the probability that the set B produced is of size βn we get that there are many distinct choices for such sets.

We complete this partial covering by placing a (nearly) perfect matching on the rest of the graph, which has $(1-2\beta)n$ vertices in each of the vertex classes, and is roughly $(1-2\beta)d$ regular. Since the rest of the graph is not exactly regular, we do not have a good lower bound on the number of its matchings. Therefore, we create a regular bipartite graph by adding some vertices and removing some edges to make it regular. This enables us to apply the lower bound in Proposition 1.3 and to estimate the number of ways to add a matching and single vertices (0-dimensional cubes) for each collection B of 2-cubes. A careful computation then yields the desired estimate.

Proof of Proposition 1.5. The goal is to prove that there is a constant $c > 1$ such that $f(d) > c^n \cdot m'(d)$. Let $\alpha > 0$ be a sufficiently small constant, $c > 1$ will depend on the choice of α. Consider the hypercube Q_d as a bipartite graph with vertex classes (U, V), (the even vertices and the odd ones). Let $A \subset V$ be a random subset of V obtained by picking each vertex of V, randomly and independently, to lie in A with probability α. Therefore the size of A is close to αn with high probability. For each $u_i \in A$ choose a vertex $v_i \in V$ uniformly and randomly among the $\binom{d}{2}$ choices so that $\{u_i, v_i\}$ together with two additional vertices from U span a copy of C_4.

This way we get a partial covering of Q_d by the 2-cubes spanned by $\{(u_i, v_i) : u_i \in A\}$. We try to complete this covering by placing a (nearly) perfect matching on the rest of the graph, which has about $(1-2\alpha)n$ vertices in each of the vertex classes, and is roughly $(1-2\alpha)d$ regular. Our gain on the number of cube partitions will come from the number of ways of choosing the set of 2-cubes. Unfortunately, the rest of the graph is not exactly regular, hence we do not have a good lower bound on the number of its matchings. To go around this we create a regular bipartite graph by adding some vertices and removing some edges to make it regular.

In case two 2-cubes are overlapping, we remove one of them. With high probability, the number of removed vertices is $\Theta(\alpha^2 n)$. We thus assume this is the case, (and count only partitions in which this holds). This way we obtain a partial C_4-covering B of Q_d (that is, B is a family of vertex disjoint C_4's), with $|B| = (\alpha - \Theta(\alpha^2))n =: \beta n$, i.e., $(\alpha - \beta)n = \Theta(\alpha^2)n$. When $\alpha > 0$ is sufficiently small, then the number of choices for B is at least

$$\frac{1}{n^2}\binom{n}{\alpha n}\cdot\binom{d}{2}^{\alpha n}\cdot\binom{n}{(\alpha-\beta)n}^{-1}\cdot\binom{d}{2}^{-(\alpha-\beta)n} \geq 2^{(h(\alpha)/2)n}\cdot d^{2\beta n}. \tag{8}$$

The first $1/n^2$ factor above is for considering only random choices of A in which $|A| = \alpha n$ and $|B| = \beta n$, meaning that B is obtained from A by removing exactly $(\alpha-\beta)n$ 2-cubes. There are at least $\frac{1}{n^2}\binom{n}{\alpha n}\cdot\binom{d}{2}^{\alpha n}$ ways to choose the collection A so that this holds, and each such A produces a collection B. The number of times a fixed collection B can be obtained this way is the number of ways to add $(\alpha-\beta)n$ 2-cubes to this fixed collection, where these added subcubes contain $(\alpha-\beta)n$ distinct vertices u_i in V and the subcube is determined by u_i and a vertex v_i of Hamming distance 2 from u_i. There are at most $\binom{n}{(\alpha-\beta)n}$ ways to choose the vertices u_i and given those, at most $\binom{d}{2}^{(\alpha-\beta)n}$ to choose the corresponding vertices v_i. Therefore, dividing by the product $\binom{n}{(\alpha-\beta)n}\cdot\binom{d}{2}^{(\alpha-\beta)n}$ ensures that each partial collection B of pairwise disjoint 2-subcubes is counted at most once. The last inequality follows from the fact that for small fixed $\alpha > 0$

$$\binom{n}{\alpha n} = 2^{(1+o(1))h(\alpha)n} \gg n^2 2^{\alpha n}\cdot 2^{h(O(\alpha^2))n} = n^2\cdot 2^{\alpha n}\cdot\binom{n}{(\alpha-\beta)n}.$$

