EXTENDED VC-DIMENSION, AND RADON AND TVERBERG TYPE THEOREMS FOR UNIONS OF CONVEX SETS

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Abstract. We define and study an extension of the notion of the VC-dimension of a hypergraph and apply it to establish a Tverberg type theorem for unions of convex sets. We also prove a new Radon type theorem for unions of convex sets and settle a well-known open problem posed by Kalai in the 1970s.

1. Introduction. Radon's theorem states that any set of $d+2$ points in $\mathbb{R}^d$ can be partitioned into two subsets whose convex hulls intersect. Formally, given a set $P = \{x_1, x_2, \ldots, x_{d+2}\} \subseteq \mathbb{R}^d$, there exists a partition of P into two disjoint subsets P_1 and P_2 ($P = P_1 \cup P_2$) such that $\mathrm{CH}(P_1) \cap \mathrm{CH}(P_2) \neq \emptyset$, where here and in what follows $CH(X)$ denotes the convex hull of a set of points X. The bound $d+2$ is tight as one can easily see by taking any set of $d+1$ affine-independent points in $\mathbb{R}^d$, that is, the vertices of the d-dimensional simplex.

This fundamental theorem is a cornerstone in discrete geometry, providing insights into the structure of point sets and their convex combinations. It implies various fundamental theorems in geometry including the classical Helly's Theorem, the center point theorem, and more. Its implications extend to machine learning, statistical learning and computational geometry, influencing algorithms for geometric separation, point location, and convex hull computations. An example of such implication is the existence of small so-called ε-nets or ε-approximations for range-spaces defined by semi-algebraic sets which are a core notion in those areas. The combinatorial notion of VC-dimension is, in fact, the analog of Radon's bound for abstract set-systems.

An equivalent formulation of Radon's theorem states that for any set P of $d+2$ points in $\mathbb{R}^d$, there exists a partition of the set into two subsets P_1 and P_2 such that any convex set containing P_1 must intersect any convex set containing P_2. This perspective highlights the strong separation properties that are impossible for sets of $d+2$ points in $\mathbb{R}^d$.

Moreover, Radon's theorem serves as a basis for numerous generalizations and related results, such as the following classical and beautiful theorem of Tverberg [20], which further enriches our understanding of geometric configurations.

THEOREM 1.1. *[Tverberg's Theorem[20]] Let $r \geq 2$ be a fixed integer and $d \geq 1$. Then for any set P of $(r-1)(d+1)+1$ points in $\mathbb{R}^d$ there exists a partition of P into r pairwise disjoint sets $P = \bigcup_{i=1}^{r} P_i$ such that $\bigcap_{i=1}^{r} CH(P_i) \neq \emptyset$.*

Note that Radon's theorem is the special case of Tverberg's theorem with $r = 2$.

Tverberg's Theorem has far-reaching implications in discrete and computational geometry and beyond. Combined with the colorful Carathéodory's theorem of Bárány [6] it implies, for example, the so-called first selection Lemma (see, e.g., [15]).

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An equivalent formulation of Tverberg's theorem states that for any set P of $(r-1)(d+1)+1$ points in $\mathbb{R}^d$, there exists a partition of the set into r pairwise disjoint sets $P = \bigcup_{i=1}^{r} P_i$ such that for any family of convex sets $C_1, \ldots, C_r$ with $P_i \subset C_i$ for every $i \in [r]$ we have that $\bigcap_{i=1}^{r} C_i \neq \emptyset$.

In this paper we settle an old open problem posed already in the 70's by Gil Kalai and reiterated in later surveys (e.g., [7, 14]) that asks for a generalization of Radon's Theorem (and Tverberg's Theorem) to sets which are not necessarily convex nor even connected, but are the union of a bounded number of convex sets.

Definition 1.2. *Let $s \geq 1$ be an integer. A set C in $\mathbb{R}^d$ is said to be s-convex if it is the union of s convex sets.*

Problem 1.3. *[[7]-problem 6.6] What is the least integer $f = f(d, s, t)$ such that for any set P of f points in $\mathbb{R}^d$ there is a partition $P = A \cup B$ such that any s-convex set containing A must intersect any t-convex set containing B.*

A more general problem is the following:

Problem 1.4. *What is the least integer $f = f_r(d, s_1, \ldots, s_r)$ such that for any set P of f points in $\mathbb{R}^d$ there is a partition into r pairwise disjoint sets $P = \bigcup_{i=1}^{r} P_i$ such that for any family of sets $C_1, \ldots, C_r$ with $P_i \subset C_i$ where C_i is an s_i-convex set for every $i \in [r]$ we have that $\bigcap_{i=1}^{r} C_i \neq \emptyset$.*

Notice that Radon's theorem is equivalent to $f(d, 1, 1) = d+2$ and more generally Tverberg's theorem is equivalent to $f_r(d, 1, \ldots, 1) = (r-1)(d+1)+1$.

Bárány and Kalai showed that $f(d, s, t)$ is always finite. However, their upper bound is roughly $twr_d(\theta(s+t))$ where twr_d is the d-fold tower function. This huge upper bound is due to their proof technique involving hypergraph Ramsey theory. Despite substantial attention, this problem remained unsolved for nearly sixty years.

