1 János Pach
Twin primes—Eli Goodman (1933-2021) and Ricky Pollack (1935–2018)

Figure 1: Eli and Ricky, the main organizers of the 2008 Discrete Geometry Conference in Oberwolfach, Germany. (From the Archives of the Mathematisches Forschungsinstitut Oberwolfach.)

In my eyes, Eli (Jacob) Goodman and Ricky Pollack were inseparable. Exactly when and where they first met was a matter of discussion between them. Was it in the alcoves of City College, playing chess or in the NYU library, listening to recordings of classical music? Eli's father was a well-known secular Jewish scholar, who published extensively in two languages: Yiddish and English. Ricky was one of the "red diaper babies," his parents were communists, constantly harassed by the authorities. Both of them were passionately interested in mathematics, in music, and in literature. Both of them played the piano, Eli at a semi-professional level. After attending the Bat Mitzva of one of Eli's daughters and listening to the band *Klezmatics*, Ricky started taking clarinet lessons from David Krakauer. He would not travel anywhere without his clarinet. Eli went even further: he got a degree in composition and he co-founded the New York Composers Circle. His pieces were performed by leading musicians and were recorded.

Both of them had brilliant supervisors, but they did not have an easy start in mathematics. Eli's supervisor was Heisuka Hironaka, who later got the Fields Medal for groundbreaking discoveries in algebraic geometry. For many years, Eli worked relentlessly on a conjecture that turned out to be false. His favorite teacher at NYU was Harold N. Shapiro, a famous number theorist, who collaborated with Paul Erdős and Richard Bellman, and loomed large in mathematical circles. Ricky became his student. They spent a lot of time together. It was easy to learn from Shapiro, but difficult to shine next to him. By the mid-seventies, both Eli and Ricky were ready to venture into a new field that they felt was their own.

They were lucky: during their sabbaticals they hit on roughly the same subject and shortly after they found out about their common interest. At McGill University, Montreal, Willy Moser told Ricky about the Happy Ending problem of Erdős, Esther Klein and George Szekeres, and he almost instantly got obsessed with it. *Is it true that any set of* $2^{n-2}+1$ *points in general position in the plane has* n *elements that form the vertex set of convex* n*-gon?* If yes, this bound would be best possible. We still do not know the answer to this question, but a few years ago Andrew Suk showed that $(2+o(1))^n$ points always suffice. Eli had been searching for a simple geometric question that can be approached by encoding the underlying configurations and translating the problem into a purely combinatorial one. The Happy Ending problem appeared to be a perfect candidate. They started to explore these ideas by introducing (rediscovering) the notions of order types and allowable sequences. They did not get any closer to the proof of the Erdős-Szekeres conjecture, but the approach quickly yielded fruits. They proved Grünbaum's conjecture that *every set of 8 pseudolines is stretchable* [7], and continued to make progress related to a number of other questions raised in Branko Grünbaum's classic 1972 treatise, "Arrangements and Spreads" [14]. However, the most elegant early application of the method of allowable sequences was found by Peter Ungar [18], a legendary problem solver of Hungarian origin. He proved that *any set of n points in the plane, not all of which are collinear, determine at least n distinct directions, provided that n is even.* His argument was reproduced by Aigner and Zieglerin their popular volume "Proofs from the Book." Erdős often said as a joke that God kept a Book with only the most elegant mathematical arguments, and he rarely allows anyone to have a glance into it.

In 1980, Ricky and Eli started a geometry seminar at Courant Institute (NYU) which was attended by faculty and students of many universities from the Greater New York area, including Rutgers, Princeton, Columbia, CUNY, Stony Brook, and Pace University, and by researchers from Bell Labs, AT&T, and IBM. Over the years, when passing through the Big Apple, almost all important figures working in combinatorics, discrete geometry, computational geometry, or convexity gave a talk in this seminar. The abstracts of these talks were widely circulated. In the pre-internet era, anyone following these announcements had a pretty good overview of the most exciting new developments in our field. It was in this seminar, where Peter Ungar learned about allowable sequences, which enabled him to prove his above mentioned theorem on directions, originally conjectured by Scott.Twenty five years later, Rom Pinchasi, Micha Sharir and myself managed to settle Scott's problem in 3 dimensions [15]. All three of us attended the meetings of this seminar for years. I also had the privilege to co-organize it in the first decade of the 21st century.

In the beginning, it was not clear whether such a seminar would ever fly. As Joe Malkevitch recalls, Ricky doubted if anyone would show up if they "put out a shingle." Yet people did come, and they came in ever growing numbers. Why? It is hard to deny that the charismatic personalities of Ricky and Eli played a big role in this. They picked the right speakers and fascinating topics, and they had a good nose for significant new developments in the subject. Sometimes they were wrong, especially Ricky, who easily

Figure 2: János Pach, Ricky Pollack, and Eli Goodman at the International Conference on Intuitive Geometry, Siófok, Hungary, 1985. (Photo by J. Pach.)

