

Fair representation by independent sets

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Abstract

For a hypergraph H let $\beta(H)$ denote the minimal number of edges from H covering $V(H)$. An edge S of H is said to represent *fairly* (resp. *almost fairly*) a partition (V_1, V_2, \dots, V_m) of $V(H)$ if $|S \cap V_i| \geq \lfloor \frac{|V_i|}{\beta(H)} \rfloor$ (resp. $|S \cap V_i| \geq \lfloor \frac{|V_i|}{\beta(H)} \rfloor - 1$) for all $i \leq m$. In matroids any partition of $V(H)$ can be represented fairly by some independent set. We look for classes of hypergraphs H in which any partition of $V(H)$ can be represented almost fairly by some edge. We show that this is true when H is the set of independent sets in a path, and conjecture that it is true when H is the set of matchings in $K_{n,n}$. We prove that partitions of $E(K_{n,n})$ into three sets can be represented almost fairly. The

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methods of proofs are topological.

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1 Introduction

1.1 Terminology and main theme

A hypergraph \mathcal{C} is called a *simplicial complex* (or just a “complex”) if it is closed down, namely $e \in \mathcal{C}$ and $f \subseteq e$ imply $f \in \mathcal{C}$. We denote by $V(\mathcal{C})$ the vertex set of \mathcal{C} , and by $E(\mathcal{C})$ its edge set. Let $\beta(\mathcal{C})$ be the minimal number of edges (“simplices”) of \mathcal{C} whose union is $V(\mathcal{C})$. For any hypergraph H we denote by $\Delta(H)$ the maximal degree of a vertex in H .

We say that $S \in \mathcal{C}$ represents a set A of vertices *fairly* if $|S \cap A| \geq \left\lfloor \frac{|A|}{\beta(\mathcal{C})} \right\rfloor$, and that it represents A *almost fairly* if $|S \cap A| \geq \left\lfloor \frac{|A|}{\beta(\mathcal{C})} \right\rfloor - 1$. We say that S represents fairly (almost fairly) a collection of sets if it does so to each set in the collection, reminiscent of the way a parliament represents fairly the voters of the different parties.

Clearly, every set A is fairly represented by some edge $S \in \mathcal{C}$. The aim of this paper is to study complexes \mathcal{C} in which for every partition V_1, \dots, V_m of $V(\mathcal{C})$ there is an edge $S \in \mathcal{C}$ representing all V_i 's fairly, or almost fairly.

In matroids, fair representation is always possible. The following can be proved, for example, by the use of Edmonds' matroids intersection theorem.

Theorem 1.1. *If \mathcal{M} is a matroid then for every partition V_1, \dots, V_m of $V(\mathcal{M})$ there exists a set $S \in \mathcal{M}$ satisfying $|S \cap V_i| \geq \left\lfloor \frac{|V_i|}{\beta(\mathcal{C})} \right\rfloor$ for all i .*

Classical examples which do not always admit fair representation are complexes of the form $\mathcal{I}(G)$, the complex of independent sets in a graph G . In this case $\beta(\mathcal{I}(G)) = \chi(G)$, the chromatic number of G , which by Brooks' theorem is at most $\Delta(G) + 1$. Indeed, there are classes of graphs for which the correct proportion of representation is $\frac{1}{\Delta(G)+1}$:

Theorem 1.2. [3] *If G is chordal and V_1, \dots, V_m is a partition of its vertex set, then there exists an independent set of vertices S such that $|S \cap V_i| \geq \frac{|V_i|}{\Delta(G)+1}$ for all $i \leq m$.*

However, in general graphs this is not always true. The following theorem of Haxell [19] pinpoints the correct parameter.

Theorem 1.3. *If $\mathcal{V} = (V_1, V_2, \dots, V_m)$ is a partition of the vertex set of a graph G , and if $|V_i| \geq 2\Delta(G)$ for all $i \leq m$, then there exists a set S , independent in G , intersecting all V_i 's.*

This was an improvement over earlier results of Alon, who proved the same with $25\Delta(G)$ [9] and then with $2e\Delta(G)$ [10]. The result is sharp, as shown in [28, 20, 27].

Corollary 1.4. *If the vertex set V of a graph G is partitioned into independent sets V_1, V_2, \dots, V_m then there exists an independent subset S of V , satisfying $|S \cap V_i| \geq \left\lfloor \frac{|V_i|}{2\Delta(G)} \right\rfloor$ for every $i \leq m$.*

Proof. For each $i \leq m$ let V_i^j ($j \leq \left\lfloor \frac{|V_i|}{2\Delta(G)} \right\rfloor$) be disjoint subsets of size $2\Delta(G)$ of V_i . By Theorem 1.3 there exists an independent set S meeting all V_i^j , and this is the set desired in the theorem. \square

1.2 The special behavior of matching complexes

Matching complexes, namely the independence complexes of line graphs, behave better than independence complexes of general graphs. For example, the following was proved in [1]:

Theorem 1.5. *If G is the line graph of a graph and V_1, \dots, V_m is a partition of $V(G)$ then there exists an independent set S such that $|S \cap V_i| \geq \lfloor \frac{|V_i|}{\Delta(G)+2} \rfloor$ for every $i \leq m$.*

This follows from a bound on the topological connectivity of the independence complexes of line graphs.

$$\eta(\mathcal{I}(G)) \geq \frac{|V|}{\Delta(G)+2} \quad (1)$$

Here $\eta(\mathcal{C})$ is a connectivity parameter of the complex \mathcal{C} (for the definition see, e.g., [1]). The way from (1) to Theorem 1.5 goes through a topological version of Hall's theorem, proved in [5]. A hypergraph version of (1) was proved in [4]. Theorem 1.2 follows from the fact that if G is chordal then $\eta(\mathcal{I}(G)) \geq \frac{|V|}{\Delta(G)+1}$.

So, matching complexes are more likely to admit fair representations. We suggest four classes of complexes as candidates for having almost fair representation of disjoint sets.

1. The matching complex of a path.
2. The matching complex of $K_{n,n}$.
3. The matching complex of any bipartite graph.
4. The intersection of two matroids.

Since the third class contains the first two and the fourth contains the third, conjecturing almost fair representation for them goes in ascending order of daring. In fact, we only dare make the conjecture for the first two. As to the fourth, let us just remark that intersections of matroids often behave unexpectedly well with respect to partitions. For example, no instance is known to the authors in which, given two matroids \mathcal{M} and \mathcal{N} , there holds $\beta(\mathcal{M} \cap \mathcal{N}) > \max(\beta(\mathcal{M}), \beta(\mathcal{N})) + 1$.

1.3 Independence complexes of paths

In Section 2 we prove that the independence complex of a path always admits almost fair representation. In fact, possibly more than that is true. Since the matching complex of a path is the independence complex of a path one vertex shorter, a conjecture in this direction (in a slightly stronger form) can be formulated as follows:

Conjecture 1.6. *Given a partition of the vertex set of a path into sets V_1, \dots, V_m there exists an independent set S and integers b_i , $i \leq m$, such that $|S \cap V_i| \geq \frac{|V_i|}{2} - b_i$ for all i , and*

1. $\sum_{i \leq m} b_i \leq \frac{m}{2}$
and

2. $b_i \leq 1$ for all $i \leq m$.

We prove the existence of sets satisfying either condition of Conjecture 1.6 (but not necessarily both simultaneously).

Theorem 1.7. *Given a partition of the vertex set of a path into sets V_1, \dots, V_m there exists an independent set S and integers b_i , $i \leq m$, such that $\sum_{i \leq m} b_i \leq \frac{m}{2}$ and $|S \cap V_i| \geq \frac{|V_i|}{2} - b_i$ for all i .*

The proof of Theorem 1.7 uses the Borsuk-Ulam theorem.