1.1. The main results. Our first main result is a fairly simple proof of the following near-optimal upper bound:

Theorem 1.5. $f(d, s, t) = O(dst \log(st+1))$

Our second main result is an extension of the theorem above to the following version of Tverberg's theorem for unions of convex sets and any $r \geq 2$:

Theorem 1.6. $f_r(d, s_1, \ldots, s_r) = O\left(dr^2 \cdot \log r\right) \cdot \Pi_{i=1}^{r} s_i \cdot \ln(1 + \Pi_{i=1}^{r} s_i)\big)$

A notable feature of our proofs is that they are mainly combinatorial and therefore work for abstract separable convexity spaces with bounded Radon numbers:

Definition 1.7. *An* abstract convexity space *is a pair $(X, \mathcal{C})$ where X is a set and $\mathcal{C} \subseteq 2^X$ is a family of subsets of X called* convex sets*, satisfying the following axioms:*

1. *$X \in \mathcal{C}$ and $\emptyset \in \mathcal{C}$.*
2. *$\mathcal{C}$ is closed under intersections, i.e., if $\mathcal{D} \subseteq \mathcal{C}$, then $\bigcap \mathcal{D} \in \mathcal{C}$.*

This generalizes the notion of convex sets in $\mathbb{R}^d$. Abstract convexity spaces arise naturally in combinatorics, geometry, and lattice theory, and provide a unifying framework to study convexity. For more on abstract convexity spaces, see, e.g., [13, 16, 21].

For a subset $P \subset X$ in an abstract convexity space $(X, \mathcal{C})$ define its convex hull $CH(P)$ to be $CH(P) = \bigcap_{P \subset S, S \in \mathcal{C}} S$.

For a set $P \subset X$ we say that $P = P_1 \cup P_2$ is a Radon partition if $CH(P_1) \cap CH(P_2) \neq \emptyset$. We say that an r partition $P = \bigcup_{i=1}^{r} P_i$ is an r-Tverberg partition of P if $\bigcap_{i=1}^{r} CH(P_i) \neq \emptyset$. The *Radon number* $r(X, \mathcal{C})$ of a space $(X, \mathcal{C})$ is the minimum integer n such that any subset $P \subset X$ with $|P| \geq n$ admits a Radon partition. If n is unbounded then the Radon number is ∞. Similarly, the r'th Tverberg number $T_r(X, \mathcal{C})$ is the minimum integer n such that any subset $P \subset X$ with $|P| \geq n$ admits an r-Tverberg partition.

A convex set $H \in \mathcal{C}$ in an abstract convexity space $(X, \mathcal{C})$ is called a *halfspace* if its complement is also a convex set i.e., $X \setminus H \in \mathcal{C}$.

We say that an abstract convexity space $(X, \mathcal{C})$ is *separable* if for any two disjoint sets there exists a separating halfspace. Formally, for any $C_1 \in \mathcal{C}$ and $C_2 \in \mathcal{C}$ if $C_1 \cap C_2 = \emptyset$ then there exists a halfspace H such that $C_1 \subset H, C_2 \subset X \setminus H$.

Our proof technique of Theorem 1.5 and Theorem 1.6 works for abstract convexity spaces which are separable and have a bounded Radon number. Let $(X, \mathcal{C})$ be a separable abstract convexity space with Radon number $d = r(X, \mathcal{C})$. Call a subset $A \subset X$ s-convex if it is the union of s convex sets in $\mathcal{C}$. Let $F = F(d, s, t)$ be the minimum integer so that any set P of cardinality at least F in a separable abstract convexity space $(X, \mathcal{C})$ with Radon number d admits a partition $P = P_1 \cup P_2$ such that any s-convex set containing P_1 must intersect any t-convex set containing P_2. We have:

THEOREM 1.8. $F(d, s, t) = O(dst \log(st + 1))$.

More generally let $F = F_r(d, s_1, \ldots, s_r)$ be the minimum integer so that any set P of cardinality at least F in a separable abstract convexity space $(X, \mathcal{C})$ with Radon number d admits a partition $P = \bigcup_{i=1}^r P_i$ into r pairwise disjoint sets $P = \bigcup_{i=1}^r P_i$ such that for any family of sets $C_1, \ldots, C_r$ with $P_i \subset C_i$ where C_i is an s_i-convex set for every $i \in [r]$ we have that $\bigcap_{i=1}^r C_i \neq \emptyset$.

THEOREM 1.9. $F_r(d, s_1, \ldots, s_r) = O\left(dr^2 \cdot \log r \cdot \Pi_{i=1}^r s_i \cdot \ln(1 + \Pi_{i=1}^r s_i)\right)$.

Our Theorem 1.9 for the special case $s_i = 1$ for all i provides an upper bound of $O(dr^2 \log r)$ on the so-called Tverberg number in separable abstract convexity spaces in terms of its Radon number d. This improves the upper bound of $c(d)r^2 \log^2 r$ by Bukh [9] who proved it for the more general setting of (not necessarily separable) abstract convexity spaces. In [18] Pálvölgyi provided an upper bound of the form $O(d^{d^{d^{\log d}}} r)$ which is linear in r but super exponential in d.

We describe here the proofs of Theorem 1.5 and Theorem 1.6. Essentially the same proofs establish the corresponding results, Theorem 1.8 and Theorem 1.9, for abstract separable convexity spaces.

1.2. Tools and preliminaries. Our proof of Theorem 1.5 is based on some simple properties of the VC-dimension and the shatter function of the relevant hypergraphs. In order to prove Theorem 1.6 we define an extended version of the VC-dimension and show how to bound it in terms of the usual dimension.

