Stability type results for hereditary properties

Noga Alon [*] Uri Stav [†]

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Abstract

The classical Stability Theorem of Erdős and Simonovits can be stated as follows. For a monotone graph property $\mathcal{P}$, let $t \geq 2$ be such that $t+1 = \min\{\chi(H) : H \notin \mathcal{P}\}$. Then any graph $G^* \in \mathcal{P}$ on n vertices, which was obtained by removing at most $(\frac{1}{t} + o(1))\binom{n}{2}$ edges from the complete graph $G = K_n$, has edit distance $o(n^2)$ to $T_n(t)$, the Turán graph on n vertices with t parts.

In this paper we extend the above notion of stability to hereditary graph properties. It turns out that to do so the complete graph K_n has to be replaced by a random graph. For a hereditary graph property $\mathcal{P}$, consider modifying the edges of a random graph $G = G(n, 1/2)$ to obtain a graph G^* that satisfies $\mathcal{P}$ in (essentially) the most economical way. We obtain necessary and sufficient conditions on $\mathcal{P}$ which guarantee that G^* has a unique structure. In such cases, for a pair of integers (r, s) which depends on $\mathcal{P}$, G^* has distance $o(n^2)$ to a graph $T_n(r, s, \frac{1}{2})$ almost surely. Here $T_n(r, s, \frac{1}{2})$ denotes a graph which consists of almost equal sized $r + s$ parts, r of them induce an independent set, s induce a clique and all the bipartite graphs between parts are quasi-random (with edge density $\frac{1}{2}$). In addition, several strengthened versions of this result are shown.

1 Introduction

1.1 Definitions and background

Given two graphs on n vertices, G and G', the **edit distance** between G and G' is the minimum number of edge additions and/or deletions that are needed in order to turn G into a graph isomorphic to G'. We denote this quantity by $\boldsymbol{\Delta(G, G')}$. A **graph property** is a set of graphs which

[*]Schools of Mathematics and Computer Science, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel. Email: nogaa@tau.ac.il. Research supported in part by the Israel Science Foundation, by the Hermann Minkowski Minerva Center for Geometry at Tel Aviv University and by a USA-Israeli BSF grant.

[†]School of Computer Science, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel. Email: uristav@tau.ac.il.

is closed under isomorphism. For a given graph property $\mathcal{P}$, the edit distance of a graph G from $\mathcal{P}$ is defined by $\boldsymbol{\Delta(G,\mathcal{P})} = \min\{\Delta(G,G') \mid G' \in \mathcal{P}, |V(G')| = |V(G)|\}$. In words, $\Delta(G,\mathcal{P})$ is the minimum edit distance of G to a graph satisfying $\mathcal{P}$.

A **monotone** graph property is closed under removal of edges and vertices. The classical Stability Theorem of Erdős and Simonovits [14] asserts the following. For any monotone graph property $\mathcal{P}$ and $t \geq 2$ such that $t+1 = \chi(\mathcal{P}) = \min\{\chi(H) : H \notin \mathcal{P}\}$, if a graph $G \in \mathcal{P}$ on n vertices contains at least $e(G) \geq (1 - \frac{1}{t} - o(1))\binom{n}{2}$ edges, then $\Delta(G, T_n(t)) = o(n^2)$, where $T_n(t)$ denotes the Turán graph [20] on n vertices with t parts. For instance, if the property $\mathcal{P}$ consists of the graphs excluding a copy of K_{t+1}, then any K_{t+1}-free graph G on n vertices that contains at least $e(G) \geq (1 - \frac{1}{t} - o(1))\binom{n}{2}$ edges, has edit distance at most $o(n^2)$ from $T_n(t)$.

Let us restate the above results as follows: whenever one removes roughly the minimal possible number of edges, namely $(\frac{1}{t} + o(1))\binom{n}{2}$, from the complete graph K_n to obtain a graph that satisfies a monotone property $\mathcal{P}$ with $\chi(\mathcal{P}) = t+1 \geq 3$, then the resulting graph has (essentially) a unique "structure" which is given by $T_n(t)$.

In this paper we consider a broader family of graph properties, namely hereditary properties. A graph property is **hereditary** if it is closed under removal of vertices (and *not* necessarily under removal of edges). Equivalently, hereditary properties are closed under taking induced subgraphs. For instance, for any graph H, the set of graphs excluding an induced copy of H, denoted $\boldsymbol{\mathcal{P}^*_H}$, is a hereditary property.

Several Turán type questions were studied with respect to hereditary graph properties (see e.g. [4], [5], [9], [13], [18], [19]). It was shown that various classical extremal results for monotone graph properties have a natural analogue for some hereditary properties. Hereditary graph properties however force some symmetry between deletion and addition of edges, and in these analogous results the role of the complete graph is taken by some random graph $G(n,p)$. In particular, some special features of $G(n, \frac{1}{2})$ were shown.

In this paper we study an analogous stability-type phenomenon for random graphs and hereditary properties. The basic question we address is the following: given a hereditary graph property $\mathcal{P}$, is there a unique structure for all the graphs in $\mathcal{P}$ that have (essentially) the minimum distance to $G(n, 1/2)$?

However, before we can formulate the meaning of a "unique structure" and the results, the presence of random graphs forces us to introduce yet another metric on graphs, namely the cut norm (defined by Frieze and Kannan [15]). For a set of vertices $A \subseteq V$, we denote by $E(A)$ the set of edges of the graph which is induced by A in G. We also denote by $e(A)$ the size of $E(A)$. For every two nonempty vertex sets A and B of a graph G, $E(A,B)$ stands for the set of edges of G between A and B, namely

$$E(A,B) = \{(x,y) \in E(G) | x \in A, y \in B\},$$

and $e(A,B)$ is the cardinality of this set $\boldsymbol{e(A,B)} = |E(A,B)|$. Note that the edges in $A \cap B$ are counted twice. When several graphs on the same set of vertices are involved, we write $e_G(A,B)$ to specify the graph G to which we refer. The following notations will be useful later: the **edge density** of the pair (A,B) is defined as $\boldsymbol{d(A,B)} = e(A,B)/|A||B|$. The degree of a vertex v, i.e. the number of vertices adjacent to v, is denoted by $\mathbf{deg}(\boldsymbol{v})$. We use the notation $x = y \pm \epsilon$ to denote the fact that $y - \epsilon \le x \le y + \epsilon$.

For a pair of labelled graphs G, G' sharing the same vertex set $\{1, \dots, n\}$, define

$$\boldsymbol{d_\square(G, G')} = \frac{1}{n^2} \max_{S,T \subseteq \{1,\dots,n\}} |e_G(S,T) - e_{G'}(S,T)|$$

For a pair of unlabelled graphs, let the **cut distance** between G and G', denoted $\boldsymbol{\Delta_\square(G, G')}$, be the minimum of $d_\square(G, G')$ over all possible overlays. For example, note that for two random graphs $G = G(n, \frac{1}{2})$ and $G' = G(n, \frac{1}{2})$, with high probability, $\Delta(G, G') = (\frac{1}{2} - o(1))\binom{n}{2}$ while $\Delta_\square(G, G') = o(1)$. Therefore, intuitively, the cut norm enables one to estimate the distance between the structures of the graphs, making random-like portions of the graph (with a common edge density) somewhat indistinguishable.