fell in love with a new problem. But this only added to the thrill of novelty and discovery. The topics covered in the seminar have opened up new avenues of research for most participants, including many established senior mathematicians and computer scientists. A touch of luck has also contributed to the remarkable success of the seminar. The early 1980s witnessed an explosive surge in computing power, which resulted in a sustained appreciation for algorithmic techniques, an appreciation that has only grown stronger over time. NYU had a Robotics Lab, co-directed by Jack Schwartz and Micha Sharir, who laid the mathematical foundations for motion planning. As a recognition of their work, in 1986, they were invited to speak at the International Congress of Mathematicians in Berkeley. Many practical questions such as the so-called Piano Movers' problem, visibility and ray-shooting problems, more or less directly, raised deep questions about arrangements of points, lines, curves, convex sets, and other geometric objects in Euclidean spaces. It turned out that some closely related questions, with deep ties to number theory, functional analysis, discrete geometry, and information theory, had been investigated before by Gauss, Hilbert, Minkowski, Fejes Tóth, Rogers, Conway, Erdős, Lovász, Spencer, Szemerédi, Trotter, and others. Many of their results proved to be applicable in the design of efficient geometric algorithms. A new field was born: computational geometry. It was also popularized by Ron Graham and Frances Yao's concise and elegantly written survey, titled 'A whirlwind tour of computational geometry,' published in the American Mathematical Monthly.

The number of graduates in computer science far surpassed the number of mathematics graduates. The new generation of computer scientists were equally well-versed in discrete mathematics and geometry as their counterparts in mathematics. They were familiar with the Happy Ending problem and the Probabilistic Method (or the Random Sampling technique, as it was called in computer science), they learned about the Szemerédi-Trotter theorem on the number of incidences between points and lines and Lovász' theorem on halving lines. And they not only knew about these results, but soon improved on them! In particular, in a seminal paper presented at the 2nd Annual Symposium on Computational Geometry in 1986, David Haussler and Emo Welzl borrowed a technique for set-systems of bounded Vapnik-Chervonenkis dimension, and applied it to a wide range of questions in geometry. This paved the way for a series of new discoveries, including far-reaching extensions of the Szemerédi-Trotter theorem and a substantial improvement of Lovász' upper bound on the number of halving lines (by Clarkson, Edelsbrunner, Guibas, Sharir and Welzl, and by Dey, all of whom are computer scientists!) The relationship between discrete geometry and computational geometry proved to be mutually beneficial and resulted in remarkable breakthroughs on both sides.

It was perhaps Ron Graham, who first suggested that the time was ripe to launch a journal devoted entirely to discrete geometry. He must have talked to some publishers, as in 1984, Cambridge University Press, Wiley, and Springer-Verlag all expressed their interest in such a project. Ricky and Eli negotiated with all of them. Forty years ago, scientific publishing was a completely different business from what it is today. Almost all mathematics editors held PhDs. They were deeply embedded in the mathematics community, they had a good sense of scholarly quality and commercial value. Striking a balance between the two was, of course, crucial, but their primary goal was the advancement of science. Finally, the project was embraced by the late Walter Kaufmann-Bühler from Springer, about whom Ricky and Eli always spoke with the greatest admiration. Confident that the marriage of the classical subject of discrete geometry and the newly emerging field of computational geometry will be fruitful and long-lasting, they named the new journal *Discrete & Computational Geometry (DCG).* This turned out to be a self-fulfilling prophecy: Through their seminar, the organization of numerous conferences, and their groundbreaking mathematical and editorial work, Ricky and Eli nurtured a thriving community that kept the subject and the journal as fresh and vibrant as ever. After all these years, it still fills me with pride to have contributed two papers to the inaugural issue of Discrete & Computational Geometry in 1986 and to have had the honor of serving for several years, alongside Ricky and Eli, as Co-Editor-in-Chief of the journal they founded.

Figure 3: Participants of the first Computational Geometry conference at Bellairs Institute of McGill University, Holetown, St. James, Barbados, in 1986. Micha Sharir in the middle, Ricky Pollack above him, in red shirt, on the top. (Courtesy of J. Pach.)

Ricky and Eli made many important discoveries in discrete and computational geometry, convexity, geometric transversal theory, and real algebraic geometry. (Some of their achievements will be mentioned below, by Micha Sharir, Noga Alon, and Andreas Holmsen.) However, they always considered their most important legacy to be the creation of the journal and a large, friendly community of researchers around it, working in the field. In their own ways, both of them were fundamentally social creatures, for whom mathematics, as Ricky's son, Danny, once put it, was an "intensely social enterprise." They were born in New York, and until the very last years of their lives, they both lived in New York. They could not imagine moving anywhere else. New York was their natural cultural and mathematical habitat. Shortly before his death, Eli completed his excellent novel, which has just appeared in print [5]. The protagonist is a professor of mathematics from New York City, who mysteriously disappears from his Manhattan apartment. From a short article published in the *Times*, one can learn that *"He was last seen there three weeks ago at a party in his honor, but failed to show up for his classes the following Monday. A police spokesman indicated that no signs of disturbance were found in his apartment and that no correspondence has turned up that might indicate his whereabouts."* Eli and Ricky have disappeared from the New York scene, but their huge social and professional footprints are destined to linger for generations to come.

Figure 4: A group of participants of the Monte Verità Conference on Discrete and Computational Geometry, in Ascona, Switzerland, 1999. In the middle, Ricky Pollack, wearing a red shirt. Eli Goodman is the 4th from the right. (Photo by Emo Welzl.)