The Vapnik-Chervonenkis dimension of a hypergraph is a measure of its complexity, which plays a central role in statistical learning, computational geometry, and other areas of computer science and combinatorics (see, e.g., [3, 2, 4, 8, 17]). Many graphs and hypergraphs that arise in geometry have bounded VC-dimension.

DEFINITION 1.10 (VC-dimension). *The* Vapnik-Chervonenkis dimension $VC(H)$ *of a hypergraph $H = (V, E)$ is the largest integer d such that there exists a subset $S \subseteq V$ (not necessarily in E) with $|S| = d$ that is* shattered *by E. A subset S is said to be shattered by E if, for every subset $T \subseteq S$, there exists a hyperedge $e \in E$ such that $e \cap S = T$.*

REMARK 1.11. *Note that in a hypergraph $H = (V, E)$ a subset S is shattered if, equivalently, for every partition $S = A \cup B$ (with $A \cap B = \emptyset$) it holds that there exists two hyperedges $e_1 \in E$ and $e_2 \in E$ such that $A \subset e_1$, $B \subset e_2$ and $e_1 \cap e_2 \cap S = \emptyset$. We will need a generalization of this notion to partitions into r sets and such a generalization is given below in Definition 1.14.*

Note that by the fact that every two disjoint convex sets in $\mathbb{R}^d$ are separable by a halfspace, Radon's Theorem implies that no set of $d+2$ points in $\mathbb{R}^d$ is shattered with halfspaces. Namely, if $\mathcal{H}_d$ is the family of all halfspaces in $\mathbb{R}^d$ then the VC-dimension of the hypergraph $(\mathbb{R}^d, \mathcal{H}_d)$ is at most $d+1$. This simple observation holds for every separable abstract convexity space $(X, \mathcal{C})$ with Radon number d. In that case the VC-dimension of the hypergraph $(X, \mathcal{H})$ where $\mathcal{H}$ is the family of all halfspaces in $\mathcal{C}$ is at most $d-1$.

Definition 1.12.

The primal shatter function *of a hypergraph $H = (V, E)$ is the following function $\pi_H : \mathbb{N} \to \mathbb{N}$:*

$$\pi_H(m) = \max_{S \subseteq V, |S| = m} |\{S \cap e : e \in E\}|.$$

The value $\pi_H(m)$ represents the maximum number of distinct subsets of a set S of cardinality m that can be realized as intersections with hyperedges in E.

The following lemma, known as the Sauer-Shelah-Perles lemma, provides an upper bound on the shatter function for hypergraphs with bounded VC-dimension (See, e.g., [15]):

Lemma 1.13 (**Sauer-Shelah-Perles**). *Let $H = (V, E)$ be a hypergraph with VC dimension d. Then*

$$\pi_H(m) \leq \sum_{i=0}^{d} \binom{m}{i}.$$

In particular, if $m > d$, then $\pi_H(m) \leq (\frac{em}{d})^d$.

In order to tackle Problem 1.4 we need to develop an analogous notion of a shattered set in hypergraphs with bounded VC-dimension to partitions with more than 2 parts. The relevant definition follows. The motivation for this definition will become clear from its application in the study of Problem 1.4

Definition 1.14. *Let $H = (V, E)$ be a fixed hypergraph. A subset $S \subset V$ is said to be* r-shattered *by E if for any partition of S into r pairwise disjoint sets $S = \bigcup_{i=1}^{r} S_i$ there exists hyperedges $e_1, \ldots, e_r \in E$ such that $S_i \subset e_i$ for all $i \in [r]$ and $S \cap \bigcap_{i=1}^{r} e_i = \emptyset$.*

The following combinatorial lemma provides an extension of the Sauer-Shelah-Perles Lemma and might be of independent interest:

Lemma 1.15. *There exists an absolute constant C such that for every integer d and any hypergraph $H = (V, E)$ with VC-dimension d and every integer $r \geq 2$, every r-shattered set has size at most $Cdr^2 \log r$. This bound is nearly optimal: for every d and r there is a hypergraph with VC-dimension d that admits an r-shattered set of size $\Omega(dr^2)$.*

The proof of the upper bound in the lemma is described in Section 3. The proof of the lower bound is given implicitly in Theorem 4.2 as explained in the remark following it.

1.3. Structure. The rest of this short paper is organized as follows. In Section 2 we describe the proof of Theorem 1.5. The proof of Theorem 1.6 is given in Section 3. After a discussion of lower and upper bounds for the relevant functions for various ranges of the parameters in Sections 4 and 5 we suggest several open problems in the final Section 6.

