Packing Ferrers Shapes

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Abstract

Answering a question of Wilf, we show that if n is sufficiently large, then one cannot cover an $n \times p(n)$ rectangle using each of the $p(n)$ distinct Ferrers shapes of size n exactly once. Moreover, the maximum number of pairwise distinct, non-overlapping Ferrers shapes that can be packed in such a rectangle is only $\Theta(p(n)/\log n)$.

1 Introduction

A *partition* p of a positive integer n is an array $p = (x_1, x_2, \cdots, x_k)$ of positive integers so that $x_1 \geq x_2 \geq \cdots \geq x_k$ and $n = \sum_{i=1}^{k} x_i$. The x_i are called the *parts* of p. The total number of distinct partitions of n is denoted by $p(n)$. A *Ferrers shape* of a partition $p = (x_1, x_2, \cdots, x_k)$ is a set of n square boxes with sides parallel to the coordinate axes so that in the ith row we have x_i boxes and all rows start at the same vertical line. The Ferrers shape of the partition $p = (4, 2, 1)$ is shown in Figure 1. Clearly, there is an obvious bijection between partitions of n and Ferrers shapes of size n.

If we reflect a Ferrers shape of a partition p with respect to its main diagonal, we get another shape, representing the *conjugate* partition of p. Thus, in our example, the conjugate of (4,2,1) is (3,2,1,1).

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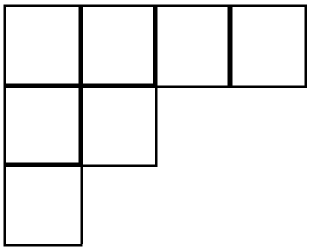

Figure 1: The Ferrers shape of (4,2,1)

Recently, Herb Wilf [6] has asked the following intriguing question: consider all distinct Ferrers shapes consisting of n boxes. Is it true that for n sufficiently large, one can always tile a rectangle of side lengths $p(n)$ and n using each of these shapes exactly once ? Obviously, in such a tiling, if one exists, the shapes cannot overlap each other. For small values of n, one gets mixed answers: for $n = 1, 2, 4$ such a tiling exists, however, for $n = 3$ there is no such tiling.

In this short paper we answer Wilf's question in the negative by showing that for n sufficiently large no such tiling exists. In fact, we prove the following stronger statement.

Theorem 1 *If n is sufficiently large, then one cannot cover an $n \times p(n)$ rectangle by using each of the $p(n)$ distinct Ferrers shapes of size n exactly once. Moreover, the maximum fraction of the area of this rectangle that can be covered by non-overlapping distinct Ferrers shapes of size n is at most $\frac{c}{\log n}$, for some absolute constant c.*

The $c/\log n$ upper bound is tight, up to the constant c, and shows that as n grows we cannot even cover a fixed fraction of the area by non-overlapping distinct shapes.

To prove the result, we use some geometric properties that are shared by the vast majority of the Ferrers shape of size n and imply that these shapes cannot be packed in an efficient way. The geometric properties we need can be derived from the extensive available information on the typical form of a Ferrers shape, given, for example, in [4], as well as in several earlier papers. However, in order to make the paper self-contained, we prefer to derive all of them directly from the Hardy-Ramanujan asymptotic formula for $p(n)$. This is done in the next section. In Section 3, we apply the geometric properties to prove our main result. Throughout the paper we assume, whenever this is needed, that the size n of the Ferrers shapes considered is sufficiently large.

2 Some geometric properties of typical Ferrers shapes

The corner of the first row and first column of a Ferrers shape will be called the *apex* of that shape. In this section we prove some asymptotic geometric properties of Ferrers shapes of size n. Our basic tool is the well-know Hardy-Ramanujan asymptotic formula for the number of shapes of size n, see, e.g., [1]. It asserts that

$$p(n) = (1 + o(1)) \frac{e^{C\sqrt{n}}}{4n\sqrt{3}}, \tag{1}$$

where $C = \pi\sqrt{\frac{2}{3}}$, and the $o(1)$-term tends to 0 as n tends to infinity.

Lemma 1 *Let $x_1 \geq x_2 \geq \ldots \geq x_k$ denote the parts of the partition corresponding to the Ferrers shape F, and let $y_1 \geq y_2 \ldots \geq y_s$ denote the parts of the conjugate partition. The following hold for all but at most $p(n)/(\log n)^2$ Ferrers shapes F of size n.*

- *I. There exists an absolute constant c_1 so that for n sufficiently large, we have $c_1\sqrt{n}\log n < x_1$ and also, $c_1\sqrt{n}\log n < y_1$.*
- *II. There exists an absolute constant $c_2 > 0$ so that F has at least $c_2\sqrt{n}$ parts of size at least $c_2\sqrt{n}$ each.*

Proof:

- I. This follows from classical results. Erdős and Lehner [2] proved that for almost all partitions of n the largest term and the number of terms differ from $\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n$ by less than $c\sqrt{n}\omega(n)$, where $\omega(n) \to \infty$ arbitrary slowly. A result of Szalay and Turán (Theorem IV in [5]) makes this information more precise by showing that this holds for all but $O(p(n)e^{-\omega(n)})$ partitions. To get the required result, set $\omega(n) = 2\log\log n$.