Theorem 1.8. *Given a partition of the vertex set of a cycle into sets V_1, \dots, V_m there exists an independent set S such that $|S \cap V_i| \geq \frac{|V_i|}{2} - 1$ for all i .*

The proof uses a theorem of Schrijver, strengthening a famous theorem of Lovász on the chromatic number of Kneser graphs. This means that it, too, uses indirectly the Borsuk-Ulam theorem, since the Lovász-Schrijver proof uses the latter. We refer the reader to Matoušek's book [21] for background on topological methods in combinatorics, in particular applications of the Borsuk-Ulam theorem.

1.4 The matching complex of $K_{n,n}$

Conjecture 1.9. *For any partition E_1, E_2, \dots, E_m of $E(K_{n,n})$ and any $j \leq m$ there exists a perfect matching F in $K_{n,n}$ satisfying $|F \cap E_i| \geq \lfloor \frac{|E_i|}{n} \rfloor$ for all $i \neq j$, and $|F \cap E_j| \geq \lfloor \frac{|E_j|}{n} \rfloor - 1$.*

We shall prove:

Theorem 1.10. *Conjecture 1.9 is true for $m = 2, 3$.*

For $m = 2$ the result is simple, and the weight of the argument is in a characterization of those cases in which there necessarily exists an index j for which $|M \cap E_j| \geq \lfloor \frac{|E_j|}{n} \rfloor - 1$. The proof of the case $m = 3$ is topological, using Sperner's lemma.

1.5 Relationship to known conjectures

Conjecture 1.6 is related to a well known conjecture of Ryser on Latin squares. Given an $n \times n$ array A of symbols, a *partial transversal* is a set of entries taken from distinct rows and columns, and containing distinct symbols. A partial transversal of size n is called simply a *transversal*. Ryser's conjecture [24] is that if A is a Latin square, and n is odd, then A necessarily has a transversal. The oddness condition is indeed necessary - for every even $n > 0$ there exist $n \times n$ Latin squares not possessing a transversal. An example is the addition table of \mathbb{Z}_n : if a transversal T existed for this Latin square, then the sum of its elements, modulo n , is $\sum_{k \leq n} k = \frac{n(n+1)}{2} \pmod{n}$. On the other hand, since every row and every column is represented in this sum, the sum is equal to $\sum_{i \leq n} i + \sum_{j \leq n} j = n(n+1) \pmod{n}$, and for n even the two results do not agree. Arsovski [13] proved a closely related conjecture, of Snevily, that every square submatrix (whether even or odd) of the addition table of an odd order abelian group possesses a transversal.

Brualdi [16] and Stein [26] conjectured that for any n , any Latin square of order n has a partial transversal of order $n-1$. Stein [26] observed that the same conclusion may follow from weaker conditions - the square does not have to be Latin, and it may suffice that the entries of the $n \times n$ square are equally distributed among n symbols. Re-formulated, this becomes a special case of Conjecture 1.9:

Conjecture 1.11. *If the edge set of $K_{n,n}$ is partitioned into sets E_1, E_2, \dots, E_n of size n each, then there exists a matching in $K_{n,n}$ consisting of one edge from all but possibly one E_i .*

Here, even for n odd there are examples without a full transversal. In matrix terminology, take a matrix M with $m_{i,j} = i$ for $j < n$, $m_{i,n} = i + 1$ for $i < n$, and $m_{n,n} = 1$.

A related conjecture to Conjecture 1.9 was suggested in [2]:

Conjecture 1.12. *If E_1, E_2, \dots, E_m are sets of edges in a bipartite graph, and $|E_i| > \Delta(\bigcup_{i \leq m} E_i) + 1$ then there exists a rainbow matching.*

Re-phrased, this conjecture reads: If H is a bipartite multigraph, $G = L(H)$ and $V_i \subseteq V(G)$ satisfy $|V_i| \geq \Delta(H) + 2$ for all i , then there exists an independent set in G (namely a matching in H) meeting all V_i 's.

Remark 1.13. 1. We know only one example, taken from [20, 28], in which $|V_i| \geq \Delta(H) + 1$ does not suffice. Take three vertex disjoint copies of C_4 , say A_1, A_2, A_3 . Number the edges of A_i cyclically as a_i^j ($j = 1 \dots 4$). Let $E_1 = \{a_1^1, a_1^3, a_1^4\}$, $E_2 = \{a_1^2, a_1^4, a_3^3\}$, $E_3 = \{a_2^1, a_2^3, a_2^4\}$ and $E_4 = \{a_2^2, a_2^4, a_3^4\}$. Then $\Delta(\bigcup_{i \leq m} E_i) = 2$, $|E_i| = 3$ and there is no rainbow matching.

2. The conjecture is false if the sets E_i are allowed to be multisets. We omit the example showing this.

An even stronger version of the conjecture is:

Conjecture 1.14. *If the edge set of a graph H is partitioned into sets E_1, \dots, E_m then there exists a matching M satisfying $|M \cap E_i| \geq \lfloor \frac{|E_i|}{\Delta(H)+2} \rfloor$ for all $i \leq m$*

1.6 Over-representation vs. under-representation and representing general systems of sets

It is easy to find examples falsifying the above conjectures when the sets that are to be fairly represented do not form a partition. Why is that? A possible explanation is that a more natural formulation of our conjectures is not in terms of over-representation, but in terms of under-representation by a large set. Here is a conjecture in this direction:

Conjecture 1.15. *For every m there exists a number $c(m)$ for which the following is true: if G is a bipartite graph and E_1, \dots, E_m are any sets of edges, then there exists a matching S in G of size at least $\frac{|E(G)|}{\Delta(G)} - c(m)$ such that*

$$|S \cap E_i| \leq \lceil \frac{|E_i|}{\Delta(G)} \rceil \quad \text{for all } i \leq m \quad (2)$$

Possibly $c(m) = \frac{m}{2}$ may suffice. When the E_i 's form a partition, condition (2) implies that all but $c(m)$ sets are fairly represented. Of course, a stronger condition is required to imply Conjecture 1.9. The reason that the under-representation formulation is natural is that if the sets E_i form a partition, the condition in (2) defines a generalized partition matroid. The conjecture thus concerns representation by a set belonging to the intersection of three matroids.

2 Fair representation by independent sets in paths: a Borsuk-Ulam approach

In this section we prove Theorem 1.7. Following an idea from the proof of the ‘‘necklace theorem’’ [8], we shall use the Borsuk-Ulam theorem. In the necklace problem two thieves want to divide a necklace with m types of beads, each occurring in an even number of beads, so that the beads of every type are evenly split between the two. The theorem is that the thieves can achieve this goal using at most m cuts of the necklace. In our case, we shall employ as ‘‘thieves’’ the sets of odd and even points, respectively, in a sense to be explained below.

We first quote the Borsuk-Ulam theorem. As usual, for $n \geq 1$, S^n denotes the set of points $\vec{x} = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$ satisfying $\sum_{i \leq n+1} x_i^2 = 1$.

Theorem 2.1 (Borsuk-Ulam). *For all $n \geq 1$, if $f : S^n \rightarrow \mathbb{R}^n$ is a continuous odd function, then there exists $\vec{x} \in S^n$ such that $f(\vec{x}) = 0$.*

Proof of Theorem 1.7. Let v_1, \dots, v_n be the vertices of P_n , ordered along the path. In order to use the Borsuk-Ulam theorem, we first make the problem continuous, by replacing each vertex v_p by the characteristic function of the p th of n intervals of length $\frac{1}{n}$ in $[0, 1]$, open on the left and closed on the right, except for the first interval which is closed on both sides. We call the interval $(\frac{p-1}{n}, \frac{p}{n}]$ ($[0, \frac{1}{n}]$ for $p = 1$) a *bead* and denote it by B_p . Let χ_i be the characteristic function of $\bigcup_{v_p \in V_i} B_p$. Let g be the characteristic function of the union of odd beads on the path, and let $h(y) = 1 - g(y)$.