2. Proof of Theorem 1.5. Before proceeding to the proof of Theorem 1.5 we need the following two easy Lemmas.

LEMMA 2.1. *Let C_1 be an s-convex set in $\mathbb{R}^d$ and C_2 a t-convex set. Assume that $C_1 \cap C_2 = \emptyset$. Then there exist s convex polytopes $K_1, \ldots, K_s$ with a total of at most st-facets whose union covers C_1 so that the complement of the union covers C_2. Namely, $C_1 \subset \bigcup_{i=1}^s K_i$ and $C_2 \subset \overline{\bigcup_{i=1}^s K_i}$.*

Proof. Since C_1 is s-convex it can be written as $C_1 = \bigcup_{i=1}^s X_i$ for some convex sets $X_1, \ldots, X_s$. Similarly $C_2 = \bigcup_{j=1}^t Y_i$ for t convex sets $Y_1, \ldots, Y_t$. Since $C_1 \cap C_2 = \emptyset$, for every $i \in [s], j \in [t]$ we have that $X_i \cap Y_j = \emptyset$ and therefore there exists a hyperplane $h_{i,j}$ strictly separating X_i and Y_j. Assume without loss of generality that the positive open halfspace $h^+_{i,j}$ bounded by $h_{i,j}$ contains X_i and the negative open halfspace $h^-_{i,j}$ contains Y_j. For every $i \in [s]$ let K_i be the convex polytope which is the intersection $\bigcap_{j=1}^t h^+_{i,j}$. Note that K_i is a convex polytope with at most t facets containing X_i and its complement $\overline{K_i}$ contains C_2. So the union of the polytopes $\bigcup_{i=1}^s K_i$ contains C_1 and its complement $\overline{\bigcup_{i=1}^s K_i}$ contains C_2. Moreover, the total number of facets of these polytopes is at most st. This completes the proof of the lemma. □

LEMMA 2.2. *Let l be an integer and let $H = (P, E)$ be a hypergraph where P is a set of points in $\mathbb{R}^d$ and $S \in E$ is a hyperedge if and only if there exists a set $K_1, \ldots, K_i$ of convex polytopes with a total of at most l facets such that $S = P \cap (\bigcup_{j=1}^i K_j)$. Namely, S can be cutoff from P by intersecting it with a set consisting of a union of convex polytopes with a total of l facets. Then the VC-dimension of H is bounded by $O(dl \log(l + 1))$.*

Proof. The proof is rather standard and follows from e.g., [15](Proposition 10.3.3). □

Proof of Theorem 1.5: Put $l = st$. Let $H = (\mathbb{R}^d, E)$ be the hypergraph as in Lemma 2.2. Let $n = O(dl \log(l + 1)) = O(dst \log(st + 1))$ be its VC-dimension. We claim that $f(d, s, t) \le n + 1$. Indeed, Let P be a set of $n + 1$ points in $\mathbb{R}^d$. Since P cannot be shattered by the hyperedges in H there exists a non-trivial subset $A \subset P$ such that no hyperedge $S \in E$ has the property that $S \cap P = A$. In other words there does not exists a set K which is the union of convex polytopes with a total of at most l facets such that $K \cap P = A$. We claim that the partition $P = A \cup (P \setminus A)$ has the property that every s-convex set containing A must intersect any t-convex set containing $P \setminus A$. Indeed, assume to the contrary that there exists an s-convex set C_1 containg A and a t-convex set C_2 containing $P \setminus A$ such that $C_1 \cap C_2 = \emptyset$. Then by Lemma 2.1 there exists a set K which is the union of convex polytopes with a total of at most $l = st$ facets containing C_1 with complement $\overline{K}$ containing C_2. In particular, K contains A and $\overline{K}$ contains $P \setminus A$ so $K \cap P = A$, a contradiction. This completes the proof. □

3. Generalized Tverberg's Theorem. In this section we tackle Problem 1.4. In what follows we provide a bound on $f_r(d, s_1, \ldots, s_r)$. For the sake of simplicity of computations, we assume that $s_1 = s_2 = \cdots = s_r = s$ and abuse the notation writing $f_r(d, s)$ for $f_r(d, s, s, \ldots, s)$. Our proof technique can be easily modified to make the bound sensitive to any r integer parameters $s_1, \ldots, s_r$ in the more general setting.

Our argument is based on the notion defined in 1.14, which is an extension of the notion of VC-dimension to partitions with more than 2 sets.

We need Lemma 1.15. We proceed with its proof.

Proof. Let $r \geq 2$ be an integer. Suppose that

$$\left(\sum_{i=0}^{d}\binom{f}{i}\right)^r < \left(\frac{r}{r-1}\right)^f \tag{3.1}$$

Then any subset S of f vertices cannot be r-shattered. Namely, there exists a partition $S = \bigcup_{i=1}^{r} S_i$ so that whenever we have r hyperedges $e_1, e_2, \ldots, e_r \in E$ such that $S_i \subset e_i$ for all $i \in [r]$ it must hold that $S \cap \bigcap_{i=1}^{r} e_i \neq \emptyset$.

Note that the inequality (3.1) holds for $f = Cdr^2 \log r$ for some absolute constant C so any r-shattered set has size at most $f-1$.

Suppose the statement of the lemma is false and there is a set S that violates the condition. For every (ordered) r-tuple of hyperedges $e_1, e_2, \ldots, e_r$ with no common intersection in S, every point of S belongs to at most $r-1$ of these hyperedges. Therefore, these fixed r hyperedges can be used to provide at most $(r-1)^f$ partitions into r sets $S_1, \ldots, S_r$. Indeed every point among the f points of S has at most $r-1$ options to decide to which part of the partition it belongs - if, for example, a point lies in all hyperedges e_1 and e_2 up to e_{r-1} it can be in the partition in either S_1 or S_2 up to S_{r-1} but not in S_r, and similarly for each other case. By the Sauer-Shelah-Perles Lemma 1.13, the number of ordered r-tuples of intersections of hyperedges with S is at most

$$\left(\sum_{i=0}^{d}\binom{f}{i}\right)^r$$

Since we have to cover all r^f ordered partitions of S into r parts we get

$$\left(\sum_{i=0}^{d}\binom{f}{i}\right)^r \cdot (r-1)^f \geq r^f$$

This contradicts the assumption (3.1) and completes the proof. □

We also need the following simple geometric result where we prove that disjoint s-convex sets can each be enclosed in an s-convex polytope such that all those s-convex polytopes are also disjoint. Moreover, we provide an upper bound on the total number of facets of each such enclosing s-convex polytope.