Denote by $T_n(r, s, \frac{1}{2})$ the graphs on n vertices which consist of almost equal sized $r + s$ parts, where r of them induce an independent set, s induce a clique and all the bipartite graphs between parts are quasi-random[1] with edge density $\frac{1}{2}$. It is important to note that for our purpose here we will only be interested in the asymptotic cut distance between $T_n(r, s, \frac{1}{2})$ and other graphs. To this end, it is straightforward to see that specifying one of the graphs in $T_n(r, s, \frac{1}{2})$ is insignificant, since all such graphs differ, in the cut norm, by $o(1)$.

1.2 The new results

In [5], it is proved that for any hereditary property $\mathcal{P}$, there exists a pair (r, s) such that by turning $G = G(n, \frac{1}{2})$ into a graph $T_n(r, s, \frac{1}{2}) \in \mathcal{P}$ the minimum possible number of modifications needed to attain $\mathcal{P}$ is applied (up to $o(n^2)$). That is, almost surely,

$$\Delta\big(G, T_n(r, s, \frac{1}{2})\big) \le \Delta(G, \mathcal{P}) + o(n^2)\ .$$

In this paper we precisely characterize the properties for which this is essentially the **only** economical way to turn $G(n, \frac{1}{2})$ into a graph that satisfies $\mathcal{P}$. In Section 2 we define, for any pair (r, s) of integers, the subfamily of hereditary properties that are (r, s)-critical. The main result of this paper is stated in the following theorem.

Theorem 1.1. *Let $\mathcal{P}$ be a hereditary graph property and let $G = G(n, \frac{1}{2})$. Then the following holds with high probability:*

(i) If $\mathcal{P}$ is an (r, s)-critical property and $\hat{G} \in \mathcal{P}$ satisfies $\Delta(G, \hat{G}) = \Delta(G, \mathcal{P}) + o(n^2)$, then $\Delta_\square(\hat{G}, T_n(r, s, \frac{1}{2})) = o(1)$.

[1]Without being too formal, we say that a bipartite graph (A, B) on n vertices is quasi-random with edge density $\frac{1}{2}$ if any pair of subsets $A' \subseteq A$ and $B' \subseteq B$ satisfies $|e(A', B') - \frac{1}{2}|A'||B'|| = o(n^2)$.

(ii) *If $\mathcal{P}$ is not (r,s)-critical for any (r,s), then there exist two graphs $G_1, G_2 \in \mathcal{P}$ such that $\Delta(G, G_i) = \Delta(G, \mathcal{P}) + o(n^2)$, $i = 1, 2$, and $\Delta_{\square}(G_1, G_2) = \Omega(1)$.*

In words, by the first part of the theorem, every graph in $\mathcal{P}$, having the minimum possible edit distance to G (up to $o(n^2)$) has the same structure as $T_n(r, s, \frac{1}{2})$. The second part shows that if $\mathcal{P}$ is not critical, then there are two graphs which are far in the cut norm from each other and have essentially the same (minimal) edit distance to G. Hence they do not exhibit a unique structure.

In fact stronger results are shown: the first part of Theorem 1.1 holds even for graphs G that are slightly modified from the random ones. The details are given in the statement of Proposition 4.2 which also implies the first part of Theorem 1.1. In addition, in some cases it is possible to obtain an even more restricted structure for the closest graph in $\mathcal{P}$ to G. We demonstrate this phenomenon in the following theorem on the property $\mathcal{P}^*_{C_4}$, namely being induced C_4-free. As we shall see, this property is $(1,1)$-critical. A similar result holds for any blow-up of a complete graph, see Section 7 for further details.

Theorem 1.2. *Let G^* be the closest graph to $G = G(n, \frac{1}{2})$ in $\mathcal{P}^*_{C_4}$. Then, with high probability, G^* is a split graph (i.e. its vertices can be partitioned into a clique and an independent set).*

1.3 Related work

- Prömel and Steger studied the class of induced H-free graphs. In [17] they showed that almost all induced C_5-free graphs are "generalized split graphs": graphs whose vertex set can be partitioned into two sets, one inducing a clique and the other inducing a disjoint union of cliques, or the complement of these graphs. It implicitly follows from their results that modifying $G(n, \frac{1}{2})$ into any such graph is essentially the most economical way to obtain a graph in $\mathcal{P}^*_{C_5}$. In [18] they estimate the number of induced H-free graphs for an arbitrary H and define critical forbidden induced subgraphs which assure a stable structure of an extremal graph for that problem. We note that our definition of graphs H for which the property $\mathcal{P}^*_H$ is a critical property is in fact more restrictive than their definition. In particular, our examples for such H in Section 2 supply many natural critical forbidden induced subgraphs for the results of [18].

- The following result was proved by Babai, Simonovits and Spencer [10]. Given a random graph $G = G(n, 1/2)$, suppose one removes edges from G in the most economical way to obtain a graph G' that does not contain a (not necessarily induced) copy of a graph H. If $\chi(H) = 3$ and H has a color critical edge, then, with high probability, by removing $O(1)$ edges from G' it is possible to obtain a bipartite graph. Some of our results here for hereditary properties are of a similar nature.

1.4 Organization

In Section 2 we define and discuss critical hereditary properties, namely the ones for which the first part of Theorem 1.1 holds. In Section 3 we introduce several definitions and results which form the basic tools for the proofs. Then, Section 4 consists of the proof of the first part of Theorem 1.1. The proof of the second part appears in Section 5. Section 6 contains the proof of Theorem 1.2. In Section 7 we mention some additional results, concluding remarks and questions for further research.

2 Critical hereditary properties

In this section we characterize the family of properties which satisfy the first part of Theorem 1.1. First we extend the usual notion of graph coloring as follows.

Definition 2.1. *For any pair of integers (r, s), such that $r+s > 0$, we say that a graph $G = (V, E)$ is* ***(r,s)-colorable*** *if there is a partition of V into $r+s$ (possibly empty) subsets $I_1, \dots, I_r, C_1, \dots, C_s$ such that each I_k induces an independent set in G, and each C_k induces a clique in G.*

For example, $(r, 0)$-colorable graphs are r-colorable graphs. We denote by $\boldsymbol{\mathcal{P}_{r,s}}$ the graph property comprising all the (r, s)-colorable graphs. Clearly, for any pair (r, s), the property $\mathcal{P}_{r,s}$ is hereditary. These properties, which were introduced in several contexts, capture important algorithmic and extremal characteristics of hereditary properties. See e.g. Prömel and Steger ([18] and [19]) and Bollobás and Thomason ([13] and [12]).

Definition 2.2. *Let $\mathcal{P}$ be a hereditary property. Define the* ***binary chromatic number*** *of $\mathcal{P}$ as the least integer $k+1$ such that for any (r, s) satisfying $r+s = k+1$ there is a graph not in $\mathcal{P}$ that is (r, s)-colorable, and denote it by $\boldsymbol{\chi_B(\mathcal{P})}$. Equivalently,*

$$\boldsymbol{\chi_B(\mathcal{P})} = 1 + \max\{r + s : \mathcal{P}_{r,s} \subseteq \mathcal{P}\}$$

This definition extends the definition of the binary chromatic number for graphs from [9] and [18] since $\chi_B(\mathcal{P}^*_H) = \chi_B(H)$.