Given a point $\vec{x} \in S^m$, let $z_k = \sum_{j \leq k} x_j^2$, for all $k = 0, \dots, m+1$ (where $z_0 = 0, z_{m+1} = 1$).

For each $i \leq m$ define a function $f_i : S^m \rightarrow \mathbb{R}$ by:

$$f_i(x_1, \dots, x_m) = \sum_{1 \leq k \leq m} \int_{z_{k-1}}^{z_k} (g(y) - h(y)) \chi_i(y) \text{sign}(x_k) dy$$

Here, as usual, $\text{sign}(x) = 0$ if $x = 0$, $\text{sign}(x) = 1$ if $x > 0$, and $\text{sign}(x) = -1$ if $x < 0$. Since the set of points of discontinuity of the sign function is discrete, the functions f_i are continuous. The sign term guarantees that $f_i(-\vec{x}) = -f_i(\vec{x})$. Hence, by the Borsuk-Ulam theorem there exists a point $\vec{w} = (w_1, \dots, w_{m+1}) \in S^m$ such that $f_i(\vec{w}) = 0$ for all $i \in [m]$, where $z_k = \sum_{j \leq k} w_j^2$, for all $k = 0, \dots, m+1$.

For $y \in [0, 1]$ such that $y \in (z_{k-1}, z_k]$ define $POS(y) = 1$ if $w_k \geq 0$ and $POS(y) = 0$ otherwise. Let $NEG(y) = 1 - POS(y)$. Let

$$J_1(y) = POS(y)g(y) + NEG(y)h(y), \quad J_2(y) = POS(y)h(y) + NEG(y)g(y).$$

For fixed $i \in [m]$, the fact that $f_i(\vec{w}) = 0$ means that

$$\int_{y=0}^1 \chi_i(y) POS(y) [g(y) - h(y)] dy = \int_{y=0}^1 \chi_i(y) NEG(y) [g(y) - h(y)] dy$$

Shuffling terms this gives:

$$\int_{y=0}^1 \chi_i(y) [POS(y)g(y) + NEG(y)h(y)] dy = \int_{y=0}^1 \chi_i(y) [POS(y)h(y) + NEG(y)g(y)] dy \quad (3)$$

Denoting the integral $\int_0^1 u(y) dy$ of a function u by $|u|$, and noting that $J_1(y) + J_2(y) = 1$ for all $y \in [0, 1]$, Equation (3) says that

$$|\chi_i J_1| = |\chi_i J_2| = \frac{|\chi_i|}{2} \quad (4)$$

for every $i \leq m$.

A bead contained in an interval $(z_{k-1}, z_k]$ is called *positive* if $w_k \geq 0$ and *negative* otherwise. For $k = 1, \dots, m$ let T_k be the bead containing z_k . The beads that are equal to T_k for some k are called *transition beads*. Let F be the set of transition beads, and let $Z = \bigcup F$. We next remove the transition beads from J_j , by defining:

$$\tilde{J}_j(y) = \min(J_j(y), 1 - \chi_Z(y))$$

Thus \tilde{J}_1 is the characteristic function of the union of those beads that are either positive and odd, or negative and even, and \tilde{J}_2 is the characteristic function of the union of those beads that are either positive and even, or negative and odd. Let I_j ($j = 1, 2$) be the set of vertices v_p on whose bead B_p the function \tilde{J}_j is positive. Since the transition beads have been removed, I_1 and I_2 are independent.

For $i \leq m$ and $j = 1, 2$ let $c(i, j)$ be the amount of loss of \tilde{J}_j with respect to J_j on beads belonging to V_i , namely beads $B_p \in F$ such that $v_p \in V_i$. Formally,

$$c(i, j) = \sum_{v_p \in V_i, B_p \in F} |\chi_{B_p} \cdot (J_j - \tilde{J}_j)|$$

Then

$$c(i, 1) + c(i, 2) = \frac{1}{n} |\{v_p \in V_i, B_p \in F\}| \quad (5)$$

and $\sum_{i \leq m} c(i, 1) + \sum_{i \leq m} c(i, 2) = \frac{m}{n}$. Hence for either $j = 1$ or $j = 2$ we have $\sum_{i \leq m} c(i, j) \leq \frac{m}{2n}$. Let $I = I_j$ for this particular j , and denote $c(i, j)$ by b_i . Then, by (4) and (5) we have $|I \cap V_i| \geq \frac{|V_i|}{2} - b_i$, while $\sum_{i \leq m} b_i \leq \frac{m}{2}$. Namely, the set I satisfies the conditions of the theorem. \square

Example 2.2. Let $P = P_4$, the path with 4 vertices $v_i, 1 \leq i \leq 4$, and let $V_1 = \{v_1, v_2, v_4\}$ and $V_2 = \{v_3\}$. Then one possible set of points given by the Borsuk-Ulam theorem is $z_1 = \frac{1}{8}$, $z_2 = \frac{5}{8}$, and $w_1 > 0$, $w_2 < 0$, $w_3 > 0$ (or with all three signs reversed), as illustrated in Figure 1. Thus, J_1 is the characteristic function of $[0, \frac{1}{8}] \cup (1, 2] \cup (\frac{5}{8}, \frac{3}{4}]$ and J_2 is the characteristic function of $(\frac{1}{8}, 1] \cup (\frac{1}{2}, \frac{5}{8}] \cup (\frac{3}{4}, 1]$. The set I_1 is obtained from J_1 by removing the z_i -infected beads, namely $I_1 = \{v_2\}$, and then $I_2 = \{v_4\}$. In this case $\sum_{i \leq m} c(i, j) = \frac{m}{2n} = \frac{2}{4} = \frac{1}{2}$ for both $j = 1$ and $j = 2$, and thus we can choose I as either I_1 or I_2 . This is what the proof gives, but in fact in this example we can do better - we can take $I = \{v_1, v_3\}$, in which only V_1 is under-represented.

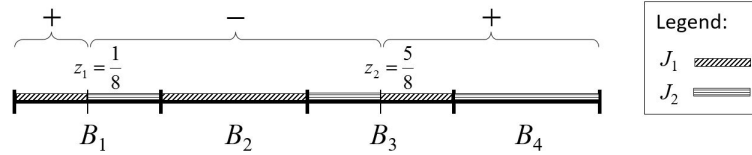


Figure 1: An example with four vertices divided into two sets.

Remark 2.3.

1. The inequality $\sum_{i \leq m} b_i \leq \frac{m}{2}$ can possibly be improved, but not much. Namely, there are examples in which the minimum of the sum $\sum_{i \leq m} b_i$ in the theorem is $\frac{m-1}{2}$. To see this, let $m = 2k + 1$, and let each V_i be of size $2k$. Consider a sequence of length $2k \times (2k + 1)$, in which the $(i-1)m + 2j - 1$ -th element belongs to V_i ($i = 1, \dots, 2k$, $j = 1, \dots, k + 1$), and the rest of the elements are chosen in any way so as to satisfy the condition $|V_i| = 2k$. For example, if $k = 2$ then the sequence is of the form:

$$1 * 1 * 1 - 2 * 2 * 2 - 3 * 3 * 3 - 4 * 4 * 4$$

where the *s can be filled in any way that satisfies $|V_i| = 4$ (namely, four of them are replaced by the symbol 5, and one is replaced by i for each symbol $i = 1, 2, 3, 4$. The dashes are there to facilitate the reference to the four stretches). If S is an independent set in the path then we may assume that S contains no more than k elements from the same V_i from each stretch (for example, in the first stretch of the example above choosing all three 1s will result in deficit of 2 in the other sets), Thus $|S| \leq 2k \times k$, which is $\frac{m-1}{2}$ short of half the length of the path.

2. It may be of interest to find the best bounds as a function of the sizes of the sets V_i and their number. Note that in the example above the size of the sets is almost equal to their number. As one example, if all V_i 's are of size 2, then the inequality can be improved to: $\sum_{i \leq m} b_i \leq \frac{m}{3}$. To

see this, look at the multigraph obtained by adding to P_n the pairs forming the sets V_i as edges. In the resulting graph the maximum degree is 3, and hence by Brooks' theorem it is 3-colorable. Thus there is an independent set of size at least $\frac{n}{3}$, which represents all V_i 's apart from at most $\frac{m}{3}$ of them.