Let H be the (random) graph spanned by $Q_d - V(B)$. Observe that if $u, v \in V(H)$ and the distance between them in Q_d is a least, say, 10, then their degrees in this graph are independent.

Claim 2.1. *With high probability, all but at most n/d vertices in H have degree in the interval $J := [(1-2\beta)d - d^{2/3}, (1-2\beta)d + d^{2/3}]$.*

Proof. It is easy to see that for each fixed vertex of H, its degree in H lies in the interval J with high probability. This follows, for example, from Azuma's martingale inequality (c.f., e.g., [3]). Note, however, that this does not suffice to imply the claim, as the events corresponding to distinct vertices are not independent.

To complete the proof of the claim partition $V(H)$ into d^{10} classes, so that in each class the vertices are at distance at least 10 from each other. In each class, the degrees of the vertices in H are independent. Applying the Chernoff bound to each class, we obtain that each class contains about the expected number of vertices whose degrees are not in the interval. The claimed result follows by the union bound, with room to spare. □

Returning to the proof of the proposition, we assume that the assertion of the claim holds (and only count such partitions). We thus assume that there are at most

n/d vertices in H with degrees not in J. As long as there is a vertex with degree larger than $(1-2\beta)d+d^{2/3}$, we remove an arbitrary subset of its edge set to make its degree $(1-2\beta)d+d^{2/3}$. This way we remove a total of at most $n/d \cdot 2\beta \cdot d = 2\beta n$ edges. The number of edges missing in each vertex class in order to get a $((1-2\beta)d+d^{2/3})$-regular graph is smaller than

$$n/d \cdot ((1-2\beta)d + d^{2/3}) + n \cdot 2d^{2/3} + 2\beta n < 2.5d^{2/3} \cdot n.$$

We now add auxiliary vertices to H, to create a $((1-2\beta)d + d^{2/3})$-regular graph F. This can be done by adding at most $2.5d^{2/3} \cdot n/(d(1-2\beta)) \leq 3d^{-1/3} \cdot n$ vertices to each class.

Consider an arbitrary fixed perfect matching M of F. By the lower bound in Proposition 1.3, the number of such M is at least

$$\left(\frac{(1-2\beta)d + d^{2/3}}{e}\right)^{(1-2\beta)n} \geq (1-2\beta)^{(1-2\beta)n} \cdot \left(\frac{d}{e}\right)^{(1-2\beta)n}.$$

Using M, we define the following cube partition of $Q_d - B$. We let the edges of M be the 1-dimensional cubes, with the (obvious) restriction that if an endpoint of an edge of M is not in Q_d, then it is not used, and if exactly one end point of the edge is in $V(Q_d)$, then it is considered as a 0-dimensional subcube of the partition. Note that the number of such 0-cubes is at most $3n/d^{1/3}$ in each class.

This way, we obtain a cube covering using subcubes of dimensions $0, 1, 2$; but we might obtain the same covering from different matchings M. The overcounting is at most the number of ways a perfect matching could be placed on the vertices of $V(F) - V(Q_d)$ and the vertices of the 0-dimensional cubes. The number of such vertices in V is at most $6n/d^{1/3}$. Hence, by the Bregman-Minc inequality, the number of ways to cover them with a perfect matching is at most $(d!)^{6n/d^{4/3}}$. To summarize, the number of tilings using subcubes of dimensions $\{0, 1, 2\}$ is at least

$$\frac{1}{2} \cdot 2^{(h(\alpha)/2)n} \cdot d^{2\beta n} \cdot (1-2\beta)^{(1-2\beta)n} \cdot \left(\frac{d}{e}\right)^{(1-2\beta)n} \cdot (d!)^{-6n/d^{4/3}} > 2^{(h(\alpha)/3)n} \cdot N > c^n \cdot m'(d).$$