Lemma 3.1. *Let $C_1, C_2, \ldots, C_r$ be r sets in $\mathbb{R}^d$ where each set is an s-convex set. Assume that $\bigcap_{i=1}^{r} C_i = \emptyset$. Then there exist r sets $K_1, \ldots, K_r$ where each K_i is the union of s convex polytopes with a total of at most s^r-facets such that $C_i \subset K_i$ for all $i \in [r]$ and $\bigcap_{i=1}^{r} K_i = \emptyset$.*

Proof. Since C_i is s-convex for any $i \in [r]$,it can be written as $C_i = \bigcup_{j=1}^{s} X_{i,j}$ for some convex sets $X_{i,1}, \ldots, X_{i,s}$. Since $\bigcap_{i=1}^{r} C_i = \emptyset$, then for every $i \in [r]$ we have that C_i is disjoint from $B_i = \bigcap_{j \neq i} C_j$. Note that each such B_i is the union of at most s^{r-1} convex sets since it is an $(r-1)$-fold intersection of unions of s-convex sets. We construct the sets $K_1, \ldots, K_r$ one by one. First, we replace C_1 by K_1 separating it from the union of at most s^{r-1} convex sets $B_1 = \bigcap_{j>1} C_j$. As before, this can be done with K_1 which is the union of s convex polytopes with a total of at most s^r facets. In particular K_1 is also s-convex. Also $C_1 \subset K_1$. Moreover, $K_1 \cap \bigcap_{j>1} C_j = \emptyset$. We then apply the same argument to C_2 as the intersection of $K_1, C_2, .., C_r$ is empty and all the sets are s-convex. So we can find a set K_2 which is s-convex and consists of the union of s convex poytopes with a total of at most s^r facets and such that $C_2 \subset K_2$ and $K_1 \cap K_2 \cap \bigcap_{j>2} C_j = \emptyset$. Continuing in the same manner we conclude that each C_i can be replaced with such a K_i so that $C_i \subset K_i$ for all $i \in [r]$, each K_i is the union of s convex polytopes with a total of s^r factes and $\bigcap_{i=1}^r K_i = \emptyset$. This completes the proof of the lemma. □

We are now ready to prove the following theorem extending Theorem 1.5 to partitions with $r > 2$ parts, and generalizing Tverberg's theorem to s-convex sets:

Proof. [**Proof of Theorem 1.6 (for $s_1 = s_2 = \ldots = s_r = s$)**] **Put $l = s^r$. Let $H = (\mathbb{R}^d, E)$ be the hypergraph as in Lemma 2.2. Let $d' = O(dl \log(l+1))$ be its VC-dimension. Let n be the maximum size of an r-shattered set. Note that by Lemma 1.15 $n = O(d'r^2 \log r) = O(dr^2 \log r \cdot s^r \cdot \log(s^r + 1))$**

We claim that $f_r(d, s) \leq n + 1$. Indeed, let P be a set of $n + 1$ points in $\mathbb{R}^d$. Since P cannot be r-shattered by the hyperedges in H there exists a partition $P = \bigcup_{i=1}^r P_i$ such that whenever we have r hyperedges $e_1, \ldots, e_r \in E$ with $P_i \subset e_i$ for each $i \in [r]$ it follows that $P \cap \bigcap_{i=1}^r e_i \neq \emptyset$. In other words there do not exist sets $K_1, \ldots, K_r$ each of which is the union of convex polytopes with a total of at most s^r facets such that $P_i \subset K_i$ for every $i \in [r]$ and $\bigcap_{i=1}^r K_i = \emptyset$. We claim that the partition $P = \bigcup_{i=1}^r P_i$ has the property that for every family of r sets $C_1, \ldots, C_r$ such that for each $i \in [r]$ $P_i \subset C_i$ and each C_i is an s-convex set it must hold that $\bigcap_{i=1}^r C_i \neq \emptyset$. Indeed, assume to the contrary that there exist $C_1, \ldots, C_r$ such that for each $i \in [r]$ $P_i \subset C_i$, each C_i is an s-convex set and such that $\bigcap_{i=1}^r C_i = \emptyset$. Then by Lemma 3.1 there exist sets $K_1, \ldots, K_r$ for which $C_i \subset K_i, \forall i \in [r]$ and each K_i is the union of convex polytopes with a total of at most $l = s^r$ facets and $\bigcap_{i=1}^r K_i = \emptyset$, a contradiction. This completes the proof. □

4. A lower bound for $f_r(d, s)$. In the title of this section as well as in its content and in the next two sections we denote the function $f_r(d, s_1, s_2, \ldots, s_r)$ in which $s_i = s$ for all i by $f_r(d, s)$. An easy lower bound for the function $f_r(d, s)$ can be proved by taking s translated copies of an extremal example for the classical theorem of Tverberg with pairwise disjoint convex hulls. This gives the following.