3 Fair representation by independent sets in cycles: using a theorem of Schrijver

In this section we shall prove Theorem 1.8. The proof uses a result of Schrijver [25], which is a strengthening of a theorem of Lovász:

Theorem 3.1 (Schrijver [25]). *For integers k, n satisfying $n > 2k$ let $K = K(n, k)$ denote the graph whose vertices are all independent sets of size k in a cycle C of length n , where two such vertices are adjacent iff the corresponding sets are disjoint. Then the chromatic number of K is $n - 2k + 2$.*

The hard part of this inequality is that the chromatic number of K is at least $n - 2k + 2$, which can be formulated as follows:

Theorem 3.2. *The family $\mathcal{I}(n, k)$ of independent sets of size k in the cycle C_n cannot be partitioned into fewer than $n - 2k + 2$ intersecting families.*

We start with a simple case, in which all V_i 's but one are odd:

Theorem 3.3. *Let m, r_1, r_2, \dots, r_m be positive integers, and put $n = \sum_{i=1}^m (2r_i + 1) - 1$. Let $G = (V, E)$ be a cycle of length n , and let $V = V_1 \cup V_2 \cup \dots \cup V_m$ be a partition of its vertex set, where $|V_i| = 2r_i + 1$ for all $1 \leq i < m$ and $|V_m| = 2r_m$. Then there is an independent set S of G satisfying $|S| = \sum_{i=1}^m r_i$ and $|S \cap V_i| = r_i$ for all $1 \leq i \leq m$.*

Proof of Theorem 3.3: Put $k = \sum_{i=1}^m r_i$ and note that $n - 2k + 2 = m + 1 > m$. Assume, for contradiction, that there is no $S \in \mathcal{I}(n, k)$ satisfying the assertion of the theorem. Then for every $S \in \mathcal{I}(n, k)$ there is at least one index i for which $|S \cap V_i| \geq r_i + 1$. Indeed, otherwise $|S \cap V_i| \leq r_i$ for all i and hence $|S \cap V_i| = r_i$ for all i , contradicting the assumption. Let \mathcal{F}_i be the family of sets $S \in \mathcal{I}(n, k)$ for which $|S \cap V_i| \geq r_i + 1$. Clearly, \mathcal{F}_i is intersecting (in fact, intersecting within V_i), contradicting the conclusion of Theorem 3.2. □

Corollary 3.4. *Let $V = V_1 \cup V_2 \cup \dots \cup V_m$ be a partition of the vertex set of a cycle C .*

(i) *For every i such that $|V_i|$ is even there exists an independent set S_i of C satisfying:*

1. $|S_i \cap V_i| = |V_i|/2$.
2. $|S_i \cap V_j| = (|V_j| - 1)/2$ for all j for which $|V_j|$ is odd.
3. $|S_i \cap V_j| = |V_j|/2 - 1$ for every $j \neq i$ for which $|V_j|$ is even. .

(ii) *If $|V_i|$ is odd for all $i \leq m$ then for any vertex v of C there is an independent set S of C not containing v and satisfying $|S \cap V_i| = (|V_i| - 1)/2$ for all i .*

Proof of Corollary 3.4: Part (i) in case all sets V_j besides V_i are of odd sizes is exactly the assertion of Theorem 3.3. If there are additional indices $j \neq i$ for which $|V_j|$ is even, choose an arbitrary vertex from each of them and contract an edge incident with it. The result follows by applying the theorem to the shorter cycle obtained. Part (ii) is proved in the same way, contracting an edge incident with v . □

4 More applications of Schrijver's theorem and its extensions

4.1 Hypergraph versions

The results above can be extended by applying known hypergraph variants of Theorem 3.1. For integers $n \geq s \geq 2$, let C_n^{s-1} denote the $(s-1)$ -th power of a cycle of length n , that is, the graph obtained from a cycle of length n by connecting every two vertices whose distance in the cycle is at most $s-1$. Thus if $s=2$ this is simply the cycle of length n whereas if $n \leq 2s-1$ this is a complete graph on n vertices. For integers n, k, s satisfying $n > ks$, let $K(n, k, s)$ denote the following s -uniform hypergraph. The vertices are all independent sets of size k in C_n^{s-1} , and a collection V_1, V_2, \dots, V_s of such vertices forms an edge iff the sets V_i are pairwise disjoint. Note that for $s=2$, $K(n, k, 2)$ is exactly the graph $K(n, k)$ considered in Theorem 3.1. The following conjecture appears in [11].

Conjecture 4.1. *For $n > ks$, the chromatic number of $K(n, k, s)$ is $\lceil \frac{n-ks+s}{s-1} \rceil$.*

This is proved in [11] if s is any power of 2. Using this fact we can prove the following.

Theorem 4.2. *Let $s \geq 2$ be a power of 2, let m and r_1, r_2, \dots, r_m be integers, and put $n = s \sum_{i=1}^m r_i + (s-1)(m-1)$. Let V_1, V_2, \dots, V_m be a partition of the vertex set of C_n^{s-1} , where $|V_i| = sr_i + s - 1$ for all $1 \leq i < m$, and $|V_m| = sr_m$. Then there exists an independent set S in C_n^{s-1} satisfying $|S \cap V_i| = r_i$ for all $1 \leq i \leq m$.*

Proof: Put $k = \sum_{i=1}^m r_i$ and note that the chromatic number of $K(n, k, s)$ is $\lceil (n-ks+s)/(s-1) \rceil > m$. Assume, for contradiction, that there is a partition of the vertex set of C_n^{s-1} with parts V_i as in the theorem, with no independent set of C_n^{s-1} of size $k = \sum_{i=1}^m r_i$ satisfying the assertion of the theorem. In this case, for any such independent set S there is at least one index i so that $|S \cap V_i| \geq r_i + 1$. We can thus define a coloring f of the independent sets of size k of C_n^{s-1} by letting $f(S)$ be the smallest i such that $|S \cap V_i| \geq r_i + 1$. Since the chromatic number of $K(n, k, s)$ exceeds m , there are s pairwise disjoint sets S_1, S_2, \dots, S_s and an index i such that $|S_j \cap V_i| \geq r_i + 1$ for all $1 \leq j \leq s$. But this implies that $|V_i| \geq sr_i + s$, contradicting the assumption on the size of the set V_i , and completing the proof. \square

Just as in the previous section, this implies the following.

Corollary 4.3. *Let $s > 1$ be a power of 2. Let V_1, V_2, \dots, V_m be a partition of the vertex set of C_n^{s-1} , where $n = \sum_{i=1}^m |V_i|$. Then there is an independent set S in C_n^{s-1} satisfying*

$$|S \cap V_i| = \left\lfloor \frac{|V_i| - s + 1}{s} \right\rfloor$$

for all $1 \leq i < m$, and

$$|S \cap V_m| = \left\lfloor \frac{|V_m|}{s} \right\rfloor.$$

The proof is by contracting edges, reducing each set V_i to one of size $s \left\lfloor \frac{|V_i| - s + 1}{s} \right\rfloor + s - 1$ for $1 \leq i < m$, and reducing V_m to a set of size $s \left\lfloor \frac{|V_m|}{s} \right\rfloor$. The result follows by applying Theorem 4.2 to this contracted graph.

4.2 The Du-Hsu-Wang conjecture

Du, Hsu and Wang [17] conjectured that if a graph on $3n$ vertices is the edge disjoint union of a Hamilton cycle of length $3n$ and n vertex disjoint triangles then its independence number is n . Erdős conjectured

that in fact any such graph is 3-colorable. Using an algebraic approach introduced in [12], Fleischner and Stiebitz [18] proved this conjecture in a stronger form - any such graph is in fact 3-choosable.