2 Micha Sharir A tribute to Ricky Pollack and Eli Goodman

Richard Pollack (Ricky) and Jacob E. Goodman (Eli) were, in many aspects, the founding fathers of Discrete and Computational Geometry, as a thriving, active, and mainly interactive research area.

Before turning to their scientific achievements, I would like to highlight two aspects of their leadership and influence on the field, which were not emphasized in the above summary.

A. The Discrete and Computational Geometry conference series. Computational Geometry was a young field in 1986 when Eli and Ricky launched the journal *DCG*. Discrete Geometry had been around for several decades, but bringing the two fields together was to a large extent the work of Eli and Ricky. Alongside with the journal, they have also launched, at the same year, a scientific conference, also called *Discrete and Computational Geometry*, at Santa Cruz, CA. It had a huge impact in defining and directing the field, as a common discipline. It was so successful that they have continued the tradition, by organizing two follow-up conferences, one at Mt. Holyoke, MA (*Discrete and Computational Geometry Ten Years Later*, 1996), and one at Snowbird, UT (*Discrete and Computational Geometry Twenty Years Later*, 2006). A fourth conference, *Discrete and Computational Geometry Thirty Years Later*, has been organized (by others) at Monte Verita, Switzerland, in 2016.

B. The Books. Each of them has contributed a major book in this area. Eli (with Joe O'Rourke) has edited the *Handbook of Discrete and Computational Geometry* [6], a monumental 1500-page collection of papers surveying basically all aspects of the field. It has later mushroomed into an even bigger collection (1937 pages), with Csaba Tóth as a third editor.

Ricky was a co-author, with Saugata Basu and Marie-Françoise Roy, of another extremely influential book, *Algorithms in Real Algebraic Geometry* [2], on which I will remark later in this note.

The major networking ventures undertaken by Eli and Ricky, and their effect on the community had been enormous. Their technical contributions to discrete and computational geometry has been equally influential. They spanned many topics, including *order types and allowable sequences*, *Helly-type results*, *algorithms in real algebraic geometry*, and a variety of applications in computational geometry.

A common thread in many of their works is the use of topological considerations in the analysis of structures in discrete geometry. Most notably, they looked for topological generalizations of standard concepts, such as pseudolines instead of lines, topological planes, and more. Perhaps the first example of such studies is their groundbreaking work on allowable sequences, which I will mention shortly.

In the remainder of this note I would like to combine a brief review of some of the major achievements of Eli and Ricky with illustrations of how these works have influenced my own research. I divide the discussion into four themes.

1. Allowable sequences and order types. Take a set P of n points in the plane, and project it onto a line ℓ. In general, we get a sequence P^* of n distinct points on ℓ. As we rotate ℓ, the sequence does not change combinatorially, except at certain critical orientations of ℓ, at which a block of consecutive elements of P^*, or several blocks simultaneously collapse into points and then are reversed. This evolution of P^* as ℓ rotates is called an *allowable sequence*; see [11]. Many interesting properties of P can be deciphered out of its allowable sequence, but the most intriguing question is whether a given allowable sequence (a sequence that obeys the evolution rule given above) is *realizable*, that is, whether it comes out of an actual set P of n points. The answer is that most sequences are not realizable, and that deciding whether a given sequence is realizable is PSPACE-complete (a computational complexity term, meaning, roughly, "intractable"). This however was not known at the time when they were working on the problem, and they were very excited about finding an effective solution to the decidability problem. As a matter of fact, the first time I ever met Ricky, in 1982, he gave me right away, very enthusiastically, a 'research announcement' (as these things were called those days) where he and Eli have obtained an effective (albeit, sadly, wrong) solution.

Let me switch to the related topic of *order types*, which generalizes these concepts to higher dimensions. For example, in the plane, the order type of a set P of n points specifies the orientation (left turn, right turn, or straight) of every ordered triple of points of P. Order types, introduced by Eli and Ricky, can be regarded as a concise purely discrete way of representing the "essence" of point configurations or, dually, of arrangements of lines (in the plane) or of hyperplanes (in higher dimensions). In fact, in this dual setting, order types can naturally be defined for arrangements of more general curves and surfaces, most notably for arrangements of *pseudolines* and pseudo-hyperplanes. Again, the question of the realizability of order types was a major topic of study. It was another manifestation of their interest in how basic discrete and combinatorial properties can be studied in a purely topological context, that had motivated many of their joint works.

Let me mention a fairly fresh result (Comput. Geom., 2023), which is based on Eli and Ricky's pioneering work on order types. We studied subquadratic algorithms for some 3Sum-hard geometric problems in the algebraic decision tree model. These are problems that are at least as hard as the 3Sum problem: determine whether a set of n real numbers has a triple that sums to zero. These problems include *collinearity testing*, i.e., determining whether a set of n points in the plane has a collinear triple. This problem is not known to have a subquadratic solution in the standard real-RAM model, and our work gave such a solution, for a restricted kind of collinearity testing (also known to be 3Sum-hard), in the algebraic decision tree model, where we only count algebraic sign tests involving the input data.