PROPOSITION 4.1. $f_r(d, s) > s(d + 1)(r - 1)$

This is, of course, tight for $s = 1$, as Tverberg's Theorem is tight. In this section we show that, somewhat surprisingly, for $s \geq 3$ the lower bound becomes quadratic in r. As we show in the next section, this is the case even in dimension $d = 1$, where the lower bound is tight up to a constant factor for all r and $s \geq 3$. For convenience we describe the proof for even r, a similar bound for odd r follows from the one for $r - 1$.

THEOREM 4.2. *For every $d \geq 1, s \geq 3$ and even $r \geq 2$,*

$$f_r(d, s) > \frac{1}{4} \left(\left\lfloor \frac{d}{2} \right\rfloor + 1 \right) \left\lfloor \frac{s-1}{2} \right\rfloor r^2$$

Proof. Put $m = (\lfloor \frac{d}{2} \rfloor + 1)r/2$, $p = \lfloor \frac{s-1}{2} \rfloor r/2$ and $n = mp$. Let P be a set of n points on the moment curve in $\mathbb{R}^d$ consisting of the n points $z_i = (t_i, t_i^2, \ldots t_i^d)$, where $0 < t_1 < t_2 < \ldots < t_n < 1$. Let $I_1, I_2, \ldots, I_p$ be p open intervals that cover $(0, 1)$ and appear on it in this order, where the left endpoint of each interval I_{j+1} is just slightly smaller than the right endpoint of I_j for each j. The intervals are chosen so that each t_i belongs to exactly one such interval, and each interval contains exactly m points t_j.

In order to complete the proof we show that for any coloring of the points of P by r colors there are s-convex sets C_i, with C_i containing all points of color i, so that the intersection of all the sets C_i is empty. Each set C_i will be defined as the union of at most s convex sets, where each of the convex sets will be the convex hull of points of color i with first coordinate lying in a union of some consecutive intervals I_q. The crucial property of the definition of these convex sets is that for each interval I_q there will be an index i so that one of the convex sets in the definition of C_i will be the set of all points $z_j = (t_j, t_j^2, \ldots, t_j^d)$ of color i for which t_j lies in this single interval I_q. Moreover, this will be done in a way that ensures that the number of such points is always at most $\lfloor d/2 \rfloor$.

Note that if such a choice indeed exists, then the intersection of the corresponding r sets C_i will be empty as needed. Indeed, otherwise the intersection is some point $z \in \mathbb{R}^d$ (not necessarily on the moment curve). Let t denote the first coordinate of this point z. Suppose first that t belongs only to one interval I_q, and let i be the color chosen for I_q so that there are at most $\lfloor d/2 \rfloor$ points of color i with their first coordinate in I_q. Then z has to lie in the convex hull of these points (as any other convex set among the ones defining C_i either has all its points with first coordinate smaller than t or all its points with first coordinate larger than t). Since any finite subset of the moment curve is $\lfloor d/2 \rfloor$-neighborly[1], this convex hull is disjoint from the convex hull of all the other points of P, implying that z cannot lie in any other set C_g besides C_i. If t belongs to two intervals, say I_{q-1} and I_q and i is defined as before, then z cannot lie in C_i, since in this case any convex set among those defining C_i either has all its points with first coordinate smaller than t or all its points with first coordinate larger than t.

It thus only remains to show that we can choose for every interval I_q a color i with the required properties, making sure that no fixed color is chosen more than $\lfloor (s-1)/2 \rfloor$ times (as this way each set C_i will be the union of at most s convex sets). To do so we go over the intervals I_q one by one, in an arbitrary order (for example, from left to right). When dealing with the interval I_q after handling several previous ones, we first note that since there are exactly $m = (\lfloor d/2 \rfloor + 1)r/2$ points with first coordinate in I_q, there are at least $r/2$ colors that appears at most $\lfloor d/2 \rfloor$ times in I_q. As so far we have selected the colors i for at most $p - 1 = \lfloor (s-1)/2 \rfloor r/2 - 1$ intervals, there are less than $r/2$ colors i that have already been chosen $\lfloor (s-1)/2 \rfloor$ times. It follows that there is at least one color i that can be chosen for the interval I_q, implying that the process terminates successfuly. This completes the proof of the theorem. □

Remark 4.3. *Note that for $d = 1$, any hypergraph whose vertices are an arbitrary set of points on the line, and whose hyperedges are the intersections of this set of points with union of s intervals, has VC-dimension at most $2s$. Indeed, it is easily verified that no set of size $2s+1$ is shattered. Assume to the contrary that there is a set $P = \{x_1, \ldots, x_{2s+1}\}$ that is shattered where the points are indexed by their increasing order along the line. Notice that every union of s-intervals that contains only the $s+1$ points with odd indices $\{x_1, x_3, \ldots, x_{2s+1}\}$ must have one of its intervals containing two consecutive such points and hence also a point with an even index, a contradiction. Therefore, Theorem 4.2 in dimension 1 provides a hypergraph with*

[1] A set of points is r-neighborly if any subset of at most r points form a face of the convex-hull of the set.

VC-dimension bounded by $D = 2s$ and an r-shattered set of size at least

$$\frac{1}{4}\left\lfloor\frac{s-1}{2}\right\rfloor r^2 = \frac{1}{4}\left\lfloor\frac{D-2}{4}\right\rfloor r^2.$$

This establishes the claimed lower bound in Lemma 1.15

5. Improved bounds in special cases. In this section we establish improved upper and lower bounds for the functions f and f_r in several special cases.