The original conjecture, in a slightly stronger form, can be derived from Theorem 3.3: omit any vertex and apply the theorem with $r_i = 1$ for all i . So, for every vertex v there exists a representing set as desired in the conjecture omitting v . The derivation of the statement of Theorem 3.3 from the result of Schrijver in [25] actually supplies a quick proof of the following:

Theorem 4.4. *Let $C_{3n} = (V, E)$ be cycle of length $3n$ and let $V = A_1 \cup A_2 \cup \dots \cup A_n$ be a partition of its vertex set into n pairwise disjoint sets, each of size 3. Then there exist two disjoint independent sets in the cycle, each containing one point from each A_i .*

Proof. Define a coloring of the independent sets of size n in C_{3n} as follows. If S is such an independent set and there is an index i so that $|S \cap A_i| \geq 2$, color S by the smallest such i . Otherwise, color S by the color $n + 1$. By [25] there are two disjoint independent sets S_1, S_2 with the same color. This color cannot be any $i \leq n$, since if this is the case then

$$|(S_1 \cup S_2) \cap A_i| = |S_1 \cap A_i| + |S_2 \cap A_i| \geq 2 + 2 = 4 > 3 = |A_i|,$$

which is impossible. Thus S_1 and S_2 are both colored $n + 1$, meaning that each of them contains exactly one element of each A_i . \square

The Fleischner-Stiebitz theorem implies that the representing set in the DHW conjecture can be required to contain any given vertex. This can also be deduced from the topological version of Hall's Theorem first proved in [5] (for this derivation see e.g [6]). The latter shows also that the cycle of length $3n$ can be replaced by a union of cycles, totalling $3n$ vertices, none being of length $1 \pmod 3$. Simple examples show that the Fleischner-Stiebitz theorem on 3-colorability does not apply to this setting.

Note that none of the above proofs supplies an efficient algorithm for finding the desired independent set.

5 Fair representation by matchings in $K_{n,n}$, the case of two parts

The case $m = 2$ of Conjecture 1.9 is easy. Here is its statement in this case:

Theorem 5.1. *If F is a subset of $E(K_{n,n})$, then there exists a perfect matching N such that $|N \cap F| \geq \lfloor \frac{|F|}{n} \rfloor - 1$ and $|N \setminus F| \geq \lfloor \frac{|E(G) \setminus F|}{n} \rfloor - 1$.*

Partitioning $E(K_{n,n})$ into n perfect matchings shows that there exist two perfect matchings, N_1 and N_2 , such that $|N_1 \cap F| \leq \frac{|F|}{n} \leq |N_2 \cap F|$. The fact that any permutation can be reached from any other by a sequence of transpositions means that it is possible to reach N_2 from N_1 by a sequence of exchanges, replacing at each step two edges of the perfect matching by two other edges. Thus, by a mean value argument, at some matching in the process the condition is satisfied.

The question remains of determining the cases in which the (-1) term is necessary. That this term is sometimes necessary is shown, for example, by the case of $n = 2$ and F being a perfect matching. Another example - $n = 6$ and $F = ([3] \times [3]) \cup (\{4, 5, 6\} \times \{4, 5, 6\})$: it is easy to see that there is no perfect matching containing precisely 3 edges from F , as required in Conjecture 1.9.

The appropriate condition is given by the following concept:

Definition 5.2. A subset F of $E(K_{n,n})$ is said to be *rigid* if there exist subsets K and L of $[n]$ such that $F = K \times L \cup ([n] \setminus K) \times ([n] \setminus L)$.

The rigidity in question is with respect to F -parity of perfect matchings:

Theorem 5.3. [7] *A subset F of $E(K_{n,n})$ is rigid if and only if $|P \cap F|$ has the same parity for all perfect matchings P in $K_{n,n}$.*

This characterization shows that when F is rigid, it is not always possible to drop the “minus 1” term in Theorem 5.1. Conversely, if F is not rigid, then the “minus 1” term can indeed be dropped, as indicated by Corollary 5.5 below.

We shall show:

Theorem 5.4. *Let $a < c < b$ be three integers and suppose that $F \subseteq E(K_{n,n})$ is not rigid. If there exists a perfect matching P_a such that $|P_a \cap F| = a$ and a perfect matching P_b such that $|P_b \cap F| = b$, then there exists a perfect matching P_c satisfying $|P_c \cap F| = c$.*

It follows from Theorem 5.4 that if a subset F of $E(K_{n,n})$ is not rigid then for every integer c such that $n - \nu(E(K_{n,n}) \setminus F) \leq c \leq \nu(F)$ there exists a perfect matching N satisfying $|N \cap F| = c$. This implies,

Corollary 5.5. *If a subset F of $E(K_{n,n})$ is not rigid, or if $n \nmid |F|$, then there exists a perfect matching N such that $|N \cap F| \geq \lfloor \frac{|F|}{n} \rfloor$ and $|N \setminus F| \geq \lfloor \frac{|E(K_{n,n}) \setminus F|}{n} \rfloor$.*

Proof of Theorem 5.4. We use the matrix language of the original Ryser conjecture (Section 1.5). Let M be the $n \times n$ matrix in which $m_{i,j} = 1$ if $(i,j) \in F$ and $m_{i,j} = 0$ if $(i,j) \notin F$. A perfect matching in G corresponds to a *generalized diagonal* (abbreviated *GD*) in M , namely a set of n entries belonging to distinct rows and columns. A GD will be called a k -GD if exactly k of its entries are 1. By assumption there exist an a -GD T^a and a b -GD T^b . Assume, for contradiction, that there is no c -GD. The case $n = 2$ is trivial, and hence, reversing the roles of 0s and 1s if necessary, we may assume that $c > 1$. Since a GD corresponds to a permutation in S_n , and since every permutation can be obtained from any other permutation by a sequence of transpositions, there exists a sequence of GD's $T^a = T_1, T_2, \dots, T_k = T^b$, where each pair T_i and T_{i+1} , $i = 1, \dots, k-1$, differ in two entries. By the contradictory assumption there exists i such that $T := T_{i+1}$ is a $(c+1)$ -GD and $T' := T_i$ is a $(c-1)$ -GD. Without loss of generality we may assume that T lies along the main diagonal, its first $c+1$ entries are 1, and the rest of its entries are 0.

Let $I = [c+1]$, $J = [n] \setminus I$ and let $A = M[I | I]$, $B = M[I | J]$, $C = M[J | I]$, $D = M[J | J]$ (we are using here a common notation - $M[I | J]$ denotes the submatrix of M induced by the row set I and column set J). We may assume that the GD T' is obtained from T by replacing the entries (c, c) and $(c+1, c+1)$ by $(c+1, c)$ and $(c, c+1)$ (Figure 2).

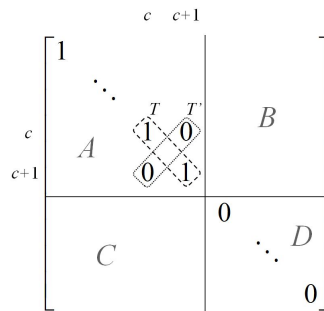


Figure 2

Claim 1. *The matrices A and D are symmetric.*

Proof of Claim 1. To prove that A is symmetric, assume, for contradiction, that there exist $i_1 \neq i_2 \in I$ such that $m_{i_1, i_2} \neq m_{i_2, i_1}$. Then, we can replace the entries (i_1, i_1) and (i_2, i_2) in T by (i_1, i_2) and (i_2, i_1) to obtain a c -GD. The proof for D is similar, applying the replacement in this case to T' .

Claim 2. *If $i \in I$ and $j \in J$ then $m_{i,j} \neq m_{j,i}$.*

Proof of Claim 2. Case I: $m_{i,j} = m_{j,i} = 0$. Replacing (i, i) and (j, j) in T by (i, j) and (j, i) results in a c -GD.