We are now ready to define critical hereditary properties. The formal definition is followed by several natural examples of such properties.

Definition 2.3. *We say that a hereditary graph property $\mathcal{P}$ is* ***(r, s)-critical*** *if*

(i) $\mathcal{P}_{r,s} \subseteq \mathcal{P}$ (i.e. every (r, s)-colorable graph satisfies $\mathcal{P}$)

(ii) For any $r', s' \geq 0$, $(r', s') \neq (r, s)$, if $r' + s' = r + s$ then $\mathcal{P}_{r',s'} \nsubseteq \mathcal{P}$ (i.e. there exists an (r', s')-colorable graph which does not satisfy $\mathcal{P}$)

(iii) There exist graphs $H_1, H_2 \notin \mathcal{P}$ and partitions of their vertices into $r+s+1$ pairwise disjoint sets such that r of them induce independent sets, s induce complete graphs and

(a) In H_1, the extra set induces an edgeless graph which is completely connected to one of the other independent sets.

(b) In H_2, the extra set induces a complete graph which is completely disconnected to one of the other cliques.

Remark 2.4: Practicing the above definition, we note that if $\mathcal{P}$ is (r,s)-critical, then $\chi_B(\mathcal{P}) = r+s+1$ since for any r', s' such that $r'+s' > r+s$ it follows from item (ii) that there is some graph $H \in \mathcal{P}_{r',s'} \setminus \mathcal{P}$.

Example 2.5. *It is easy to verify that for any (r,s) the property $\mathcal{P}_{r,s}$ is (r,s)-critical. Another easy example is that for any monotone property $\mathcal{P}$, if $t+1 = \chi(\mathcal{P}) = \min_{H \notin \mathcal{P}}(\chi(H))$, then $\mathcal{P}$ is $(t,0)$-critical. To see this, assume $H \notin \mathcal{P}$ is some graph such that $\chi(H) = t+1$ and let $V(H) = h$. Then $K_{t+1}(h)$, a complete $(t+1)$-partite graph with h vertices in each part, contains a (not necessarily induced) copy of H and hence does not satisfy $\mathcal{P}$. Therefore, it can play the role of H_1 while $K_h \notin \mathcal{P}$ can play the role of H_2, showing that indeed all the requirements are fulfilled.*

The following theorem provides an infinite non-trivial family of natural critical hereditary properties.

Theorem 2.6. *For any pair (r,s) of integers $(r+s>0)$, let $H = K_{r+1}(s+1)$ be the complete $(r+1)$-partite graph with $(s+1)$ vertices in each part. Then $\mathcal{P}_H^*$ is (r,s)-critical.*

Proof. We first show that any (r,s)-colorable graph G does not contain an induced copy of H. We use a pigeonhole argument: consider some (r,s)-coloring of G, and suppose it contains an induced copy of H. Each of the r independent sets, consists of vertices from a single part of H. Therefore, one of the $r+1$ parts of H has no vertices at all within any independent set. Yet each of that part's $s+1$ vertices has to be in a different clique, which is impossible since there are only s cliques.

This shows that item (i) in Definition 2.3 holds. It is easy to see that if $s' \geq s+1$ or $r' \geq r+1$ then there are (r',s')-colorable graphs which contain an induced copy of H, thus showing that item (ii) also holds. As for item (iii), consider a partition of the vertices of $K_{r+1}(s+1)$ into r independent sets with $s+1$ vertices and $s+1$ sets of size 1. This partition witnesses that $K_{r+1}(s+1)$ can play the role of both H_1 and H_2, as every set of size 1 is completely connected to the independent sets, and completely disconnected to each of the other sets of size 1 (which may be treated as cliques).

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Example 2.7. *For $H = K_2(2) = C_4$, by Theorem 2.6 the property $\mathcal{P}_{C_4}^*$ is $(1,1)$-critical. In particular, if $G = (V,E)$ is $(1,1)$-colorable, then G is also induced C_4-free. $(1,1)$-colorable graphs are sometimes referred to as* ***split graphs****.*

In addition to the critical properties established in Example 2.5 and Theorem 2.6, for any pair of integers such that $p < q-1$, the property $\mathcal{P}^*_{K_{p,q}}$ is $(0, q-1)$-critical. Note that if $p = q$ then $K_{p,q} = K_2(q)$, hence this falls into the cases of Theorem 2.6 and $\mathcal{P}^*_{K_{q,q}}$ is $(1, q-1)$-critical. The case $p = q-1$ is discussed later, see Section 7.

3 Preliminaries

3.1 Partitions and colored regularity graphs

The following definitions suggest a modelling for partitions of graphs which appear to be a useful tool for handling induced subgraphs. The usage of these definitions is to model regular partitions of graphs. This is done via Lemma 3.4 which is proved in [4]. This powerful lemma allows us to omit some of the tedious details that usually accompany the usage of the regularity lemma. However, the following definitions are necessary before we can state this result.

Definition 3.1. *A* ***colored regularity graph*** *K is a complete graph whose vertices are colored black or white, and whose edges are colored black, white or grey.*

Note that neither the vertex nor the edge colorings are assumed to be legal in the standard sense. We denote the sets of black, white and grey edges of K by $EB(K)$,$EW(K)$ and $EG(K)$ respectively. Similarly, we write $VB(K)$ and $VW(K)$ for K's black and white vertices. The definition of colored regularity graphs should be considered with respect to (regular) vertex partitions.

Definition 3.2. *For a graph $G = (V, E)$, and a colored regularity graph K on k vertices $v_1, \ldots, v_k$, we say that* ***G conforms to K*** *if there exists a partition $\mathcal{V} = \{V_1, \ldots, V_k\}$ of V, such that*

1. *For any $i = 1, \ldots, k$, if $v_i \in VW(K)$ is white then V_i is an independent set in G, and otherwise $v_i \in VB(K)$ is black and V_i induces a complete graph in G.*
2. *For any $1 \le i < j \le k$, if $(v_i, v_j) \in EB(K)$ is a black edge then (V_i, V_j) induce a complete bipartite graph in G. If $(v_i, v_j) \in EW(K)$ is white then (V_i, V_j) induce an empty bipartite graph. Otherwise, if $(v_i, v_j) \in EG(K)$ is grey then there is no restriction on the edges of (V_i, V_j).*

In this case, $\mathcal{V}$ is said to ***witness*** *that G conforms to K.*

The connection between colored regularity graphs and induced subgraphs is established by the following definition.

