The Grothendieck constant of random and pseudo-random graphs

To the memory of George Dantzig

Noga Alon *   Eli Berger †

Abstract

The *Grothendieck constant* of a graph $G = (V, E)$ is the least constant K such that for every matrix $A : V \times V \to R$:

$$\max_{f:V \to S^{|V|-1}} \sum_{\{u,v\} \in E} A(u, v) \cdot \langle f(u), f(v) \rangle \leq K \max_{\epsilon:V \to \{-1,+1\}} \sum_{\{u,v\} \in E} A(u, v) \cdot \epsilon(u)\epsilon(v).$$

The investigation of this parameter, introduced in [2], is motivated by the algorithmic problem of maximizing the quadratic form $\sum_{\{u,v\} \in E} A(u, v)\epsilon(u)\epsilon(v)$ over all $\epsilon : V \to \{-1, 1\}$, which arises in the study of correlation clustering and in the investigation of the spin glass model. In the present note we show that for the random graph $G(n, p)$ the value of this parameter is, almost surely, $\Theta(\log(np))$. This settles a problem raised in [2]. We also obtain a similar estimate for regular graphs in which the absolute value of each nontrivial eigenvalue is small.

1 Introduction

The *Grothendieck constant* of a graph $G = (V, E)$, denoted by $K(G)$, is the least constant K such that for every matrix $A : V \times V \to R$:

$$\max_{f:V \to S^{|V|-1}} \sum_{\{u,v\} \in E} A(u, v) \cdot \langle f(u), f(v) \rangle \leq K \max_{\epsilon:V \to \{-1,+1\}} \sum_{\{u,v\} \in E} A(u, v) \cdot \epsilon(u)\epsilon(v).$$

This notion was introduced and investigated in [2]. The motivation, besides an interesting connection to a classical inequality of Grothendieck proved in [6], is mainly algorithmic. For various algorithmic applications we are interested in solving an integer program of the form

$$\max_{\epsilon:V \to \{-1,+1\}} \sum_{\{u,v\} \in E} A(u, v) \cdot \epsilon(u)\epsilon(v), \tag{1}$$

*Schools of Mathematics and Computer Science, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel. Email: nogaa@tau.ac.il. Research supported in part by the Israel Science Foundation, by a USA-Israel BSF grant, and by the Hermann Minkowski Minerva Center for Geometry at Tel Aviv University.

†Department of Mathematics, University of Haifa, Haifa 31905, Israel. Email:berger@math.haifa.ac.il.

for a given input matrix A that assigns real values to each edge of G. See [2] and its references for various specific applications, such as correlation clustering and the estimation of the energy of a ground state in the spin glass model, in which the value of such a maximum arises.

One natural way to get an approximation of this maximum, is to consider the natural semidefinite relaxation of the problem, which is:

$$\max_{f:V\rightarrow S^{|V|-1}} \sum_{\{u,v\}\in E} A(u,v)\cdot\langle f(u)f(v)\rangle. \tag{2}$$

This relaxation can be solved efficiently, using the methods of [7], up to any desired accuracy. It is obvious that the value of (2) is larger or equal than that of (1). The Grothendieck constant of the corresponding graph is thus the integrality gap of (1), measuring the maximum possible ratio between the value of (1) and that of its semidefinite relaxation (2), where the maximum is taken over all real matrices A.

It is proved in [2] that for every graph G, $\Omega(\log w(G)) \leq K(G) \leq O(\log\theta(\overline{G}))$, where $w(G)$ is the clique number of G, and $\theta(\overline{G})$ is the Lovász θ-function of the complement of G.

One of the open problems mentioned in [2] is to decide whether for the random graph $G(n, 1/2)$, almost surely, $K(G) = \Theta(\log n)$. Here we show that this is indeed the case by proving the following more general result.

Proposition 1.1 *There are two absolute positive constants c_1, c_2 such that almost surely, that is, with probability that tends to 1 as n tends to infinity, the Grothendieck constant of the random graph $G = G(n,p)$, where $p = p(n) \leq 1$ is such that $np > 1$, satisfies*

$$c_1 \log(np) \leq K(G) \leq c_2 \log(np).$$

A similar argument applies to (n, d, λ)-graphs. An (n, d, λ)-graph is a d-regular graph on n vertices, so that each eigenvalue of its adjacency matrix, besides the first one, is bounded, in absolute value, by λ.