The first $1/2$ factor takes care of the fact that we count only partitions in which the required high probability events hold. The constant c can be chosen, for example, to be $c = 2^{(h(\alpha)/4)} > 1$, where $\alpha > 0$ is a sufficiently small absolute constant. $\square$

3 Upper bound on $f(d)$ and related estimates

Proof of Proposition 1.6. We need to show that $f(d) \le (d+1)^n$. Let $v_1, v_2, \ldots, v_n$ be an enumeration of all the n even vertices of the hypercube Q_d (for example, lexicographically), and fix a similar enumeration of all odd vertices. For each partition P of the hypercube into subcubes we construct a sequence $S = S(P) = (s_1, s_2, \ldots, s_n)$ of length n over the alphabet $\{0, 1, \ldots, d\}$ so that $S(P)$ completely defines the partition P. This is done as follows. Each element s_i of $S(P)$ corresponds to the vertex v_i. If v_i lies in a subcube D of the partition P of positive dimension, and it is a neighbor of the lexicographically first odd vertex u of D, then $s_i = j$, where j is the coordinate in which v_i and u differ. If v_i lies in such a subcube D and is not a neighbor of u then $s_i = 0$, and this is also the case if v_i forms a subcube of P of dimension 0. This completes the definition of $S = S(P)$. It is easy to see that given $S = S(P)$, we can reconstruct all the odd vertices which are the lexicographically smallest ones in subcubes in the partition P of positive dimension. For each of those, we can reconstruct the corresponding subcubes, and then we also get all the remaining vertices which are subcubes of dimension 0. This shows that $f(d) \le (d+1)^n$ and completes the proof of the proposition. □

Proof of Proposition 1.7. First we prove the lower bound. A slight modification of the Frankl-Rödl [11] nibble method, as pointed out by Grable and Phelps [13], gives an estimate on the number of almost perfect matchings in r-uniform hypergraphs. We use a version from Asratian and Kuzjurin [4]:

Theorem 3.1. *The following holds for every fixed small $\delta > 0$. Let r be fixed and let $\mathcal{H}_{2n}$ be an r-uniform D-regular hypergraph on $2n$ vertices, where both D and n are tending to infinity. Furthermore, assume that the maximum codegree is $o(D)$, where the codegree of a pair of vertices is the number of hyperedges containing both. Then, the number of matchings of $\mathcal{H}_{2n}$ with at least $(1-\delta)2n/r$ hyperedges is at least $D^{(1-2\delta)(2n/r)}$.*

To prove the proposition, define a 2^r-uniform hypergraph $\mathcal{H}_{2n}$ on the vertex set of Q_d, where the hyperedges are the r-subcubes of Q_d. The hypergraph $\mathcal{H}_{2n}$ is $D = \binom{d}{r}$-regular, with maximum codegree $\binom{d-1}{r-1}$. A matching M of size $(1-\delta)n/2^{r-1}$ corresponds to a $\{0, r\}$-covering of Q_d, where if a vertex is not covered by M, then we cover it by a 0-cube, and the hyperedges of M correspond to r-cubes. As δ can be chosen to be arbitrarily small, this implies that

$$f_{0,r}(d) \ \ge \ \binom{d}{r}^{(2^{1-r}-o(1))n} = N^{r/2^{r-1}+o(1)}.$$

To prove the upper bound we apply the same encoding that appears in the proof of Proposition 1.6 to each of the relevant partitions here. The improved bound is obtained since every sequence that appears in this encoding contains only a small number of nonzero entries. Indeed, note that each partition here contains no subcubes of dimensions $1, 2, \ldots, r-1$, and for each subcube D of dimension $k > 0$ (which must be at least r) we get exactly k nonzero elements of the sequence, and $2^{k-1} - k$ zeros. All other elements of the sequence that correspond to subcubes of dimension 0 (consisting of an even vertex) are also 0. This gives sequences of length n in which the fraction of non-zeros is at most $r/2^{r-1}$ providing an upper bound of

$$\sum_{j \le rn/2^{r-1}} \binom{n}{j} d^j = N^{r/2^{r-1}+o(1)},$$

that provides the required result. □

Proof of Proposition 1.8. We estimate the number of cube partitions which do not use 2-dimensional subcubes. To do so we bound the number of partitions in which exactly T vertices in each of the two vertex classes, the even (U) and the odd (V), are covered by cubes of dimension at least 3, where here $0 \le T \le n$:

– there are at most $\binom{n}{T}$ ways to choose a set A of T even vertices covered by cubes of dimension at least 3,

– given such choices, we can construct a sequence of length T over the alphabet $\{0, 3, 4, \ldots, d\}$ by following the procedure in the previous proofs, but here it is applied only to the set A of T chosen even vertices. More precisely, the sequence produced corresponds to the enumeration of the vertices of A induced by the lexicographic order. For each subcube D of the partition of dimension at least 3 let u be its lexicographically first odd vertex. If a vertex v of A belongs to D and is a neighbor of u then the corresponding element of the sequence is j, where u and v differ in coordinate j, and if v is in D but is not a neighbor of u then the corresponding element is 0. This produces a sequence of length T over $\{0, 1, 2, \ldots, d\}$ in which the fraction of nonzero elements is at most $3/4$. Note that the sequence enables one to reconstruct all the subcubes of dimension at least 3 in the partition. Therefore, for each fixed choice of A there are less than $2^T \cdot d^{3T/4}$ ways to place the cubes of dimension at least 3.

– Using Proposition 1.4 (the statement holds for maximum degree a, instead of a-regular) with $a = d$ and $b = n - T$, given the above choices, there are at most $\left(\frac{d}{e}\right)^{n-T} \cdot 2^{\Theta((n-T)/\sqrt{d})}$ ways to place a perfect matching on the rest of the vertices.

Hence, for a fixed T the number of these cube partitions is at most

$$\binom{n}{T} \cdot 2^T \cdot d^{3T/4} \cdot \left(\frac{d}{e}\right)^{n-T} \cdot 2^{\Theta((n-T)/\sqrt{d})}.$$

Summing up over all choices of T, and using the binomial theorem, we obtain the upper bound

$$(1 + 2ed^{-1/4})^n \cdot 2^{\Theta(n/\sqrt{d})} \cdot N \leq \exp(20n/d^{1/4}) \cdot N.$$

$\square$

Proof of Proposition 1.11. The construction is recursive. By induction on the dimension d we construct many irreducible tight partitions of Q_d, all containing one specific subcube. We start in some fixed dimension d with at least 3 irreducible tight partitions of Q_d all containing some subcube. Given two distinct partitions B_1, B_2 among those consider the partition $B = B_1 0 \cup B_2 1$ of Q_{d+1} in which the partition B_1 is used in the first hyperplane in which the last coordinate is 0 and the partition B_2 is used in the second hyperplane in which the last coordinate is 1. Since B_1 and B_2 have the same subcube D we replace $D0$ and $D1$ in B by their union $D' = D_0 \cup D_1$. By replacing all (the nonzero number) of such D we obtain this way an irreducible tight partition of Q_{d+1} from each ordered pair of partitions B_1, B_2 we had, and all these new partitions contain the same subcube D'. It is not difficult to see that all the partitions obtained are tight since B_1, B_2 are tight and different. Each partition obtained is also irreducible since no subcube in which the last coordinate is fixed can be spanned by a subfamily, as B_1, B_2 are irreducible. Similarly, if a subcube in which the last coordinate is not fixed is spanned by a subfamily, then the construction ensures that its two subcubes consisting of all vertices with a fixed last coordinate are also spanned by a subfamily of B_1 and a subfamily of B_2. As all the partitions obtained are distinct, this shows that if the number of irreducible tight partitions we get for dimension d in this way is x_d, then $x_{d+1} = x_d(x_d - 1)$, implying the desired lower bound.

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