Theorem 5.1.

$$f(2, s, 1) = 2s + 2$$

and for every $d \geq 4$

$$f(d, s, 1) = \Theta(ds \log(s + 1))$$

Proof. Let l denote the VC-dimension of a hypergraph defined by points in $\mathbb{R}^d$ with respect to convex polytopes with at most s facets. We first show that $f(d, s, 1) = l+1$. Combined with the known bounds on the VC-dimension of such hypergraphs we get the claimed bounds. If l is the VC-dimension of such a hypergraph then there exists a shattered set P of size l. Then for any parition $P = P_1 \cup P_2$ there is a polytope K with at most s facets such that $P_2 \subset K$ and $K \cap P_1 = \emptyset$. In particular P_1 is contained in the union of the at most s complement halfspaces supporting the facets of K, which is an s-convex set (consisting of the at most s complement halfspaces), and P_2 lies in its complement. Since this holds for any partition it follows that $f(d, s, 1) \geq l + 1$. In order to show equality we need to show that every $l + 1$ point set in $\mathbb{R}^d$ admits a partition that cannot be realized by intersections with disjoint convex and s-convex sets. This follows by the same argument from the fact that any set P of $l + 1$ points cannot be shattered in the corresponding hypergraph and therefore there is at least one partition $P = P_1 \cup P_2$ so that no polytope with at most s facets can contain P_2 while being disjoint from P_1. Hence any s-convex set C containing P_1 must intersect the convex hull $CH(P_2)$ for otherwise by separation arguments as above there would be a polytope which is the intersection of s half spaces containing P_2 and disjoint from P_1, a contradiction. It is a simple exercise to see that in the plane ($d = 2$), the VC-dimension is $l = 2s + 1$ and thus $f(2, s, 1) = l + 1 = 2s + 2$. The statement for any fixed $d \geq 4$ follows from the results in [10]. □

Theorem 5.2.

$$f(2, s, s) \geq 4s$$

Proof. Let P be a set of $4s - 1$ points placed along a cycle U in the plane. We have to show that for any partition of P into two disjoint sets A and B there are disjoint s-convex sets C_1 and C_2 so that $A \subset C_1$ and $B \subset C_2$. Partition the set of points P into disjoint subsets, where each subset is a maximal subset of P consisting of consecutive points along U that are all in A or all in B. Let this partition be $A_1, B_1, A_2, B_2, \ldots, A_t, B_t$, where each A_i is a subset of A and each B_i is a subset of B, and the sets appear in this order along U. Clearly $t \leq 2s - 1$, since each of the sets A_i, B_j is nonempty. If $t < 2s - 1$ define $A_j = B_j = \emptyset$ for all $t < j \leq 2s - 1$. Let C_1 consist of the union of the following s convex sets (some of which may be empty): the convex hull of $A_1 \cup A_2 \cup \ldots \cup A_s$, the convex hull of A_{s+1}, the convex hull of A_{s+2},

..., the convex hull of A_{2s-1}. Then C_1 is s-convex and contains A. Similarly, let C_2 be the union of the following s convex sets: the convex hull of $B_s \cup B_{s+1} \cup \ldots \cup B_{2s-1}$, the convex hull of B_1, the convex hull of B_2, ..., the convex hull of B_{s-1}. Then C_2 is clearly s-convex and contains B. It is easy to check that C_1 and C_2 are disjoint, completing the proof. □

Theorem 5.3.

$$f(3, s, 1) \leq 4s + 1$$

Proof. We need to prove that for every set P of $4s+1$ points there exists a partition $P = A \cup B$ such that any convex set containing A must intersect any s-convex set containing B. Notice that by the above arguments it is enough to prove that the VC-dimension of the hypergraph $H = (P, E)$ where E is the family of all intersections of P with a convex polytope with at most s facets is bounded by $4s$. This fact is proved in [11]. For completeness we include a proof based on an argument in [19]. Assume to the contrary there there exists a set $P \subset \mathbb{R}^3$ of size $4s+1$ that is shattered by H. In particular, for any partition $P = A \cup B$ if A can be separated from B by a convex polytope with at most s facets, then there exists a set of s half spaces whose union contains B but none of the points in A.

Next, we need the following fact which was proved in [19]: There exists a 4-coloring of the points of P such that no halfspace that contains at least two points of P is monochromatic.

Consider such a coloring. By the pigeonhole principle there is a monochromatic set $B \subset P$ of size at least $s+1$. We claim that B cannot be separated from $A = P \setminus B$ with a union of only s halfspaces. Indeed, if such s halfspaces exist then one of them must contain at least 2 points of B and none of the points in A so such a halfspace cuts off a monochromatic set of points, a contradiction. This completes the proof.
□

We next show that $g(3, s, s)$ is super-linear in s.

Theorem 5.4.