  For self-containment, however, we include a short direct proof for this lemma. The inequalities for x_1 and y_1 are clearly equivalent by taking conjugates. Thus it suffices to prove the statement for x_1.

  We need to prove that for almost all partitions we have $x_1 > c_1\sqrt{n}\log n$, for some positive constant c_1. Let S be the set of partitions of n violating this constraint, and attach two additional parts x_0 and x_{-1} in all possible ways to all partitions in S so that the following hold:

  $$x_0 + x_{-1} = 3 \cdot [c_1\sqrt{n}\log n],$$

  and

  $$x_{-1} \geq x_0 \geq c_1\sqrt{n}\log n.$$

  Let S' be the set of partitions obtained this way. It then follows that x_{-1} and x_0 are the two largest parts in all partitions in S', and that S' contains partitions of the integer $n+3\cdot[c_1\sqrt{n}\log n]$. As $x_{-1} \geq x_0$, we must have $1.5 \cdot c_1\sqrt{n}\log n \leq x_{-1} \leq 2 \cdot c_1\sqrt{n}\log n$, so we have $0.5 \cdot c_1\sqrt{n}\log n$ choices for x_{-1}. This implies

  $$|S'| = |S| \cdot 0.5 \cdot c_1\sqrt{n}\log n \leq p(n + 3 \cdot [c_1\sqrt{n}\log n]),$$

  which yields

  $$|S| = \frac{|S'|}{0.5 \cdot c_1\sqrt{n}\log n} \leq \frac{p(n + 3 \cdot [c_1\sqrt{n}\log n])}{0.5 \cdot c_1\sqrt{n}\log n} \leq \frac{p(n) \cdot e^{1.5Cc_1\log n}}{0.5 \cdot c_1\sqrt{n}\log n} \leq \frac{p(n) \cdot n^{1.5Cc_1}}{0.5 \cdot c_1\sqrt{n}\log n},$$

  as $\sqrt{n + 3c_1\sqrt{n}\log n} < \sqrt{n} + 1.5c_1\log n$. By choosing a sufficiently small c_1 (e.g., $c_1 = 1/1000$), we see that all but $p(n)/n^{0.49}$ partitions satisfy $x_1 \geq c_1\sqrt{n}\log n$. (Note that it is not difficult to show that in fact for every fixed r there is some $c_1 > 0$ so that for all but at most $p(n)/n^r$ partitions of n, $x_1 \geq c_1\sqrt{n}\log n$. This can be done by adding to each partition that does not satisfy the above more than 2 parts with a prescribed sum, and by repeating the above argument. For our purpose here, however, the above estimate suffices.)

- II. It is known ([5], Theorem II) that for any λ satisfying

$$11\log n \leq \lambda \leq \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n - 3\sqrt{n}\log\log n,$$

the number of terms exceeding λ is

$$\left(1 + O(\frac{1}{\log n})\right)\frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-e^{-\pi\lambda/\sqrt{6n}}},$$

with the exception of $O(p(n)n^{-7/4})$ partitions, and our claim follows.

Here, too, we inlcude a self-contained, short proof relying only on (1). Let $S = \{\ p :\ \ p = (x_1, \ldots, x_k)$ is a partition of n so that $|\{i|x_i \geq c_2\sqrt{n}\}| < c_2\sqrt{n}\}$. Let $\mathcal{F}$ be a family of at least, say, $2^{0.1c_2\sqrt{n}}$ subsets of a set of cardinality $10c_2\sqrt{n}$ in which the Hamming distance between any two subsets is larger than $2c_2\sqrt{n}$. (The existence of such a family is easy and follows from the Gilbert-Varshamov bound, see, e.g., [3].) Let the underlying set of $\mathcal{F}$ be the set $\{c_2\sqrt{n}+1, c_2\sqrt{n}+2, \cdots, 11c_2\sqrt{n}\}$.