Definition 3.3. *A* ***colored-homomorphism*** *from a (simple) graph H to a colored regularity graph K is a mapping $\varphi : V(H) \mapsto V(K)$, which satisfies the following:*

1. *If $(u,v) \in E(H)$ then either $\varphi(u) = \varphi(v) = t$ and t is colored black, or $\varphi(u) \neq \varphi(v)$ and $(\varphi(u), \varphi(v))$ is colored black or grey.*

2. *If $(u,v) \notin E(H)$ then either $\varphi(u) = \varphi(v) = t$ and t is colored white, or $\varphi(u) \neq \varphi(v)$ and $(\varphi(u), \varphi(v))$ is colored white or grey.*

*If such a homomorphism does not exist we say that K is **H-homomorphism-free**. If the latter holds for each member of a set of graphs $\mathcal{H}$, then we say that K is **$\mathcal{H}$-homomorphism-free**.*

One should observe at this point that for any set of graphs $\mathcal{H}$ and colored regularity graph K, if a graph G conforms to K, and K is $\mathcal{H}$-homomorphism-free, then G is induced $\mathcal{H}$-free. That is, G does not contain an induced copy of any member of $\mathcal{H}$. The following lemma shows that every induced $\mathcal{H}$-free graph is close to conforming to some $\mathcal{H}$-homomorphism-free colored regularity graph. This lemma is implicitly proved in [4], following the method of [2].

Lemma 3.4. **([4])** *For any hereditary property $\mathcal{P}$, integer k_0 and $\gamma > 0$ there exist $T_{3.4}(\gamma, \mathcal{P}, k_0) > 0$ and $n_{3.4}(\gamma, \mathcal{P}, k_0) > 0$ which satisfy the following property. Let $G = (V, E)$ be a graph on $n > n_{3.4}$ vertices which satisfies $\mathcal{P}$, and let $\mathcal{H}$ denote the set of graphs that do not satisfy $\mathcal{P}$ (hence G does not contain an induced copy of any member of $\mathcal{H}$). Then there exists a colored regularity graph K with $k_0 \leq |V(K)| \leq T_{3.4}(\gamma, \mathcal{P}, k_0)$ such that*

1. *There exists a graph G' such that $\Delta(G, G') \leq \gamma n^2$ and G' conforms to K*

2. *K is $\mathcal{H}$-homomorphism free*

The proof of Lemma 3.4 is attained by first applying the strengthened version of the Szemerédi Regularity Lemma of [1] to G, then applying the regularity lemma to each of the clusters of the first partition. An appropriate choice of the parameters, depending on $\mathcal{P}$, together with Ramsey Theorem, then determines the choice of colors for the vertices and edges of K for which the lemma holds. The details of the proof appear as steps 1 to 3 in the proof of Theorem 1.1 in [4].

3.2 The distance of $G(n, \frac{1}{2})$ from hereditary properties

Generalizing a result of Axenovich, Kézdy and Martin [9], it is proved in [5] that the binary chromatic number determines the asymptotic edit distance of $G(n, \frac{1}{2})$ from any hereditary property.

Theorem 3.5. **([5])** *Let $\mathcal{P}$ be an arbitrary hereditary property, then with high probability*

$$\Delta\big(G(n, \frac{1}{2}), \mathcal{P}\big) = \Big(\frac{1}{2(\chi_B(\mathcal{P}) - 1)} \pm o(1)\Big)\binom{n}{2}$$

Let r, s be the pair achieving $\chi_B(\mathcal{P})$, namely $r + s = \chi_B(\mathcal{P})$ and $\mathcal{P}_{r,s} \subseteq \mathcal{P}$. Note that by arbitrarily partitioning the vertices of $G(n, \frac{1}{2})$ into $r + s$ sets of nearly equal sizes, then modifying its edges such that r of the sets induce independent sets and s induce complete subgraphs, one obtains a graph that satisfies $\mathcal{P}$. By Theorem 3.5, with high probability, this way (essentially) the minimum possible number of modifications is applied.

In addition, we shall need the following lemma, stating that in a random graph, with high probability, the edge density between and within any two large enough sets of vertices is close to the edge density of the whole graph. The proof (see e.g. [4]) is a standard application of Chernoff's inequality.

Lemma 3.6. *Assume $0 \leq p \leq 1$, and $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(n) = \omega(n^{1.5})$. Then for a sufficiently large n, with high probability, $G = G(n,p)$ satisfies*

1. *For any set $A \subseteq V(G)$: $\mid e(A) - p\binom{|A|}{2} \mid \; < f(n)$*
2. *For any pair of disjoint sets $A, B \subseteq V(G)$: $\mid e(A,B) - p|A||B| \mid \; < f(n)$.*

4 Stability for critical properties

In this section we prove the first part of Theorem 1.1. However, this will be an easy corollary of yet a stronger result which holds: if $\mathcal{P}$ is critical, then even if one modifies arbitrarily a small number of edges in the random graph $G = G(n, \frac{1}{2})$, then the structure of any graph in $\mathcal{P}$ that is essentially the closest to G is unique. The restriction on the type of modifications for which this result holds is specified in the following definition.

Definition 4.1. *Let $G = (V, E)$ be a graph on $n = |V|$ vertices, and $\lambda > 0$. We say that a graph $G' = (V, E')$ is a* ***λ-modification of G*** *if E' was obtained by applying arbitrary modifications to the edges of G such that for any vertex $v \in V$ the number of modifications applied to edges incident to v is at most λn. Equivalently, if $D = (V, E \triangle E')$, then for any $v \in V$: $d_D(v) \leq \lambda n$.*

For instance, any graph with minimum degree $(1-\lambda)n$ is a λ-modification of the complete graph K_n. We are now ready to state the main technical result of this section.

Proposition 4.2. *Let $\delta > 0$ and $\mathcal{P}$ be an (r,s)-critical hereditary property. Then there is $\lambda = \lambda_{4.2}(\mathcal{P}, \delta) > 0$ which satisfies the following. Let $G = (V, E)$ be a λ-modification of $G_r = G(n, \frac{1}{2})$. Assume $\hat{G} = (V, \hat{E})$ is a graph in $\mathcal{P}$ such that $\Delta(G, \hat{G}) \leq \Delta(G, \mathcal{P}) + o(n^2)$, and let $D = (V, E \triangle \hat{E})$ denote the modifications graph. Then with high probability there is an equipartition $\mathcal{V}$ of V into $r+s$ sets $\{V_1, \ldots, V_{r+s}\}$ such that*

1. *$\mathcal{V}$ witnesses that $\Delta(\hat{G}, \mathcal{P}_{r,s}) \leq \delta n^2$, namely $\sum_{i=1}^{r} e_{\hat{G}}(V_i) + \sum_{i=r+1}^{r+s} (\binom{|V_i|}{2} - e_{\hat{G}}(V_i)) \leq \delta n^2$*
2. *$\sum_{i<j} e_D(V_i, V_j) \leq \delta n^2$.*

Before getting into details, let us briefly overview the proof of the proposition. The proof is implicitly based on Szemerédi's Regularity Lemma. We apply Lemma 3.4 to $\hat{G}$, and obtain a colored regularity graph K. We then analyze K. First, the condition on the distance of $\hat{G}$ from G (and hence also from G_r) guarantees that K contains sufficiently many grey edges. Moreover, since K is H-homomorphism free for any graph $H \notin \mathcal{P}$, then by Definition 2.3 it does not contain a complete subgraph on $r+s+1$ vertices in which all the edges are grey. This then enables us to apply the classical Stability Theorem of Erdős and Simonovits [14] to the (simple) graph of the grey edges in K. We only need the simplest case in which the forbidden graph is a complete graph, namely Theorem 4.3 below. Recall that we denote by $T_n(t)$ the Turán graph on n vertices with t parts.