We worked in the dual plane, where we have a set L of n lines (or other curves), and we want to preprocess the arrangement $\mathcal{A}(L)$ for fast point location. This is of course a well known problem, which can be solved with $O(n^2)$ storage and $O(\log n)$ query time, using line sweep and persistent search trees. However, to achieve this performance, one needs, among other things, to sort the vertices of $\mathcal{A}(L)$ by their x-order, and each comparison in this sorting involves *four* input lines, two for each of the two vertices that are being compared. For our application, we wanted to obtain algebraic comparisons involving the input data that depend on a smaller number of elements, and for that it was crucial to reduce the number of lines involved in a comparison. The theory of order types was the tool that we needed. The order type information gives us the order of the vertices of $\mathcal{A}(L)$ along each line of L, and each comparison that these sortings perform involves only *three* lines. This seemingly unimportant difference was crucial in improving the running time of our algorithm.

This is just one, personal application, among many others, of the beautiful theory of order types; see [11].

2. k-sets. Eli and Ricky's papers on this topic have opened up a rich area of research on *k-sets* in configurations of points, and of levels in arrangements of curves and surfaces, in the plane and in higher dimensions. A k-set of a set P of n points in the plane, say, is a subset of size k that can be cut off its complement by a half-plane. In a dual setting, the *k-level* in an arrangement of a set L of n lines (or other curves) in the plane, say, is the set of all vertices and edges of the arrangement $\mathcal{A}(L)$ of L that have exactly k lines below them.

What Eli and Ricky had shown was that the number of at-most-k-sets, namely the overall number of j-sets, for $j = 0, 1, \dots, k$, is $O(nk)$, which is an asymptotic worst-case tight bound; another proof with a tighter bound was given later by Alon and Győri.

As it turned out, this notion plays a crucial role in the analysis of randomized algorithms in computational geometry, and in many other computational and combinatorial problems in geometry; such as the celebrated probabilistic analysis technique of Clarkson and Shor.

3. Hadwiger-type theorems and geometric permutations. Hadwiger's theorem gives a necessary condition for a finite collection of pairwise disjoint convex sets in the plane to have a *line transversal*, namely a line that crosses of all of them: If there exists a linear ordering of the sets such that every triple of sets are met by a directed line in the corresponding order, then the entire collection has a line transversal. In a remarkable work, Eli and Ricky have extended this result to arbitrary dimensions, giving a condition for the existence of a hyperplane transversal in terms of the multi-dimensional order type of the input sets, replacing the one-dimensional sorted order.

Eli and Ricky were also interested in line transversals in higher dimensions; see [13]. A line transversal to any collection of disjoint convex sets meets all the sets in a given order or its reverse, depending on the direction of the transversal. This pair of orders (permutations) is called a *geometric permutation*. The study of geometric permutations, mainly to derive upper and lower bounds on the number of such permutations, took off from their pioneering work, and I have been involved in some of these studies. There are still many open challenges to understand better the structure of geometric permutations.

4. Algorithms in real algebraic geometry. A real *semi-algebraic* set is a region in $\mathbb{R}^d$ that is defined by a Boolean combination of a finite number of polynomial equalities and inequalities. Given such a set A, how can we process it algorithmically? For example, how do we determine whether A is empty? Compute its connected components? Find a point in each connected component? These and many other basic problems in *Computational Real Algebraic Geometry* have been studied by Ricky, together with his colleague Marie-Françoise Roy and his student Saugata Basu. These works have culminated in their monumental book *Algorithms in Real Algebraic Geometry* [2], which has become, in a sense, the bible of this area, containing all the basic tools, techniques, and algorithms in computational real algebra and algebraic geometry. It is remarkable that the book is publicly available via open access, as demanded by the authors.

I would like to finish this note by mentioning some of my joint works with Eli and Ricky, most of them are with Ricky only. An exception is a work with both of them on the space of hyperplane transversals to a family of n separated and strictly convex sets in $\mathbb{R}^d$, where we show that the maximum combinatorial complexity of this space is $\Theta\left(n^{\lfloor d/2 \rfloor}\right)$. A main feature of the analysis, mainly contributed by Eli and Ricky, is the analysis of the topology of the space of common tangents, of a special kind, of a collection of such sets. It is yet another manifestation of their interest in studying topological aspects of discrete geometry.

Of the works with Ricky, I would like to mention the one on counting and cutting cycles of lines in space. This was a notoriously difficult problem, which has been solved only much later, where the goal was to break the lines in a set of n lines in $\mathbb{R}^3$ into the smallest (or, at least, a small) number of pieces, so as to eliminate all the depth cycles between them. The newly derived upper bound is close to $n^{3/2}$, which is nearly tight in the worst case. However, back when the paper with Ricky appeared, only very partial results were known. The paper was accompanied by another work of Pach, Pollack and Welzl, in which they have shown that a 4×4 pattern of lines in space cannot be completely *weaving*, namely that it is impossible for each line to alternate between passing above and below the lines in the other set in order. To experiment with this finding, they went out to buy some toy sticks to physically test how they can weave. Without thinking too much, they naturally bought $4 \times 4 = 16$ sticks...

Several other works with Ricky are on quasi-planar graphs, on arrangement of Jordan arcs with three intersections per pair, and various problems on simple polygons. All this goes to show that Ricky and Eli had been very curious and open-minded, and were interested in basically everything. Working with them was both fun and very inspiring.

The community at large, and I personally, are still reeling from the loss of both of them, and sorely miss their leadership and great science, not to mention friendship.