Define $S' = \{P \cup F | P \in S, F \in \mathcal{F}\}$. It is not too difficult to check that all $|S| \cdot |\mathcal{F}|$ partitions in S' are pairwise distinct; indeed, if two such unions have the same P or the same F then they clearly differ. On the other hand, for distinct P, P' in S and distinct F, F' in $\mathcal{F}$, $P \cup F$ and $P' \cup F'$ do not have the same sets of parts of size bigger than $c_2\sqrt{n}$, by the definition of S and the choice of $\mathcal{F}$. Therefore, we have

$$|S'| = |S| \cdot |\mathcal{F}|$$

which yields

$$|S| \leq \frac{|S'|}{2^{0.1c_2\sqrt{n}}}. \tag{2}$$

All the elements of S' are partitions of integers not larger than $n + 110c_2^2 n$. As these are all distinct it follows that

$$|S'| \leq \sum_{k \leq n+110c_2^2 n} p(k) \leq e^{C\sqrt{n(1+110c_2^2)}} \leq e^{C\sqrt{n}+55C\cdot c_2^2\sqrt{n}},$$

and therefore, by inequality (2),

$$|S| \leq \frac{e^{C\sqrt{n}}e^{55C\cdot c_2^2\sqrt{n}}}{2^{0.1c_2\sqrt{n}}}$$

which gives the desired result, as by choosing, say, $c_2 = 0.001$, the second term of the numerator becomes much smaller than the denominator.

This completes the proof of Lemma 1. $\diamond$

3 The proof of the main result

In this section we prove Theorem 1. Let us call a partition having properties I and II *regular*. Assume that $Kp(n)/\log n$ disjoint Ferrers shapes are packed in an $n \times p(n)$ rectangle for n large enough. We may and will assume, without loss of generality, that all the apexes are in the left upper corner of the Ferrers shapes used, and that all the partitions are regular.

Define a *diamond* as follows. The diamond is a plane region which is the union of a disk and two triangles. The disk is centered at the origin of the xy plane and has radius $\frac{c_2}{4}\sqrt{n}$. One triangle is the convex hull of the vertices

$$(0, \frac{c_2}{4}\sqrt{n}), (\frac{c_2}{4}\sqrt{n}, 0), (\frac{c_1}{\sqrt{2}}\sqrt{n}\log n, \frac{c_1}{\sqrt{2}}\sqrt{n}\log n).$$

The second triangle is the mirror image of the first triangle with respect to $(0, 0)$. The point $(0, 0)$ is the center of the diamond.

For every apex of a Ferrers shape in the packing, draw a translated copy of the diamond centered at the apex. Note that the diamonds associated with the apexes are pairwise disjoint. Indeed, if the disk of a diamond intersects with the disk of another one, then looking at the corresponding partitions, we see that their Durfee squares are overlapping. In case of any other intersection, either the squares are ovelapping, or the first row of one of the partitions intersects the first column of the other.

Figure 2: A diamond

For the last step, define the *exceptional region* as the four quarterdisks around the four corners of the $n \times p(n)$ rectangle, with radius $\frac{c_2}{2}\sqrt{n}\log n$ each. It is easy to check that diamonds centered in the rectangle but not in the exceptional region have at least half of their areas in the rectangle. The exceptional region may contain very few ($O(\log^2 n)$) apexes of partitions, since apexes have distance at least $c_2\sqrt{n}$ from each other. Each diamond not located in the exceptional region covers at least $(\frac{c_1}{\sqrt{2}}\sqrt{n}\log n \times \sqrt{2}\frac{c_2}{4}\sqrt{n})/2 = \frac{c_1 c_2}{8} n \log n$ from the rectangle, and those pieces are disjoint. Since the area of the rectangle is $np(n)$, this gives an absolute constant upper bound for K, completing the proof. $\square$

It is not difficult to see that the assertion of the theorem is tight, up to the multiplicative constant c. Indeed, one can first omit all shapes for which, in the notation of Lemma 1, either $x_1 > C\sqrt{n}\log n$ or

$y_1 > C\sqrt{n}\log n$, where C is an absolute constant chosen to ensure that there are less than $p(n)/\log n$ such shapes (it is easy to see that such a C exists.) Then it is possible to pack the remaining shapes along diagonals, where the apex of each shape touches the furthest point on the main diagonal of the previous shape. An illustration appears in Figure 3.

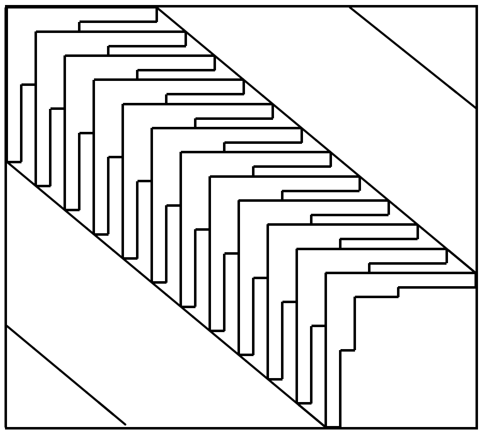

Figure 3: An efficient packing

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