Theorem 4.3. **(*The Stability Theorem*, [14] see also [11], pp 340)** *For any $\delta > 0$ and $t \geq 2$, there exist $\gamma = \gamma_{4.3}(t,\delta) > 0$ and $n_{4.3}(t,\delta)$, such that any graph G on $n > n_{4.3}(t,\delta)$ vertices that contains at least $e(G) \geq (1 - \frac{1}{t} - \gamma)\binom{n}{2}$ edges, and does not contain a copy of K_{t+1}, has edit distance at most δn^2 from $T_n(t)$: $\Delta(G, T_n(t)) \leq \delta n^2$.*

To proceed with the proof of Proposition 4.2 we then further analyze the structure of K, using the assertions of Definition 2.3. Going back to the original graph $\hat{G}$, which is close to conforming to K, completes the proof of the proposition. The detailed proof follows.

Proof of Proposition 4.2:
Let $\hat{\delta} = \frac{1}{5(r+s)}\delta$ and set $\gamma = \min\{\gamma_{4.3}(r+s, \hat{\delta}^2), \frac{1}{2}\delta\}$ and $\lambda = \gamma/10$. Assume $G_r = G(n, \frac{1}{2})$ satisfies the conditions of Lemma 3.6 and Theorem 3.5, and that G, $\hat{G}$ and D are as described in the statement of the proposition. We apply Lemma 3.4 to $\hat{G}$ with $\mathcal{P}$, $n_{4.3}(r+s, \hat{\delta}^2)$ and $\gamma/20$, and obtain a graph $\hat{G}_c$ and a colored regularity graph K on $k = |V(K)|$ vertices that satisfy the following:

1. K is $\mathcal{H}$-homomorphism free.
2. $\Delta(\hat{G}, \hat{G}_c) < \frac{\gamma}{20}n^2$
3. $\hat{G}_c$ conforms to K.
4. $k \geq n_{4.3}(r+s, \hat{\delta}^2)$

We now analyze the colored regularity graph K. First, only consider the grey edges in K: denote by K_{grey} the (simple) graph on k vertices spanned by the grey edges in K.

Claim 4.4. *The number of grey edges in K is at least $e(K_{grey}) \geq (1 - \frac{1}{r+s} - \gamma)\binom{k}{2}$.*

Proof of Claim 4.4: By Theorem 3.5, for a large enough n, with high probability, $\Delta(G, \hat{G}) \leq \Delta(G, \mathcal{P}) + o(n^2) \leq (\frac{1}{2(r+s)} + \frac{\gamma}{100})\binom{n}{2}$. Thus,

$$\Delta(\hat{G}_c, G) \leq \Delta(G, \hat{G}) + \Delta(\hat{G}_c, \hat{G}) < (\frac{1}{2(r+s)} + \frac{\gamma}{20})\binom{n}{2} + \frac{\gamma}{100}n^2 \quad (1)$$

On the other hand, for *any* mapping of the vertices of G to the vertices of $\hat{G}_c$, we have the following differences. By Lemma 3.6, for any $1 \leq i < j \leq k$, $d_{G_r}(V_i, V_j) = \frac{1}{2} \pm \gamma/10$, and hence

$d_G(V_i,V_j) = \frac{1}{2} \pm (\frac{1}{10}\gamma + \lambda) = \frac{1}{2} \pm \gamma/5$. If (v_i, v_j) is a black edge in K, then since $d_{\hat{G}_c}(V_i,V_j) = 1$ there were at least $(\frac{1}{2} - \gamma/5)\frac{n^2}{k^2}$ edge modifications between G and $\hat{G}_c$. The same holds in case (v_i, v_j) is white. This implies a lower bound on the distance $\Delta(\hat{G}_c, G)$, and by (1) we have

$$\left(\binom{k}{2} - e(K_{grey})\right)(\frac{1}{2} - \frac{\gamma}{5})\frac{n^2}{k^2} \leq (\frac{1}{2(r+s)} + \frac{\gamma}{20})\binom{n}{2} + \frac{\gamma}{100}n^2$$

Multiplying by $\frac{k^2}{n^2}$ and reorganizing we get

$$\binom{k}{2} - e(K_{grey}) \leq \left(\frac{\frac{1}{2}(\frac{1}{r+s} + \frac{\gamma}{5})}{\frac{1}{2} - \gamma/5}\right)\binom{k}{2} < \left[(1 + \frac{3\gamma}{5})(\frac{1}{r+s} + \frac{\gamma}{5})\right]\binom{k}{2} < \left(\frac{1}{r+s} + \gamma\right)\binom{k}{2}$$

thus proving the claim.

■

Note that in case there is a complete graph on $r+s+1$ vertices in K_{grey}, then regardless of the colors of the vertices, by item *(ii)* in the definition of critical properties we get a contradicting colored homomorphism to K. It therefore follows from Theorem 4.3 that

$$\Delta(K_{grey}, T_k(r+s)) \leq \hat{\delta}^2 k^2 \tag{2}$$

Consider a labelling of the vertex set of $T_k(r+s)$ by $V(K)$ which witnesses this proximity, and let $V_1, \ldots, V_{r+s}$ be the equipartition of the vertices of $V(K)$ into $r+s$ sets corresponding to the $r+s$ parts of $T_k(r+s)$. With respect to this labelling, let $\hat{D} = (V(K), E(K_{grey}) \triangle E(T_k(r+s)))$. It follows from (2) that $e(\hat{D}) \leq \hat{\delta}^2 k^2$, hence there are at most $2\hat{\delta}k$ vertices with $\deg_{\hat{D}}(v) \geq \hat{\delta}k$.

We pick some complete graph $\hat{K}$ on $r+s$ vertices which is a subgraph of K_{grey} consisting of exactly one representative vertex from each part, and such that any vertex v in $\hat{K}$ satisfies $\deg_{\hat{D}}(v) < \hat{\delta}k$. It is not difficult to verify that choosing one vertex uniformly at random from each set V_i will result in an appropriate graph with positive probability.

Let $V(\hat{K}) = \{v_1, \ldots, v_{r+s}\}$ and consider $\hat{K}$ as a subgraph of the colored regularity graph K. By item *(ii)* in the definition of critical properties, exactly r of the vertices of $\hat{K}$ are white and the other s are black. Without loss of generality, assume that $v_i \in V_i$ $(1 \leq i \leq r+s)$, and $v_1, \ldots, v_r$ are white vertices in K while $v_{r+1}, \ldots, v_{r+s}$ are colored black.

Let $L \subseteq V(K)$ denote the set of vertices $v \in V(K)$ that satisfy the following condition: suppose $v \in V_i$, then $v \in L$ if the $r+s-1$ edges connecting v to $V(\hat{K}) \setminus \{v_i\}$ are grey. By the construction of $\hat{K}$, any vertex $v_i \in V(\hat{K})$ rules out at most $\deg_{\hat{D}}(v_i) \leq \hat{\delta}k$ vertices in K, hence $|L| \geq (1-(r+s)\hat{\delta})k$.