Figure 5: Ricky Pollack, Eli Goodman, and Peter McMullen at the AMS-IMS-SIAM Summer Research Conference on Discrete and Computational Geometry, Santa Cruz, CA, July 1986. (Photo by J. Pach.)

3 Noga Alon
Eli Goodman, Ricky Pollack, and Semivarieties

The friendship and collaboration between Eli Goodman and Ricky Pollack has been rare and productive, spanning decades of joint work and including the foundation of a leading journal and the organization of meetings and an active research seminar. Their joint papers stimulated a considerable amount of follow-up work. In this section we focus on one of their beautiful contributions and describe some of its many subsequent developments. The basic idea appears in a remarkable short note [9], where Goodman and Pollack observed that a Theorem of Milnor in real algebraic geometry can be used to provide an elegant nearly tight asymptotic estimate for the number of (simplicial) polytopes with n vertices in $\mathbb{R}^d$. Their approach paved the way to a significant amount of additional results in combinatorics, discrete and computational geometry and related areas, obtained by applying powerful tools from real algebraic geometry. It is natural to speculate that the background of Goodman in algebraic geometry demonstrated by his influential early work in the subject [3] helped in the early development of this approach. Below I describe the background to Goodman and Pollack's work on convex polytopes, followed by their results, and two research directions inspired by their work.

1. Connected components and sign patterns. There are several known results that provide upper bounds for the number of connected components of real varieties or semivarieties. Following such estimates by Oleĭnik–Petrovski (1949), Milnor (1964) and Thom (1965), Warren [19] proved that if $m \geq \ell \geq 2$, and $\{P_i(x_1, \ldots, x_\ell),\ 1 \leq i \leq m\}$ is a set of m polynomials in ℓ real variables, then the number of connected components of the semivariety

$$V = \{(x_1, \ldots, x_\ell) \in \mathbb{R}^\ell,\ P_i(x_1, \ldots x_\ell) \neq 0 \\ \text{for all } \ 1 \leq i \leq m\}$$

is at most $(4ekm/\ell)^\ell$.

For each point $x = (x_1, x_2, \ldots x_\ell) \in V$, the sign pattern of the polynomials P_i at the point x is the vector $(\text{sign}(P_1(x)), \ldots, \text{sign}(P_m(x))) \in \{-1, 1\}^m$. Let $s(P_1, \ldots, P_m)$ denote the total number of distinct sign patterns of the polynomials P_i, as x ranges over all points of V. Since the sign of each polynomial cannot change in any connected component of V, it follows that for m and ℓ as above, $s(P_1, \ldots, P_m) \leq (4ekm/\ell)^\ell$. In applications it is sometimes desirable to bound the number of sign patterns $(\text{sign}(P_1(x)), \ldots, \text{sign}(P_m(x)) \in \{-1, 0, 1\}^m$, where here x ranges over all points of $\mathbb{R}^\ell$ (including those in which some of the polynomials P_i vanish). It is not difficult to show (see [1]) that the result of Warren implies that this number does not exceed $(8ekm/\ell)^\ell$.

2. Counting polytopes and configurations. Let $c(n, d)$ denote the number of (combinatorial types of) d-polytopes on n labeled vertices and let $c_s(n, d)$ denote the number of simplicial d-polytopes on n labeled vertices (that is, polytopes in which all facets are simplices). The problem of determining or estimating these two functions (especially for 3-polytopes) has been the subject of much effort and frustration of nineteenth-century geometers as described, for example, in the book *Convex Polytopes* of Grünbaum. Despite these efforts, for any $d \geq 4$ (and large n) the best known upper bound for both $c_s(n, d)$ and $c(n, d)$ has been exponential in $n^{\lfloor d/2 \rfloor} \log n$. This estimate follows from the upper bound theorem for convex polytopes. The remarkable result of Goodman and Pollack [8, 9] improved it dramatically to a bound exponential in $d^2 n \log n$. They started by bounding the number of order types of configurations of n (labeled) points in $\mathbb{R}^d$, defined in what follows.

If $(p_0, p_1, \ldots, p_d)$ is a sequence of $d+1$ points in $\mathbb{R}^d$, with $p_i = (x_{i1}, \ldots, x_{id})$ for each i, we say they have a positive orientation if the determinant of the matrix $(x_{ij})_{0 \leq i,j \leq d}$ where $x_{i0} = 1$ for each i, is positive. If the determinant is negative they have a negative orientation, and if the determinant is zero they lie on a common hyperplane. The order type of a configuration C of n labeled points $p_1, p_2, \ldots, p_n$ in $\mathbb{R}^d$ is a function w from the set of all $(d+1)$-subsets of $[n] = \{1, 2, \ldots, n\}$ to $\{0, \pm 1\}$, where for $S = \{i_0, i_1, \ldots, i_d\}$ with $1 \leq i_0 < i_1 < \ldots < i_d \leq n, w(S)$ is $+1, -1$ or 0 according to the orientation of the points $p_{i_0}, \ldots, p_{i_d}$.