Proposition 1.2 *There is an absolute positive constant c_3 such that the Grothendieck constant of any (n, d, λ)-graph G with $d/\lambda > 1$ satisfies*

$$K(G) \geq c_3 \log(d/\lambda).$$

As it is known (see [5]) that a random d-regular graph is, almost surely, an (n, d, λ)-graph for $\lambda = (2 + o(1))\sqrt{d-1}$ it follows that the Grothendieck constant of a random d-regular graph is, almost surely, $\Theta(\log d)$.

The above results will be deduced from the following general theorem. In its statement, we assume that n is divisible by m, but this is not crucial, and is assumed here only in order to simplify the notation and avoid non-essential floor and ceiling signs.

Theorem 1.3 *There is an absolute positive constant c such that the following holds. Let $G = (V, E)$ be a graph on n vertices. Suppose that there are integers m and s, such that $n = ms$, and there are positive reals $\gamma < p/2 < p \leq 1$ so that for every two disjoint sets of vertices $X, Y \subset V$, each of size at most s, the number of edges $e(X, Y)$ between X and Y deviates from $p|X||Y|$ by at most γs^2. Suppose, further, that $m \leq p/\gamma$. Then $K(G) \geq c \log m$.*

2 Proofs

Let $G = (V, E)$, n, m, s, p and γ satisfy the assumptions of Theorem 1.3. Fix an m by m symmetric matrix $A = A(i, j)$ and unit vectors $x_1, x_2, \ldots, x_m$ in S^{m-1} such that

$$\Delta \geq c_5 \log m \cdot \delta, \tag{3}$$

where c_5 is an absolute positive constant,

$$\Delta = \Delta(A) = \max_{y_1, y_2 \ldots, y_m \in S^{m-1}} \sum_{1 \leq i \neq j \leq m} A(i, j) \langle y_i, y_j \rangle = \sum_{1 \leq i \neq j \leq m} A(i, j) \langle x_i, x_j \rangle$$

and

$$\delta = \delta(A) = \max_{\epsilon_1, \epsilon_2, \ldots, \epsilon_m \in \{-1, 1\}} \sum_{1 \leq i \neq j \leq m} A(i, j) \epsilon_i \epsilon_j.$$

The existence of such an A follows from the result of [2] that $K(G) \geq \Omega(\log m)$.

By a simple lemma, proved in [4],

$$\delta(A) \cdot m \geq \sum_{1 \leq i \neq j \leq m} |A(i, j)|. \tag{4}$$

Let $V = V_1 \cup V_2 \cdots \cup V_m$ be an arbitrary partition of V into m pairwise disjoint sets, each of size s. Define a real matrix $B = B(u, v)_{u, v \in V}$ as follows. For each two vertices u, v that lie in distinct sets, $u \in V_i, v \in V_j, i \neq j$, put $B(u, v) = \frac{A(i,j)}{e(V_i, V_j)}$. In any other case, $B(u, v) = 0$. Define

$$\Delta(B) = \max_{f : V \to S^{|V|-1}} \sum_{u, v} B(u, v) \langle f(u), f(v) \rangle,$$

and

$$\delta(B) = \max_{\epsilon : V \to \{-1, 1\}} \sum_{u, v} B(u, v) \epsilon(u) \epsilon(v).$$

Our objective is to show that $\Delta \geq \Omega(\log m) \cdot \delta$. Note, first that

$$\Delta(B) \geq \Delta(A). \tag{5}$$

Indeed, this simply follows by defining $f(u) = x_i$ for each $u \in V_i$ and by noticing that with this choice

$$\sum_{u, v} B(u, v) \langle f(u), f(v) \rangle = \sum_{1 \leq i \neq j \leq m} \sum_{u \in V_i, v \in V_j} B(u, v) \langle f(u), f(v) \rangle$$

$$= \sum_{1\le i\neq j\le m} \frac{A(i,j)}{e(V_i,V_j)} e(V_i,V_j)\langle x_i, x_j\rangle = \Delta(A).$$

To bound $\delta(B)$ consider an arbitrary function $\epsilon : V \to \{-1,1\}$. For each fixed i, $1 \le i \le m$, express the vector $\epsilon_i = \epsilon(v)_{v\in V_i}$ as a sum of a constant vector and a vector with sum of coordinates zero. If $X_i = \{v \in V_i : \epsilon(v) = 1\}$, $Y_i = \{v \in V_i : \epsilon(v) = -1\}$, and $\mathbf{1}_Z$ denotes the characteristic vector of a set $Z \subset V_i$, then

$$\epsilon_i = \frac{|X_i| - |Y_i|}{s}\mathbf{1}_{V_i} + \frac{2|Y_i|}{s}\mathbf{1}_{X_i} - \frac{2|X_i|}{s}\mathbf{1}_{Y_i}.$$

For each $1 \le i \neq j \le m$, let B_{ij} denote the s by s submatrix $(B(u,v)_{u\in V_i, v\in V_j}$. Then.

$$\sum_{u,v} B(u,v)\epsilon(u)\epsilon(v) = \sum_{i\neq j} \epsilon_i^t B_{ij}\epsilon_j,$$

where the vectors ϵ_i are considered as column vectors.