Consider first $i < r$. If there is a black vertex in $L \cap V_i$, then K contains a graph with $s+1$ black and $r-1$ white vertices, which spans only grey edges. Nevertheless, by item (ii) in the definition of critical properties we obtain a contradiction (since there is a colored homomorphism from some $H \notin \mathcal{P}$ to K). Moreover, for a pair of white vertices in $L \cap V_i$, if the edge between them is either grey or black, then item (iii) in the definition implies a contradicting homomorphism. Hence $L \cap V_i$

spans a graph which consists solely of white edges and vertices. Similarly, for $r+1 \le i \le r+s$, all the edges and vertices spanned by $L \cap V_i$ are black.

The above observations can be restated as follows. By modifying the colors of at most $|V(K) \setminus L| \le (r+s)\hat{\delta}k$ vertices and less than $(r+s)\hat{\delta}k^2$ edges in K it is possible to obtain a graph that has an equipartition into $r+s$ sets such that r of them are totally white, s are totally black, and all the edges between sets are grey.

However, going back to the original graph $\hat{G}_c$, this partition of K induces an equipartition $\mathcal{V}$ of the vertices in $\hat{G}_c$. Modifying the color of each edge or vertex in K is translated to at most $\frac{n^2}{k^2}$ edges modifications in $\hat{G}_c$ to obtain a graph $G_{r,s}$ that conforms to the modified K. Altogether, by applying at most $2(r+s)\hat{\delta}k^2\frac{n^2}{k^2} = 2(r+s)\hat{\delta}n^2$ edge modifications in $\hat{G}_c$, we obtain an (r,s)-colorable graph $G_{r,s}$ in which the color classes are given by $\mathcal{V}$. Thus the distance $\Delta(\hat{G}, G_{r,s})$ is less than $\frac{\gamma}{20}n^2 + 2(r+s)\hat{\delta}n^2 < \frac{1}{2}\delta n^2$ and the first condition of the proposition indeed holds.

Lemma 3.6, together with the conditions of the proposition, imply that for all $1 \le i \le k$: $d_G(V_i) = \frac{1}{2} \pm (\gamma/10 + \lambda) = \frac{1}{2} \pm \gamma/5$. Therefore $\sum_i e_D(V_i) \ge (\frac{1}{2} - \gamma/5)\frac{1}{r+s}\binom{n}{2} - \frac{1}{2}\delta n^2$. On the other hand, $\sum_i e_D(V_i) + \sum_{i<j} e_D(V_i, V_j) = \Delta(G, \hat{G}) \le (\frac{1}{2(r+s)} + \frac{\gamma}{20})\binom{n}{2}$. Altogether this yields that $\sum_{i<j} e_D(V_i, V_j) \le \gamma\binom{n}{2} + \frac{1}{2}\delta n^2 < \delta n^2$ completing the proof of Proposition 4.2.

■

Proof of Theorem 1.1, Part I:
For any $\delta > 0$ applying Proposition 4.2 to G and $\hat{G}$ we obtain the partition $\mathcal{V}$ which witnesses that indeed with high probability $\Delta_\square(\hat{G}, T_n(r, s, \frac{1}{2})) \le \delta$, thus completing the proof.

■

5 Necessary conditions for stability

The detailed definition of critical hereditary properties is somewhat unnatural. In this section we prove the second part of Theorem 1.1, thus showing that indeed all the details are necessary. The basic method is showing that for a specific pair (r,s), if any of the conditions for (r,s)-criticality is not fulfilled, then it is possible to obtain two graphs in $\mathcal{P}$ which are: (i) far in the cut norm from each other; and (ii) have essentially the same asymptotically optimal distance to G.

Proof of Theorem 1.1, Part II: Let $\mathcal{P}$ be an arbitrary hereditary property and $G = G(n, \frac{1}{2})$. Fix some pair (r,s) such that $\chi_B(\mathcal{P}) = r+s+1$ and $\mathcal{P}_{r,s} \subseteq \mathcal{P}$. Since $\mathcal{P}$ is not (r,s)-critical then either item (ii) or item (iii) in the definition of critical properties are not valid.

Assume first that item (ii) is not valid. Namely, there are $(r', s') \ne (r,s)$ such that $r'+s' = r+s$ and $\mathcal{P}_{r',s'} \subseteq \mathcal{P}$. In this case, by Theorem 3.5, with high probability $\Delta(G, \mathcal{P}) = \left(\frac{1}{2(r+s)} \pm o(1)\right)\binom{n}{2}$. Therefore, consider turning G into a graph G_1 in $\mathcal{P}$ by arbitrarily partitioning the vertices into $r+s$ equal sized sets, then removing all the edges inside r of them and adding all the missing edges inside the other s. We thus turned G into $G_1 \in \mathcal{P}$, since $G_1 \in \mathcal{P}_{r,s}$. Moreover, by Lemma 3.6, with high probability, we modified $(\frac{1}{2(r+s)} \pm o(1))\binom{n}{2}$ edges. Yet, similarly, we could turn G into an (r', s')-colorable graph G_2 by applying essentially the same number of modifications. Hence indeed $\Delta(G, G_1) \le \Delta(G, \mathcal{P}) + o(n^2)$ and $\Delta(G, G_2) \le \Delta(G, \mathcal{P}) + o(n^2)$. We will now show that the cut distance between these two graphs G_1 and G_2 is bounded from below by a constant independent of n (we make no attempt to optimize the constants).

Indeed, assume w.l.o.g. that $r' > r$ and $s > s'$. Note that the number of edges in G_1 is

$$e_{G_1}(V) = \left(\frac{1}{2}(1 - \frac{1}{r+s}) + \frac{s}{r+s} \cdot \frac{1}{r+s} \pm o(1)\right)\binom{n}{2}$$

whereas, for G_2, the number of edges is

$$e_{G_2}(V) = \left(\frac{1}{2}(1 - \frac{1}{r+s}) + \frac{s'}{r+s} \cdot \frac{1}{r+s} \pm o(1)\right)\binom{n}{2}.$$

Since we assumed that $s > s'$ we get that $|e_{G_1}(V) - e_{G_2}(V)| \ge (\frac{s-s'}{r+s} - o(1))\binom{n}{2}$. Therefore, taking $S = T = V$ in the definition of the cut distance we get that

$$\begin{aligned}
\Delta_{\square}(G_1, G_2) &\ge \frac{1}{n^2}\Big|e_{G_1}(V, V) - e_{G_2}(V, V)\Big| \\
&\ge \frac{2}{n^2}\Big|e_{G_1}(V) - e_{G_2}(V)\Big| \\
&\ge \frac{2}{n^2}\left(\frac{s-s'}{r+s} - o(1)\right)\binom{n}{2} \\
&\ge \frac{s-s'}{r+s} - o(1)
\end{aligned}$$

which concludes the proof of this case.