Let $t(n, d)$ denote the number of distinct order types of configurations of n labeled points in $\mathbb{R}^d$. Note that $t(n, d)$ is the number of sign patterns of $\binom{n}{d+1}$ polynomials of degree d in the dn real variables $(x_{i1}, \ldots, x_{id})$, $i = 1, \ldots, n$, which are the coordinates of the points. The polynomials are just all the determinants $\det(x_{i_k j}), 0 \leq k, j \leq d$, where $x_{i_k 0} = 1$ for all k and $1 \leq i_0 < i_1 < \cdots < i_d \leq n$. Therefore, the estimate of Warren (and its slight extension for the total number of sign patterns) shows that $t(n, d) \leq n^{(1+o(1))d^2 n}$.

This supplies immediately a similar bound for the number $c(n, d)$ of d-polytopes on n points. Indeed, the order type of a configuration that spans $\mathbb{R}^d$ determines which sets of its points lie on supporting hyperplanes of its convex hull. Hence, the order type of a configuration on a set of n points in $\mathbb{R}^d$ which is the set of vertices of a convex polytope P determines its facets and its complete combinatorial type.

3. Signrank. The sign-pattern of an m by n real matrix A with nonzero entries $(a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ is an m by n matrix $Z(A) = (z_{ij})$ of $1, -1$ entries where $z_{ij} = \text{sign } a_{ij}$. For an m by n matrix Z of $1, -1$ entries, let $r(Z)$ denote the minimum possible rank of a matrix A such that $Z(A) = Z$. Define $r(n, m) = \max\{r(Z) : Z \text{ is an } m \text{ by } n \text{ matrix over } \{1, -1\}\}$. The problem of determining or estimating $r(n, m)$, and in particular $r(n, n)$, was raised by Paturi and Simon in the early 80s, motivated by the study of the so-called unbounded-error probabilistic communication complexity of a Boolean function of $n + n$ bits. Alon, Frankl and Rödl (cf. [1]) proved in 1985 that

$$\frac{n}{16} \le r(n,n) \le \frac{n}{2} + 3\sqrt{n}$$

and that if $m/n^2 \to \infty$ and $(\log_2 m)/n \to 0$ then

$$r(n,m) = \left(\frac{1}{2} + o(1)\right) n \ .$$

The lower bounds in both estimates are derived from the estimate of Warren by a simple counting argument.

4. Semialgebraic properties. A graph property is any family of graphs $\mathcal{F}$ closed under isomorphism. Such a family is called semialgebraic if every vertex is a point in a real space of bounded dimension, and the adjacency of two vertices is determined by the signs of a finite set of bounded degree polynomials in the coordinates of the corresponding points. This can be extended to hypergraphs too, but for simplicity we focus here only on the case of graphs. Natural special cases of such properties are intersection graphs of simple geometric objects, like segments or disks in the plane, boxes in $\mathbb{R}^3$ and more. The speed of a family $\mathcal{F}$ is the function $f(n) = |\mathcal{F}_n|$, where $\mathcal{F}_n$ is the set of all graphs with n vertices in the family. The results about the number of sign patterns of polynomials described here imply that the speed $f(n)$ of any semialgebraic family of graphs satisfies $f(n) \le 2^{cn \log n}$, where $c = c(\mathcal{F})$ is a constant that depends on the dimension and the degrees of the polynomials in the definition of the property. Several specific examples can be found in [1] and the references therein. A result of Sauermann [17] shows that under some mild conditions the estimate obtained for the constant $c = c(\mathcal{F})$ by applying Warren's Theorem is tight. Besides their modest speed functions, it turns out that semialgebraic graph properties are simpler than general families of graphs in many respects. The study of their Ramsey properties and the investigation of additional extremal questions for such families received a considerable amount of attention in the last decade. It will surely keep being the subject of future research, like additional related topics initiated by the work of Eli Goodman and Ricky Pollack.

4 Andreas Holmsen
Eli Goodman, Ricky Pollack, and Geometric Transversal Theory

A specific branch of combinatorial geometry in which the work of Eli and Ricky had a tremendous impact is what we call *"geometric transversal theory"*. This line of research, an offshoot of *Helly's theorem*, was initiated in the 1930's by Vincensini and Santaló, and explored further in the 50's and 60's by a number of prominent geometers such as Grünbaum, Hadwiger, Klee, and Danzer. The famous survey, *"Helly's theorem and its relatives"*, gives a detailed account of the state of affairs in 1963, and motivated further study throughout the 70's and 80's.

Eli and Ricky's 1988 paper *"Hadwiger's transversal theorem in higher dimensions"* stands as one of the milestones of geometric transversal theory. Their beautiful result related the (at the time) novel notion of *order types* to the classical study of geometric transversals, and paved the way for research directions in discrete and computational geometry which bear fruit to this day.

The study of geometric transversals originated with Helly's theorem, which asserts that for a family of at least $d+1$ compact convex sets in $\mathbb{R}^d$, if every $d+1$ members can be intersected by a point, then the entire family can be intersected by a point. Can a similar theorem be true if the property "intersected by a point" is replaced with "intersected by a line", or by a plane, or more generally by a k-dimensional affine flat?