Fix admissible $i \neq j$, and define μ_i by $|X_i| = \mu_i s$. Then $|Y_i| = (1-\mu_i)s$ and

$$\epsilon_i = (2\mu_i - 1)\mathbf{1}_{V_i} + 2(1-\mu_i)\mathbf{1}_{X_i} - 2\mu_i\mathbf{1}_{Y_i}.$$

Therefore

$$\epsilon_i^t B_{ij}\epsilon_i = (2\mu_i-1)(2\mu_j-1)\mathbf{1}_{V_i}^t B_{ij}\mathbf{1}_{V_j} + (2\mu_i-1)2(1-\mu_j)\mathbf{1}_{V_i}^t B_{ij}\mathbf{1}_{X_j} - (2\mu_i-1)2\mu_j\mathbf{1}_{V_i}^t B_{ij}\mathbf{1}_{Y_j}$$
$$+2(1-\mu_i)(2\mu_j-1)\mathbf{1}_{X_i}^t B_{ij}\mathbf{1}_{V_j} + 2(1-\mu_i)2(1-\mu_j)\mathbf{1}_{X_i}^t B_{ij}\mathbf{1}_{X_j} - 2(1-\mu_i)2\mu_j\mathbf{1}_{X_i}^t B_{ij}\mathbf{1}_{Y_j}$$
$$-2\mu_i(2\mu_j-1)\mathbf{1}_{Y_i}^t B_{ij}\mathbf{1}_{V_j} - 2\mu_i 2(1-\mu_j)\mathbf{1}_{Y_i}^t B_{ij}\mathbf{1}_{X_j} + 2\mu_i 2\mu_j\mathbf{1}_{Y_i}^t B_{ij}\mathbf{1}_{Y_j}.$$

The first term among these nine, summed over all pairs $i \neq j$, gives

$$\sum_{i\neq j}(2\mu_i-1)(2\mu_j-1)\frac{A(i,j)}{e(V_i,V_j)}e(V_i,V_j) = \sum_{i\neq j} A(i,j)(2\mu_i-1)(2\mu_j-1) \le \delta,$$

where the last inequality follows from the fact that for every i , $-1 \le 2\mu_i - 1 \le 1$, and the fact that as the function $\sum_{i\neq j} A(i,j)\nu_i\nu_j$ is linear in every ν_j, it attains its maximum over $[-1,1]^m$ at a vertex.

Each other term among the remaining eight ones can be bounded using the fact that the number of edges between any two subsets X, Y of V, of size at most s each, deviates from its expectation $p|X||Y|$ by at most γs^2. Therefore, for example,

$$(2\mu_i-1)2(1-\mu_j)\mathbf{1}_{V_i}^t B_{ij}\mathbf{1}_{X_j} = (2\mu_i-1)2(1-\mu_j)\frac{A(i,j)}{e(V_i,V_j)}e(V_i,X_j)$$

$$\le (2\mu_i-1)2(1-\mu_j)\frac{A(i,j)}{e(V_i,V_j)}p\cdot s\mu_j\cdot s + (2\mu_i-1)2(1-\mu_j)\frac{|A(i,j)|}{e(V_i,V_j)}\gamma s^2$$

$$\le p\frac{A(i,j)s^2}{e(V_i,V_j)}(2\mu_i-1)2(1-\mu_j)\mu_j + O(|A(i,j)|\frac{\gamma}{p}),$$

where in the last inequality we used the fact that $e(V_i, V_j) \geq (p-\gamma)s^2 \geq \frac{p}{2}s^2$ and that the absolute value of each of the terms $(2\mu_i - 1), 2(1-\mu_j)$ is at most 2.

Each of the other terms among the eight terms above can be bounded in the same manner, expressing each of them as a sum of two summands, the main term (like the term $p\frac{A(i,j)s^2}{e(V_i,V_j)}(2\mu_i - 1)2(1-\mu_j)\mu_j$ above), and the error term (like the $O(|A(i,j)|\frac{\gamma}{p})$ above).