There exists a function $g(s)$ tending to infinity as s tends to infinity so that

$$f(3, s, s) \geq sg(s)$$

Proof. We make no attempt to optimize the function $g(s)$, and only show that it can be chosen as a function tending to infinity with s. Following the approach in [1], the proof applies the result of Furstenberg and Katznelson [12] known as the density Hales-Jewett Theorem. For an integer $k \geq 2$, put $[k] = \{1, 2, \ldots, k\}$ and let $[k]^d$ denote the set of all vectors of length d with coordinates in $[k]$. A *combinatorial line* is a subset $L \subset [k]^d$ so that there is a set of coordinates $I \subset [d] = \{1, 2, \ldots, d\}$, $I \neq [d]$, and values $k_i \in [k]$ for all $i \in I$ for which L is the following set of k members of $[k]^d$:

$$L = \{\ell_1, \ell_2, \ldots, \ell_k\}$$

where

$$\ell_j = \{(x_1, x_2, \ldots, x_d) \; : \; x_i = k_i \text{ for all } i \in I \text{ and } x_i = j \text{ for all } i \in [d] \setminus I\}.$$

Thus a combinatorial line is a set of k vectors all having some fixed values in the coordinates in I, where the jth vector has the value j in all other coordinates. In this notation, the Furstenberg-Katznelson Theorem is the deep result that for any fixed integer k and any fixed $\delta > 0$ there exists an integer $d_0 = d_0(k, \delta)$ so that for any $d \geq d_0$, any set Y of at least δk^d members of $[k]^d$ contains a combinatorial line.

Fix a small positive real δ, a large integer k and a huge integer $d = d_0(k, \delta)$ defined as above. View the points of $[k]^d$ as points in the d-dimensional real space $\mathbb{R}^d$ and call a (geometric) line in this space *special* if it contains all k points of a combinatorial line.

By the claim in the proof of Theorem 1.3 in [1] the only points that belong to at least two special lines are the points of $[k]^d$. Equivalently, if two such lines do not have a common point of $[k]^d$ then the full geometric lines are disjoint.

Now project all the configuration of the $n = k^d$ points above randomly to the 3-dimensional space $\mathbb{R}^3$. This gives a set P of n points in $\mathbb{R}^3$. With probability 1, the condition about the intersection of the projected special lines still holds: if two of them do not have a point of P in common, then they are disjoint.

Fix a partition of P into two disjoint sets A and B. The following procedure partitions each of these sets into less than $\delta k^d + k^{d-1}$ pairwise disjoint subsets, which also have pairwise disjoint convex hulls.

Starting with the full set A, as long as it contains at least δk^d points choose a combinatorial line in it. Define a subset consisting of the projected images of the points of this line, and remove all these points from A. Once the remaining size of A is smaller than δk^d take every single point as a subset. Handle B in the same way. Defining s as $s = \delta k^d + k^{d-1}$, this shows that $f(3, s, s) > n = k^d$. Since δ is arbitrarily small and k is arbitrarily large, this shows that the ratio $g(s) = f(3, s, s)/s$ tends to infinity as s tends to infinity, completing the proof. It is worth noting that the estimate for the growth of $g(x)$ can be improved using the same reasoning together with the results in [5], but this will still leave a large gap between the upper and lower bounds we know for $f(3, s, s)$. □

The following simple result shows that the lower bound for $f_r(d, s)$ proved in Section 4 is tight up to a constant factor in dimension $d = 1$.

Theorem 5.5.

$$f_r(1, s) \leq r(r-1)(s+1) + 1$$

Proof. Put $n = r(r-1)(s+1) + 1$ and let $0 < p_1 < p_2 < \ldots < p_n < 1$ be a set of n points on the line. We have to show that there is a coloring of these points by r colors $0, 1, 2, \ldots, r-1$, so that for any r sets C_i, where each C_i is a union of at most s intervals that covers all points of color i, there is a point that lies in all sets C_i. Naturally, the coloring we choose colors the points periodically, that is, the color of p_i is defined to be i mod r. Let C_i be collections of intervals as above. Note that each C_i must contain all points of P besides at most $(s+1)(r-1)$. Indeed, since it contains all points colored i, the gap between any two consecutive intervals in it contains at most $r-1$ points, and the same holds for the gap between 0 and its leftmost point and between 1 and its rightmost point. (We note that here we can slightly improve the bound since, for example, the gap between 0 and the leftmost point of C_i can contain at most i points p_i). It follows that if $n > r(r-1)(s+1)$ then there is a point (of P, although that's not needed) that belongs to all sets C_i. This completes the proof. □

6. Concluding remarks and open problems. We established extensions of Radon's Theorem and Tverberg's Theorem for unions of convex sets. The main tools in the proofs are upper bounds for the shatter functions of range spaces with a bounded

VC-dimension as well as an extension of these results. This extension, defined and studied here, is useful in the study of partitions with more than 2 parts, which are the ones considered in the classical definition of the VC-dimension.

As mentioned in the introduction, already for $s_i = 1$ for all i the upper bound provided in our Theorem 1.9 is $O(dr^2 \log(r+1))$. It improves the upper bound of $c(d)r^2 \log^2(r+1)$ by Bukh [9] who proved it for the more general setting of (not necessarily separable) abstract convexity spaces. In [18] Pálvölgyi provided an upper bound of the form $O(d^{d^{d^{\log d}}} r)$ which is linear in r but super exponential in d. It will be interesting to decide if one can get rid of the separability assumption.

While our upper and lower bounds for the functions $f(d,s,t)$ and $f_r(d, s_1, s_2, \ldots, s_r)$ proved here are not very far from each other, the problem of determining them precisely remains open for most values of the parameters. It will be interesting to close the gap between the upper and lower bounds. The following specific questions are particularly intriguing.

Problem 6.1.

Is $f_2(s,s)$ linear in s ?

Problem 6.2.

Is $f_r(d,s)$ upper bounded by a polynomial in r,d and s ?

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