Suppose now that the first assertion of item (iii) fails. In this case, any graph that can be $r+s+1$ colored, with $r+1$ independent sets and s cliques, such that some pair of independent sets is completely connected, belongs to $\mathcal{P}$. Hence, we can again construct an (r,s)-colorable graph G_1 as before, and a graph G_2 as follows. We partition the vertices of G into $r+s$ equal sized sets, and then modify its edges such that $r-1$ of the sets induce independent sets and s induce cliques. The remaining set is arbitrarily partitioned into two equal sized sets and its edges are modified to span a complete bipartite graph. As before, it is not difficult to verify that G_1 and G_2 have essentially the same asymptotically optimal edit distance from G. In addition, $\Delta_{\square}(G_1, G_2)$ can be lower bounded by a constant which only depends on r and s, since again the edge density of G_1 and G_2 essentially different. The other option for the invalidity of item (iii) is treated similarly, this time turning the $(r+s)$'th vertex set into a disjoint union of two equal sized cliques.

■

6 Proof of Theorem 1.2

Let D denote the modifications graph $D = (V(G), E(G)\triangle E(G^*))$. Recall that by Theorem 2.6, $\mathcal{P}^*_{C_4}$ is $(1,1)$-critical. Hence, by Proposition 4.2, for an arbitrarily small $0 < \delta < \frac{1}{10^6}$ with high probability there is a partition $\mathcal{V} = (V_1, V_2)$ of $V(G^*)$ such that

(i) $|V_1|, |V_2| = \frac{n}{2} \pm 1$

(ii) $e_{G^*}(V_1) + \binom{|V_2|}{2} - e_{G^*}(V_2) \leq \frac{1}{20}\delta^2 n^2$

(iii) $e_D(V_1, V_2) \leq \frac{1}{20}\delta^2 n^2$.

With respect to some partition of $V(G^*)$ into two sets $\mathcal{U} = (U_1, U_2)$, we say that a pair of vertices in U_1 is *violating* if they are connected, and a pair of vertices in U_2 is violating if they are not connected. In other words, the violating pairs correspond to the edge modifications that are needed in order to turn G^* into a split graph which is witnessed by the given partition $\mathcal{U}$. We denote by $f(\mathcal{U})$ the number of violating vertex pairs for $\mathcal{U}$: $f(\mathcal{U}) = e_{G^*}(U_1) + \binom{|U_2|}{2} - e_{G^*}(U_2)$. Note that by item (ii) above $f(\mathcal{V}) \leq \frac{1}{20}\delta^2 n^2$.

We change the partition $\mathcal{V}$ as follows: as long as there is some vertex x such that shifting x to the other set decreases the value of f on the current partition, shift it. Note that we only change the partition, without modifying any of the edged of G^*. Since f is non-negative, this process clearly ends. Denote the partition that is obtained at the end of this process, and hence (locally) minimizes f, by $\mathcal{V}' = (V_1', V_2')$.

Claim 6.1. *The partition $\mathcal{V}'$ satisfies that $|V_1'|, |V_2'| = \frac{n}{2} \pm \delta n$ and $e_D(V_1', V_2') \leq \delta n^2$.*

Proof. Denote the vertices that were shifted from V_1 to V_2' by $A = V_2' \cap V_1$ and assume towards a contradiction that $|A| \geq \delta n$. In this case, note that since $A \subseteq V_1$ then $e(A) \leq f(\mathcal{V}) \leq \frac{1}{20}\delta^2 n^2$, and since $A \subseteq V_2'$ then $\binom{|A|}{2} - e(A) \leq f(\mathcal{V}') \leq f(\mathcal{V}) \leq \frac{1}{20}\delta^2 n^2$. Summing the two inequalities we get $\binom{|A|}{2} \leq \frac{1}{10}\delta^2 n^2$, and hence $|A| \leq \frac{1}{2}\delta n$. The same holds for $B = V_1' \cap V_2$, and altogether indeed $|V_1'|, |V_2'| = \frac{n}{2} \pm \frac{1}{2}\delta n = \frac{n}{2} \pm \delta n$. This also implies that

$$e_D(V_1', V_2') \leq e_D(V_1, V_2) + e_D(A, V_1) + e_D(B, V_2) \leq \frac{1}{20}\delta^2 n^2 + |A||V_1'| + |B||V_2'| < \delta n^2$$

proving the claim.

■

Fix $\varepsilon = \frac{1}{100}$ $(< \frac{1}{64})$. With respect to the partition $\mathcal{V}' = (V_1', V_2')$, call a vertex $x \in V_1'$ *good* if there were less than εn modifications in edges connecting x to V_2': $e_D(\{x\}, V_2') < \varepsilon n$. Otherwise, we say x is *bad*. We also define analogously good and bad vertices in V_2'. By Claim 6.1, both in V_1' and in V_2', there are at most $\frac{\delta}{\varepsilon} n = 100\delta n$ bad vertices.

Claim 6.2. *In G^*, the good vertices in V_1' induce an independent set, and the good vertices in V_2' induce a clique.*

Proof. Assume first that a_1, a_2 are good vertices in V_1'. We randomly pick (independently, with uniform distribution) two vertices b_1, b_2 in V_2'. Note that with probability at least $(1 - 200\delta)$ both b_1 and b_2 are good. In this case they are connected in G^* and by our definition of good vertices, the probability that any of the four edges connecting $\{a_1, a_2\}$ and $\{b_1, b_2\}$ was modified is at most 4ε. Moreover, since $G = G(n, \frac{1}{2})$, the probability that the bipartite graph induced by $\{a_1, a_2\}$ and $\{b_1, b_2\}$ in G is a perfect matching is $\frac{1}{8}$. Therefore, in G^*, with probability at least $\frac{1}{8} - \frac{4}{100} - 200\delta > 0$, the edge b_1b_2 exists, and the graph between $\{a_1, a_2\}$ and $\{b_1, b_2\}$ induce a matching. In this case, if there was an edge a_1a_2 in G^*, an induced copy of C_4 would exist in G^* which leads to a contradiction. We conclude that there are no edges in G^* between good vertices in V_1'.

Similarly, for a pair of good vertices $b_1, b_2 \in V_2'$, if b_1b_2 is not an edge, then a random choice of a pair $a_1, a_2 \in V_1'$ yields an induced copy of C_4 in G^* with positive probability. This time we need $4\varepsilon < \frac{1}{16}$, since we want a complete bipartite graph on $\{a_1, a_2\}$ and $\{b_1, b_2\}$ in G. ■

In words, Claim 6.2 implies that if some pair is violating $\mathcal{V}'$ then at least one of its endpoints is bad.

Claim 6.3. *Any bad vertex $a \in V_1'$ has at most $\frac{1}{4}\varepsilon n$ good neighbors in V_1' and any bad vertex $b \in V_2'$ has at most $\frac{1}{4}\varepsilon n$ good non-neighbors in V_2'.*

Proof. Let a be some bad vertex in V_1'. Denote by N_1 the set of good neighbors of a in V_1', and assume towards a contradiction that $|N_1| \geq \frac{1}{4}\varepsilon n$. Let $\overline{N}_2$ denote the set of good vertices in V_2' that are **not** connected to a. Since the partition $\mathcal{V}'$ minimizes f, it follows that $|N_1| \leq |\overline{N}_2|$, since otherwise shifting a from V_1' to V_2' would decrease the value of $f(\mathcal{V}')$.