This was the problem, posed by Vincensini in 1935, that initiated the study of geometric transversals, but it did not take long before Santaló realized that no such "Helly-type" theorem can exist for k-flats when $k > 0$. While this situation may seem somewhat discouraging, it did not prevent further study of geometric transversals. Indeed, Santaló showed that if we restrict ourselves to families of *axis parallel boxes* in $\mathbb{R}^d$, then the "Helly number" for *line transversals* becomes $2^{d-1}(2d-1)$, and for *hyperplane transversals* it becomes $2^{d-1}(d+1)$. In fact, there has been ex-

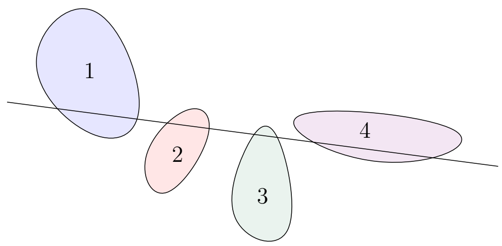

Figure 6: The line transversal induces the ordering $1 \prec 2 \prec 3 \prec 4$.

tensive work on geometric transversals which investigates Helly-type theorems under various restrictions on the geometric shapes of the sets in the family.

Rather than focusing on the geometric shape of the sets, Hadwiger's approach has a more combinatorial flavor. Suppose a finite family of *pairwise disjoint* convex sets admits a line transversal. By orienting this line, it induces an ordering on the family, namely the order in which it meets the sets. In particular, any three members of the family are met by a line which is *consistent* with the given ordering. (See Figure 6.) What Hadwiger showed is that, in the plane, this obvious necessary condition is also sufficient:

Theorem. *A finite family $\mathcal{F}$ of pairwise disjoint convex sets in the plane admits a line transversal if and only if there exists a linear ordering of $\mathcal{F}$ such that every three members of $\mathcal{F}$ are met by a line consistent with the ordering.*

By the mid 1980's, Eli and Ricky had been investigating order types of point configurations in $\mathbb{R}^d$ for several years, when it dawned upon them that the linear ordering in Hadwiger's transversal theorem was simply a 1-dimensional order type. They noticed that by making a bijection between a finite set and a point configuration in $\mathbb{R}^k$, then the order type of the point configuration induces what they called a *k-ordering* of the set, which for $k = 1$ is precisely a linear ordering. This meant that they had just the right tool to generalize Hadwiger's transversal theorem to higher dimensions! All that was missing was the right analogue of pairwise disjointness, and the natural condition they found was to define a family of at least $k + 1$ convex sets in $\mathbb{R}^d$ to be $(k-1)$-*separated* if no $k + 1$ of them admit a $(k - 1)$-transversal. In particular, being 0-separated means that no two members have a 0-transversal, i.e., the members are pairwise disjoint. A consequence of the definition is the following: If a $(k - 1)$-separated family of convex sets in $\mathbb{R}^d$ admits a k-transversal, then by choosing one point from each set within the given k-transversal, we obtain a point configuration in $\mathbb{R}^k$, and the order type of this configuration is independent of the choice of points. In this way, a k-transversal naturally induces a k-ordering of the family. These observations led Eli and Ricky to their celebrated generalization of Hadwiger's theorem:

Theorem ([10])**.** *A finite $(d-2)$-separated family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ admits a hyperplane transversal if and only if there is a $(d-1)$-ordering of $\mathcal{F}$ such that every $d + 1$ members of $\mathcal{F}$ are met by a hyperplane consistent with the $(d - 1)$-ordering.*

Another important development was made a couple years later by Wenger, a PhD student of Ricky, who was able to remove the disjointness assumption from Hadwiger's transversal theorem. This requires a clarification of what it means for a line to intersect a family of convex sets consistently with an ordering, since such an ordering may no longer be uniquely determined. Wenger showed that it suffices for *some choice* of points to be consistent with the ordering, and shortly after, Pollack and Wenger extended this to higher dimensions as well. By an elegant proof from *"the book"*, which combines order types, combinatorial convexity, and the Borsuk–Ulam theorem, they proved what we now call the Goodman–Pollack–Wenger theorem:

Theorem ([16])**.** *A finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ admits a hyperplane transversal if and only if for some k, $0 \leq k < d$, there is a k-ordering of $\mathcal{F}$ such that every $k + 2$ of the sets are met by some k-flat consistent with the k-ordering.*

Over the years there have been a number of further generalizations and extensions of Eli and Ricky's breakthrough result. Some of the highlights include:

▷ Anderson and Wenger (1996): Replaces the k-ordering by the more general concept of an *acyclic oriented matroid*.

▷ Arocha et al. (2003): Shows that not only do we get a single hyperplane transversal, but in fact "many" of them, captured by what they call a *virtual k-transversal*.

▷ Arocha et al. (2008): Gives a *colorful version* of Hadwiger's transversal theorem in the spirit of the Bárány–Lovász "colorful Helly theorem".

▷ Cheong et al. (2023): Proves the colorful version of the Goodman–Pollack–Wenger theorem conjectured by Arocha et al.

▷ McGinnis (2023): Establishes an analogue of the Goodman–Pollack–Wenger theorem for hyperplane transversal in $\mathbb{C}^d$.

For nearly two decades, Eli and Ricky (with various collaborators) continued working on geometric transversals, exploring Helly-type theorems, the topological structure and combinatorial complexity of the space of transversals, and convexity on the affine Grassmannian. Their survey [13] joint with Wenger, documented the explosion of work in geometric transversal theory that had taken place in the years following their breakthrough papers on the generalizations of Hadwiger's transversal theorem.