The sum of all those eight main terms is

$$p\frac{A(i,j)s^2}{e(V_i,V_j)}[\ (2\mu_i-1)2(1-\mu_j)\mu_j - (2\mu_i-1)2\mu_j(1-\mu_j)$$

$$+2(1-\mu_i)(2\mu_j-1)\mu_i + 2(1-\mu_i)2(1-\mu_j)\mu_i\mu_j - 2(1-\mu_i)2\mu_j\mu_i(1-\mu_j)$$

$$-2\mu_i(2\mu_j-1)(1-\mu_i) - 2\mu_i 2(1-\mu_j)(1-\mu_i)\mu_j + 2\mu_i 2\mu_j(1-\mu_i)(1-\mu_j) = 0.$$

The sum of all eight error terms for a fixed admissible i, j is at most $O(\frac{|A(i,j)|\gamma}{p})$, and summed over all $i \neq j$ it can be bounded, using (4) and the assumption that $m \leq p/\gamma$, by $O(\frac{\delta m \gamma}{p}) = O(\delta)$.

Altogether, we conclude that

$$\sum_{u,v} B(u,v)\epsilon(u)\epsilon(v) \leq \delta + O(\delta) = O(\delta),$$

which, together with (5), completes the proof of the theorem. □

The two propositions are simple consequences of Theorem 1.3.

Proof of Proposition 1.1 A simple application of Chernoff's bound (c.f., e.g., [3], Appendix A) implies that if $G = G(n,p)$ and $np > 1$ then, almost surely, the number of edges between any two sets of vertices X, Y of size at most s each deviates from its expectation $p|X||Y|$ by at most $O(\sqrt{p}s^{3/2}\sqrt{\log(en/s)}.)$ Therefore, as $n/s = m$, one can define here

$$\gamma = O(\frac{\sqrt{p\log m}}{\sqrt{s}}) = O(\frac{m^{1/2}p^{1/2}(\log m)^{1/2}}{n^{1/2}}).$$

Thus, if m satisfies

$$m \leq O(\frac{p^{1/2}n^{1/2}}{m^{1/2}(\log m)^{1/2}}),$$

the assumptions in the theorem will hold, and we can thus choose, for example, $m = \Theta((pn)^{1/4})$ to obtain the desired lower bound for $K(G)$. The upper bound follows from the results of [2], as the chromatic number of G (which is an upper bound for $\theta(\overline{G})$) is well known to be, almost surely, $\Theta(np/\log(np))$. □

Proof of Proposition 1.2 By a well known lemma (see, e.g., [1] or [3]), in any (n,d,λ)-graph the number of edges between any two sets X and Y of size at most s each deviates from $\frac{d}{n}|X||Y|$ by less than $\lambda\sqrt{|X||Y|} \leq \lambda s$. Thus we can take here $p = \frac{d}{n}$, $\gamma = \frac{\lambda}{s}$ and $m = (d/\lambda)^{1/2}$, and apply Theorem 1.3. □

In particular, if $\lambda = O(\sqrt{d})$ then the Grothendieck constant is $\Theta(\log d)$ (as the chromatic number of any d-regular graph cannot exceed d). We note that this can be used to give, for each fixed g and for infinitely many values of n, an explicit example of a graph G on n vertices whose girth exceeds g for which $K(G) \geq c(g) \log n$. That is, the Grothendieck constant is, up to a constant factor, as large as it is in a complete graph of the same size.

References

[1] N. Alon and F. R. K. Chung, Explicit construction of linear sized tolerant networks, Discrete Math. 72(1988), 15-19; (Proc. of the First Japan Conference on Graph Theory and Applications, Hakone, Japan, 1986.)

[2] N. Alon, K. Makarychev, Y. Makarychev and A. Naor, Quadratic forms on graphs, Proc. of the 37^{th} ACM STOC, Baltimore, ACM Press (2005), 486-493. Also: Inventiones Mathematicae, to appear.

[3] N. Alon and J. H. Spencer, **The Probabilistic Method**, Second Edition, Wiley, New York, 2000.

[4] M. Charikar and A. Wirth, Maximizing quadratic programs: extending Grothendieck's Inequality, FOCS 2004, 54-60.

[5] J. Friedman, A Proof of Alon's Second Eigenvalue Conjecture, to appear.

[6] A. Grothendieck, Résumé de la théorie métrique des produits tensoriels topologiques, Bol. Soc. Mat. Sao Paulo 8 (1953), 1–79.

[7] M. Grötschel, L. Lovász and A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, Combinatorica 1 (1981), 169–197.