Case II: $m_{i,j} = m_{j,i} = 1$.

Subcase II₁: $i \notin \{c, c+1\}$. Replacing in T' the entries (i, i) and (j, j) by (i, j) and (j, i) results in a c -GD.

Subcase II₂: $i \in \{c, c+1\}$. Without loss of generality we may assume $i = c+1$ and $j = c+2$ (Figure 3). If $m_{k,\ell} = m_{\ell,k} = 0$ for some $1 \leq k < \ell \leq c$ then replacing in T the entries (k, k) , (ℓ, ℓ) , $(c+1, c+1)$ and $(c+2, c+2)$ by (k, ℓ) , (ℓ, k) , $(c+1, c+2)$ and $(c+2, c+1)$ results in a c -GD (Figure 3). Thus, we may assume that $m_{k,\ell} = m_{\ell,k} = 1$ for all $k, \ell \leq c$.

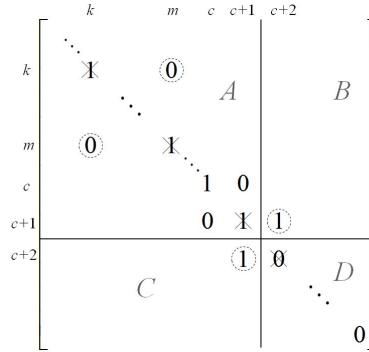


Figure 3: Subcase II₂. Removed entries are struck out by \times and added entries are circled.

We now consider three sub-subcases:

(i) $m_{c,c+2} = 0, m_{c+2,c} = 1$. In this case we may replace the entries (c, c) , $(c+1, c+1)$ and $(c+2, c+2)$ in T by $(c, c+2)$, $(c+1, c)$ and $(c+2, c+1)$ and obtain a c -GD (Figure 4(a)).

(ii) $m_{c,c+2} = 1, m_{c+2,c} = 0$. Replace the same entries as in Case (i) by $(c, c+1)$, $(c+1, c+2)$ and $(c+2, c)$, again obtaining a c -GD (Figure 4(b)).

(iii) $m_{c,c+2} = m_{c+2,c} = 1$. If $m_{c-1,c+1} = 0$ then, remembering that $m(c-1, c-1) = 1$, we can replace $(c-1, c-1)$, (c, c) , $(c+1, c+1)$ and $(c+2, c+2)$ in T by $(c-1, c+1)$, $(c, c+2)$, $(c+1, c-1)$, $(c+2, c)$ and obtain a c -GD (Figure 5(a)). If $m_{c-1,c+1} = 1$, we can replace $(c-1, c-1)$, (c, c) and $(c+1, c+1)$ in T by $(c-1, c+1)$, $(c, c-1)$ and $(c+1, c)$ and obtain a c -GD (Figure 5(b)). This proves Claim 2.

For a matrix K indexed by any set of indices X and indices $i, j \in X$, denote by $K_{(i)}$ the row of K indexed by i , and by $K^{(j)}$ the column of K indexed by j .

Claim 3. *For any $j \in J$, the submatrix A is the addition table modulo 2 of the row $C_{(j)}$ and the column $B^{(j)}$ (See illustration in Figure 6).*

Proof of Claim 3. We need to show that for any $i_1, i_2 \in I$ and $j \in J$ we have $m_{i_1, i_2} = m_{j, i_2} + m_{i_1, j} \pmod{2}$. We may assume that $i_1 \neq i_2$ since the case $i_1 = i_2$ follows from Claim 2 and the fact that A

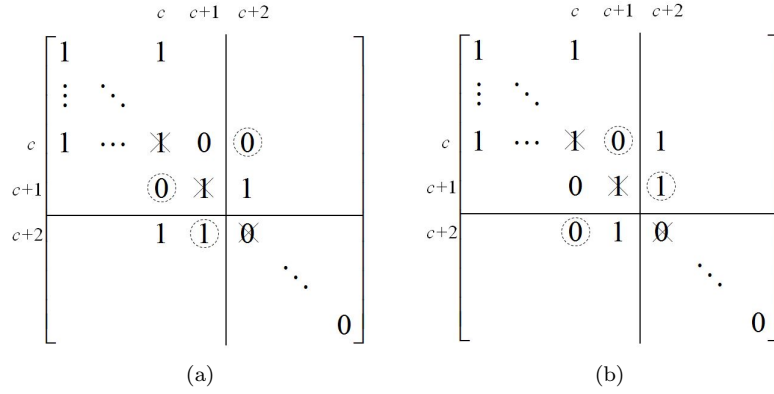


Figure 4

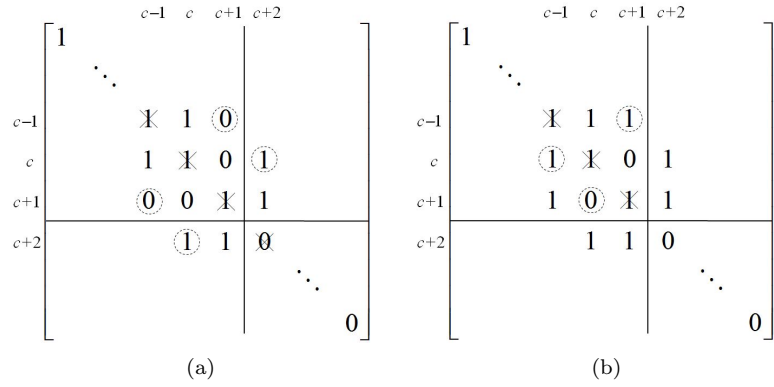


Figure 5

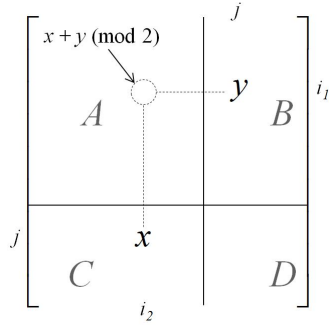


Figure 6

has 1's in the main diagonal. Let $x = m_{j,i_2} \in C_{(j)}$ and $y = m_{i_1,j} \in B^{(j)}$. We consider three cases: (i) $x \neq y$, (ii) $x = y = 0$, and (iii) $x = y = 1$.

(i) Assume, for contradiction, that $m_{i_1,i_2} = 0$. Then, by Claim 1, $m_{i_2,i_1} = 0$ and we can replace $(i_1, i_1), (i_2, i_2)$ and (j, j) in T by $(i_2, i_1), (i_1, j)$ and (j, i_2) and obtain a c -GD (Figure 7(a)). (ii) Assume,

for contradiction, that $m_{i_1, i_2} = 1$. We perform the same exchange as in Case (i) and, again, obtain a c -GD (Figure 7(b)). (iii) By Claim 2, we have $m_{i_2, j} = m_{j, i_1} = 0$. Assume, for contradiction, that $m_{i_1, i_2} = 1$. We replace $(i_1, i_1), (i_2, i_2)$ and (j, j) in T by $(i_1, i_2), (i_2, j)$ and (j, i_1) and obtain a c -GD (Figure 7(c)). This proves Claim 3.