If a vertex $b \in \overline{N}_2$ is connected to some two vertices $a_1, a_2 \in N_1$, then $\{a, a_1, b, a_2\}$ induce a C_4 in G^* (recall that a_1 and a_2 are good, and therefore are not connected in G^*). Hence, any vertex in $\overline{N}_2$ is connected to at most one vertex in N_1. Therefore, $e_{G^*}(N_1, \overline{N}_2) \leq |\overline{N}_2| < n$. On the other hand, since $\frac{1}{4}\varepsilon n \leq |N_1| \leq |\overline{N}_2|$, Lemma 3.6 implies that with high probability $e_G(N_1, \overline{N}_2) = (\frac{1}{2} \pm o(1))|N_1||\overline{N}_2| \geq (\frac{1}{32}\varepsilon^2 - o(1))n^2$. Altogether

$$e_D(V_1', V_2') \geq e_D(N_1, \overline{N}_2) \geq e_G(N_1, \overline{N}_2) - e_{G^*}(N_1, \overline{N}_2) \geq (\frac{1}{32}\varepsilon^2 - o(1))n^2 > \delta n^2$$

contradicting the corollary of Claim 6.1.

A similar argument shows that any bad vertex $b \in V_2'$ has at most εn good non-neighbors in V_2'. In this case we define N_1 and $\overline{N}_2$ as before, yet now $\frac{1}{4}\varepsilon n \leq |\overline{N}_2| \leq |N_1|$. The rest of the proof repeats the above discussion. ■

Claim 6.4. *Each bad vertex is incident to less than $\frac{1}{3}\varepsilon n$ violating edges with respect to $\mathcal{V}'$ in G^*.*

Proof. By Claim 6.3 any bad vertex has at most $\frac{1}{4}\epsilon n$ violating edges to good vertices in its part. Moreover, there are at most $100\delta n < \frac{1}{100}\varepsilon n$ bad vertices in each part. Hence, altogether, the maximum possible number of violating edges is less than $\frac{1}{3}\varepsilon n$. ■

Claim 6.5. *There are no bad vertices in $V(G^*)$.*

Proof. Suppose there are some bad vertices. In this case, we may obtain a split graph $G_{1,1}$ from G as follows: for any pair of good vertices, we modify it as for G^*. However, for any bad vertex - we modify all its violating edges with respect to $\mathcal{V}'$ (to other vertices in its part of the partition $\mathcal{V}'$), but we do not apply any modifications to edges between the two parts of the partition. This construction, together with Claim 6.2, imply that $G_{1,1}$ is a split graph, which in addition is induced C_4-free. Nevertheless, by Claim 6.4 and the definition of bad vertices, we reduce the number of modifications: $\Delta(G, G_{1,1}) < \Delta(G, G^*)$ which contradicts the fact that G^* is the closest graph in $\mathcal{P}^*_{C_4}$ to G.

■

Hence, all the vertices in G^* are good and by Claim 6.2 the partition $\mathcal{V}'$ witnesses that indeed G^* is a split graph, completing the proof of Theorem 1.2.

■

7 Concluding remarks and future work

- **Types of instability:** It may be interesting to investigate what happens when the property is not critical. As a toy example, consider the property containing all edgeless and complete graphs. Clearly this property is hereditary, however it is neither $(0,1)$ nor $(1,0)$ critical since the second item in the definition is not satisfied. Clearly, in this case, turning $G = G(n, 1/2)$ into a graph in $\mathcal{P}$ by either removing all edges or adding all missing edges requires essentially the same number of modifications. In this case, $\hat{G}$ has one of two possible structures.

  However, in some cases even more options exist. Consider the graph property consisting of all disjoint unions of complete graphs. Equivalently, this is the class $\mathcal{P}^*_{K_{1,2}}$. In this case, for any $1 \leq k \leq n$, arbitrarily partitioning the vertices of the graph $G = G(n, 1/2)$ into k sets, and modifying its edges so that each vertex set induces a complete graph and is isolated from the other sets results in a graph that satisfies the property. Moreover, with high probability, for each of those partitions, roughly $(\frac{1}{2} \pm o(1))\binom{n}{2}$ edge modifications are needed. Since this is also the distance $\Delta(G, \mathcal{P}^*_{K_{1,2}})$, here the number of possible structures grows together with the size of the graph. A similar phenomena holds, e.g., for $\mathcal{P}^*_{C_5}$ (see [17]) and for the property $\mathcal{P}^*_{K_{s,s+1}}$ for any $s \geq 1$.

- **Generalizing Theorem 1.2:** It is in fact possible to further generalize Theorem 1.2. The next statement extends Theorem 1.2 by addressing a broad family of forbidden induced subgraphs and, in addition, by allowing the starting point to be a λ-modification of the random graph (for a sufficiently small λ).

Theorem 7.1. *For any pair of integers (r, s), there is a positive $\lambda = \lambda_{7.1}(r, s)$ which satisfies the following. Let G^* be the closest graph in $\mathcal{P}^*_{K_r(s)}$ to G, where G is a λ-modification of $G_r = G(n, \frac{1}{2})$. Then, with high probability, G^* is (r, s)-colorable.*

Most of the details of the proof are a straightforward generalization of the proof of Theorem 1.2. However, for proving an analogue of Claim 6.3 some properties of induced subgraphs in a random graph are needed. We omit the details.

- **Quasi-random graphs:** It can be shown that Theorems 1.1 and 1.2, which apply to the random graph $G(n, \frac{1}{2})$, in fact hold for any quasi-random graph with edge density $\frac{1}{2}$. Lemma 3.6 provides a sufficient condition for quasi-randomness (see e.g. the survey [16] for other equivalent conditions). Note that for the proof of Theorem 1.2 some additional well known facts on the density of subgraphs in quasi-random graphs are needed.

- **Hardness of edge modification problems:** Combining the previous items, Theorem 7.1 can be proved also for certain pseudo-random graphs. In a subsequent work [6] we describe how this extremal result may be used to show that it is NP-hard to approximate the edit distance $\Delta(G, \mathcal{P}^*_{K_r(s)})$ of a graph G on n vertices within an additive error of $n^{2-\eta}$ for any positive η and pair (r, s) (s.t. $r + s > 2$). The basic idea follows the method applied by Alon, Shapira and Sudakov in [3] to prove a similar hardness result for (almost all) monotone properties. They base the proof on another, somewhat similar, extremal result which holds for monotone properties [7] and extends an earlier result of Andrásfai, Erdős and V. Sós [8]. Nevertheless, similar hardness results for various hereditary properties can be proved by some alternative methods as well.

- **Random graphs with different edge densities:** It should be interesting to investigate the structure of the closest graphs in a hereditary $\mathcal{P}$ to $G(n, p)$ for $p \neq \frac{1}{2}$. One such result implicitly follows, as a byproduct, from [5] where it is shown that $\Delta(G(n, \frac{1}{3}), \mathcal{P}^*_{K_{1,3}}) = (\frac{1}{3} - o(1))\binom{n}{2}$ (in this case $G(n, \frac{1}{3})$ is essentially the furthest graph from the property).

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