In the paper *"Foundations of a theory of convexity on affine Grassmann manifolds"*, Eli and Ricky asked whether there is a convex hull operator, $\text{conv}_k(\cdot)$, on the space of k-dimensional affine flats in $\mathbb{R}^d$, which naturally extends the standard convex hull operator for points, and satisfies general properties such as: *monotonicity*, *idempotence*, *antiexchange*, and *affine invariance*. (For precise definitions see [12] or the expository article [4].)

Their solution was as natural as the question: Fix an integer $0 \leq k < d$. For a set $\mathcal{L}$ of k-flats in $\mathbb{R}^d$, define its *dual*, $\mathcal{L}^*$, to be the family of all convex (point) sets which meets every flat in $\mathcal{L}$. For a family $\mathcal{F}$ of convex (point) sets, define its dual, $\mathcal{F}^*$, to be the set of all k-transversals to the family $\mathcal{F}$. Now define the convex hull of a set $\mathcal{L}$ of k-flats in $\mathbb{R}^d$ to be its double dual, that is, $\text{conv}_k \mathcal{L} = \mathcal{L}^{**}$.

It turns out that this notion of convexity satisfies the four properties stated above, and indeed, when restricted to the case $k = 0$, it reduces to the standard convexity. For $k > 0$ a rich theory emerges which is closely tied to central questions in geometric transversal theory, and their paper explores a number of interesting examples ranging from rulings on a hyperboloid to certain Schubert varieties. In fact, many sophisticated constructions and counterexamples in geometric transversal theory can be traced back to this convexity structure.

Today geometric transversal theory is an active area of research. In the last decade, we have witnessed an emergence of new and exciting directions motivated by recent trends in discrete and computational geometry, as well as developments on research problems dating back to Eli and Ricky's seminal work in the area.

References

[1] N. Alon. Tools from higher algebra, In R.L. Graham, M. Grötschel, and L. Lovász, eds, Handbook of Combinatorics, North Holland (1995), Chapter 32, 1749–1783.

[2] S. Basu, R. Pollack, and M.-F. Roy. Algorithms in Real Algebraic Geometry. Algorithms and Computation in Mathematics 10, 2nd edition, Springer-Verlag, Berlin, 2006.

[3] J. E. Goodman. Affine open subsets of algebraic varieties and ample divisors. Ann. of Math. (2) 89 (1969), 160–183.

[4] J. E. Goodman. When is a set of lines in space convex? Notices Am. Math. Soc. 45 (1998), 222–232.

[5] J. E. Goodman. The Mathematician. FriesenPress, Altona, Canada, 2023.

[6] J. E. Goodman and J. O'Rourke (editors). Handbook of Discrete and Computational Geometry, 2nd edition. Chapman and Hall / CRC, 2004.

[7] J. E. Goodman and R. Pollack. Proof of Grünbaum's conjecture on the stretchability of certain arrangements of pseudolines. J. Combin. Theory Ser. A 29 (1980), no. 3, 385–390.

[8] J. E. Goodman and R. Pollack. Upper bounds for configurations and polytopes in $\mathbb{R}^d$. Discrete Comput. Geom. 1 (1986), 219–227.

[9] J. E. Goodman and R. Pollack. There are asymptotically far fewer polytopes than we thought. Bull. Amer. Math. Soc. (N.S.) 14 (1986), no. 1, 127–129.

[10] J. E. Goodman and R. Pollack. Hadwiger's transversal theorem in higher dimensions. J. Am. Math. Soc. 1 (1988), 301–309.

[11] J. E. Goodman and R. Pollack. Allowable sequences and order types in discrete and computational geometry. In J. Pach (editor), New Trends in Discrete and Computational Geometry, Algorithms and Combinatorics, vol. 10. Springer-Verlag, Berlin, Heidelberg, 1993, pp. 103–134.

[12] J. E. Goodman and R. Pollack. Foundations of a theory of convexity on affine Grassmann manifolds. Mathematika 42 (1995), 305–328.

[13] J. E. Goodman, R. Pollack, and R. Wenger. Geometric transversal theory. In J. Pach (editor), New Trends in Discrete and Computational Geometry, Algorithms and Combinatorics, vol. 10. Springer-Verlag, Berlin, Heidelberg, 1993, pp. 163–198.

[14] B. Grünbaum. Arrangements and Spreads. Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 10. American Mathematical Society, Providence, RI, 1972.

[15] J. Pach, R. Pinchasi, and M. Sharir. Solution of Scott's problem on the number of directions determined by a point set in 3-space. Discrete Comput. Geom. 38 (2007), no. 2, 399–441.

[16] R. Pollack and R. Wenger. Necessary and sufficient conditions for hyperplane transversals. Combinatorica 10 (1990), 307–311.

[17] L. Sauermann. On the speed of algebraically defined graph classes. Adv. Math. 380 (2021), Paper No. 107593, 55 pp.

[18] P. Ungar. $2N$ noncollinear points determine at least $2N$ directions, J. Combin. Theory Ser. A 33 (1982), no. 3, 343–347.

[19] H. E. Warren. Lower bounds for approximation by nonlinear manifolds. Trans. Amer. Math. Soc. 133 (1968), 167–178.