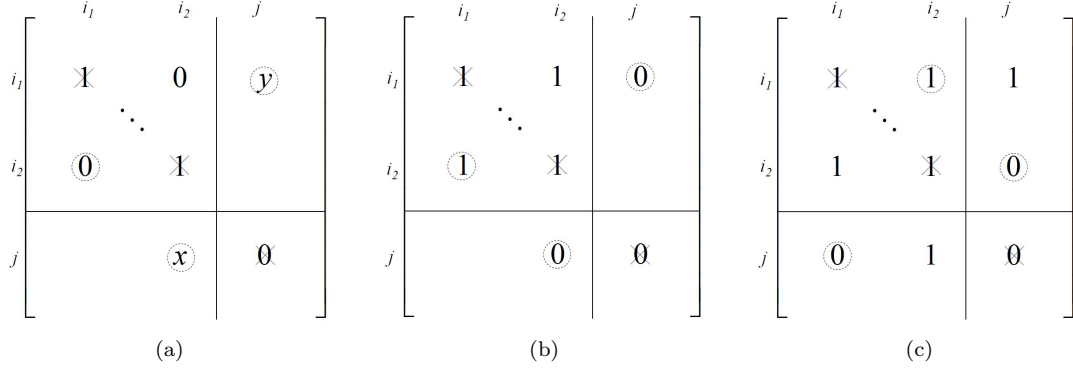


Figure 7

We say that two $(0,1)$ -vectors u and v of the same length are *complementary* (denoted $u \bowtie v$) if their sum is the vector $(1, 1, \dots, 1)$. By Claim 3, for every $i_1, i_2 \in I$, if for some $j \in J$, it is true that $m_{i_1, j} = m_{i_2, j}$ then the two rows $A_{(i_1)}, A_{(i_2)}$ are identical, and if $m_{i_1, j} \neq m_{i_2, j}$ then these two rows are complementary. Furthermore - the rows $M_{(i_1)}, M_{(i_2)}$ are identical or complementary. We summarize this in:

Claim 4. *Any two rows in $M[I \mid [n]]$ are either identical or complementary.*

Next we show that the property in Claim 4 holds for any two rows in M .

For $x, y \in \{0, 1\}$ we define the operation $x \circ y = x + y + 1 \pmod{2}$ (Figure 8).

Claim 5. *The submatrix D is the \circ -table between the column $C^{(i)}$ and the row $B_{(i)}$, for any $i \in I$.*

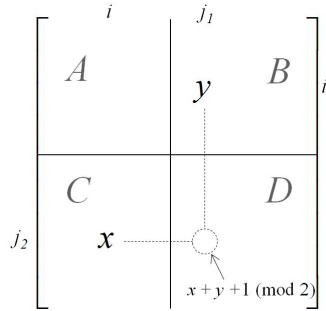


Figure 8

Proof of Claim 5. We first consider i such that $1 \leq i \leq c - 1$ (we assumed $c > 1$). Let $j_1, j_2 \in J$. We may assume that $j_1 \neq j_2$ since the case $j_1 = j_2$ follows from Claim 2 and the fact that D has 0's in the

diagonal. Let $x = m_{j_2, i}$ and $y = m_{i, j_1}$. We consider three cases: (i) $x = y = 0$, (ii) $x = y = 1$, and (iii) $x \neq y$.

(i) Assume, for contradiction, that $m_{j_2, j_1} = 0$. By Claim 1, $m_{j_1, j_2} = 0$, and we can replace (i, i) , (j_1, j_1) and (j_2, j_2) in T by (i, j_1) , (j_1, j_2) and (j_2, i) and obtain a c -GD (Figure 9(a)). (ii) By Claim 2, $m_{j_1, i} = m_{i, j_2} = 0$, and we can replace the same entries as in Case 1 by (i, j_2) , (j_1, i) and (j_2, j_1) and obtain a c -GD (Figure 9(b)). (iii) Here is where we need the assumption $i \leq c - 1$. We perform the same replacement as in Case 1, but this time on the GD T' , and obtain a c -GD (Figure 9(c)). Recall that T' is a $(c - 1)$ -GD).

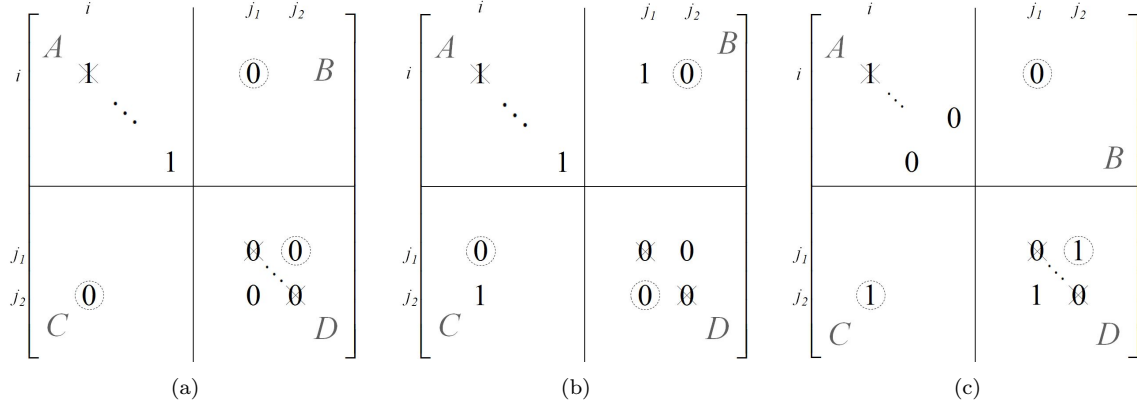


Figure 9

It remains to prove the claim for $i = c, c + 1$. It follows from Claim 4 that any two rows of B are either identical or complementary. Thus, by Claim 2, any two columns of C are either identical or complementary. If there exists $j < c$ such that $B_{(c)} = B_{(j)}$, then $C^{(c)} = C^{(j)}$. Since D is the \circ -table between $C^{(j)}$ and $B_{(j)}$, it is also the \circ -table between $C^{(c)}$ and $B_{(c)}$. If all $j < c$ satisfy $B_{(c)} \bowtie B_{(j)}$, then for any such j , we have $C^{(c)} = B_{(j)}^T$ and $C^{(j)} = B_{(c)}^T$ by Claim 2. Since \circ is commutative we again have that D is the \circ -table between $C^{(c)}$ and $B_{(c)}$. A similar argument holds for $i = c + 1$.

Claim 6. *Any two rows of M are either identical or complementary.*

Proof of Claim 6. The fact that any two rows in $M[J][n]$ are either identical or complementary follows in the same manner as Claim 4. Now, assume $i \in I, j \in J$. We want to show that $M_{(i)}$ is either identical or complementary to $M_{(j)}$. From Claim 3 we know that $A_{(i)}$ is either identical or complementary to $C_{(j)}$ and from Claim 5 we have that $B_{(i)}$ is either identical or complementary to $D_{(j)}$. We need to show that $A_{(i)}$ is identical to $C_{(j)}$ if and only if $B_{(i)}$ is identical to $D_{(j)}$. Note that $m_{ii} = 1, m_{jj} = 0$ and $m_{ij} \neq m_{ji}$. So, if $m_{ji} = 1$ we have identity in both cases and if $m_{ji} = 0$ we have complementarity in both cases.

Suppose all the rows of M are identical. Then, the first $c + 1$ columns are all-1 and the rest of the columns are all-0. So, any GD has exactly $c + 1$ 1s. So, $a = b = c + 1$, which is obviously not the case. Thus, by Claim 6, we can permute the rows and columns to obtain a matrix M' consisting of four submatrices M_1, M_2, M_3 and M_4 of positive dimensions, where M_1 and M_4 are all-1, and M_2 and M_3 are all-0 (Figure 10).

Thus, F is rigid (Definition 5.2), contrary to the hypothesis. We conclude that there must be a c -GD in M . \square

In the case that the partition $E(G) = F \cup (E(G) \setminus F)$ is rigid, if there exists a partition P_{c+1} such that $|P_{c+1} \cap F| = c + 1$, then clearly there is no partition P_c such that $|P_c \cap F| = c$. The proof of Theorem 5.4 shows that in this case, for any c between a and b there is a partition $P_{c'}$ such that $0 \leq |P_{c'} \cap F| - c \leq 1$.

$$M' = \left[\begin{array}{c|c} M_1 & p \ M_2 \\ \hline q & \\ \hline M_3 & M_4 \end{array} \right]$$

Figure 10

Corollary 5.6. *Let $G = K_{n,n}$ and assume the partition $E(G) = F \cup (E(G) \setminus F)$ is not rigid. Then, there exist perfect matchings P_1 and P_2 such that $|P_1 \cap F| = \left\lfloor \frac{|F|}{n} \right\rfloor$ and $|P_2 \cap F| = \left\lceil \frac{|F|}{n} \right\rceil$.*

6 Fair representation by perfect matchings in $K_{n,n}$, the case of three parts

In this section we prove Conjecture 1.9 for $m = 3$, namely:

Theorem 6.1. *Suppose that the edges of $K_{n,n}$ are partitioned into sets E_1, E_2, E_3 . Then, there exists a perfect matching F in $K_{n,n}$ satisfying $\left\lfloor \frac{|E_i|}{n} \right\rfloor + 1 \geq |F \cap E_i| \geq \left\lceil \frac{|E_i|}{n} \right\rceil - 1$ for every $i = 1, 2, 3$.*

It clearly suffices to prove the theorem for partitions of $E(K_{n,n})$ into sets E_1, E_2, E_3 such that $|E_i| = k_i n$, for k_i integers ($i = 1, 2, 3$). Assuming negation of Theorem 6.1 there is no perfect matching with exactly k_i edges from each E_i . As already mentioned, the theorem is patently true if one of the sets E_i is empty, so we may assume $k_1, k_2, k_3 \in \{1, \dots, n-2\}$.

We identify perfect matchings in $K_{n,n}$ with permutations in S_n . For $\sigma, \tau \in S_n$, the *Hamming distance* (or plainly *distance*) $d(\sigma, \tau)$ between σ and τ is $|\{i \mid \sigma(i) \neq \tau(i)\}|$. We write $\sigma \sim \tau$ if $d(\sigma, \tau) \leq 3$. Let \mathcal{C} be the simplicial complex of the cliques of this relation. So, the vertices of \mathcal{C} are the permutations in S_n and the simplexes are the sets of permutations each two of which have distance at most 3 between them. The core of the proof of the theorem will be in showing that \mathcal{C} is simply connected, which will enable us to use Sperner's lemma.

Here is a short outline of the proof of the theorem. Clearly, for each $i \leq 3$ there exists a matching F_i representing E_i fairly, namely $|F_i \cap E_i| \geq \left\lfloor \frac{|E_i|}{n} \right\rfloor$. We shall connect every pair F_i, F_j ($1 \leq i < j \leq 3$) by a path consisting of perfect matchings representing fairly $E_i \cup E_j$, in such a way that every two adjacent matchings are \sim -related. This generates a triangle D that is not necessarily simple (namely it may have repeating vertices), together with a triangulation T of its circumference, and an assignment A of matchings to its vertices. We shall then show that there exists a triangulation T' extending T and contained in \mathcal{C} (meaning that there is an assignment A' extending A of perfect matchings to the vertices of T'), such that the perfect matchings assigned to adjacent vertices are \sim -related. We color a vertex v of T' by color i if $A'(v)$ represents fairly the set E_i . By our construction, this coloring satisfies the conditions of the 2-dimensional version of Sperner's lemma, and applying the lemma we obtain a multicolored triangle. We shall then show that at least one of the matchings assigned to the vertices of this triangle satisfies the condition required in the theorem.

6.1 Topological considerations

Let us recall the 2-dimensional version of Sperner's lemma:

Lemma 6.2. *Let T be a triangulation of a triangle ABC and suppose that the vertices of T are colored 1, 2, 3. Assume that*

- *The vertex A has color 1.*
- *The vertex B has color 2.*
- *The vertex C has color 3.*
- *Every vertex in the subdivision of the edge AB has either color 1 or color 2.*
- *Every vertex in the subdivision of the edge BC has either color 2 or color 3.*
- *Every vertex in the subdivision of the edge CA has either color 3 or color 1.*

Then T contains a region triangle with three vertices colored 1, 2 and 3.

We shall need a "hexagonal" version of the lemma:

Lemma 6.3. *Let T be a triangulation of a hexagon, whose outer cycle is the union of six paths p_1, \dots, p_6 (which are, in a cyclic order, subdivisions of the six edges of the hexagon). Suppose that the vertices of T are colored 1, 2, 3, in such a way that*

- *No vertex in p_1 has color 1.*
- *No edge in p_2 is between two vertices of colors 1 and 2.*
- *No vertex in p_3 has color 2.*
- *No edge in p_4 is between two vertices of colors 2 and 3.*
- *No vertex in p_5 has color 3.*
- *No edge in p_6 is between two vertices of colors 3 and 1.*

Then T contains a region triangle with three vertices colored 1, 2 and 3.

Proof. Add three vertices to T outside the circumference of the hexagon in the following way. Add a vertex A of color 1 adjacent to all vertices in p_4 , a vertex B of color 2 adjacent to all vertices in p_6 and a vertex of color 3 adjacent to all vertices in p_2 . Using Sperner's Lemma on this augmented triangulation yields the lemma. \square

Our strategy for the proof of Theorem 6.1 is the following. First we form a triangulation of a hexagon and assign a permutation in S_n to each vertex of the triangulation, where adjacent permutations are \sim related. Afterwards we color each permutation σ with some color i , where E_i is fairly represented in σ . We then apply Lemma 6.3 to get three permutations $\sigma_1, \sigma_2, \sigma_3$ which are pairwise \sim related, and fairly represent E_1, E_2, E_3 respectively. We then show that how to use this to construct a permutation almost fairly representing all three sets E_1, E_2, E_3 , simultaneously.

For $i \in [n]$ let $shift_i : S_n \rightarrow S_n$ be a function defined as follows. For every $\sigma \in S_n$, if $\sigma(i) = j$ then

$$shift_i(\sigma)(k) = \begin{cases} i & \text{if } k = i \\ j & \text{if } \sigma(k) = i \\ \sigma(k) & \text{otherwise} \end{cases}$$

Remark 6.4. Note that if $\sigma(i) = i$ then $shift_i(\sigma) = \sigma$.

Lemma 6.5. *If $\sigma \sim \tau$ then $shift_i(\sigma) \sim shift_i(\tau)$.*

Proof. Without loss of generality let $i = 1$. If $shift_1(\sigma) = \sigma$ and $shift_1(\tau) = \tau$ then we are done.

Case I: $shift_1(\tau) = \tau$ and $shift_1(\sigma) \neq \sigma$. Without loss of generality $\tau = I$, the identity permutation. For every $k \in [n]$, if $\sigma(k) = k$ then also $shift_1(\sigma)(k) = k$ and thus the distance between $shift_1(\sigma)$ and I is at most the distance between σ and I , yielding $shift_1(\sigma) \sim I = shift_1(\tau)$.

Case II: $shift_1(\sigma) \neq \sigma$ and $shift_1(\tau) \neq \tau$. Without loss of generality $\tau = (12)$ and hence $shift_1(\tau) = I$. As in the previous case, for every $k \in [n]$ if $\sigma(k) = k$ then also $shift_1(\sigma)(k) = k$. We also note that $shift_1(\sigma)(1) = 1$ but $\sigma(1) \neq 1$ (since $shift_1(\sigma) \neq \sigma$). Therefore $d(shift_1(\sigma), I) < d(\sigma, I)$. If $d(\sigma, I) \leq 4$ then $shift_1(\sigma) \sim I = shift_1(\tau)$ and we are done. Since $\sigma \sim \tau$, we have $d(\sigma, I) \leq 5$ so we may assume that $d(\sigma, I) = 5$. Note that if $\sigma(1) = j \neq 2$, then σ and τ differ on $1, 2$ and j , and thus $\sigma(k) = k$ for all $k \notin \{1, 2, j\}$, so $d(\sigma, I) \leq 3$, contrary to the assumption that this distance is 5. Thus, we must have that $\sigma(1) = 2$. It follows that $A := \{i \in [n] : \sigma(i) \neq \tau(i)\}$ is a set of size 3 disjoint from $\{1, 2\}$. But then also $\{i \in [n] : shift_1(\sigma(i)) \neq shift_1(\tau(i))\} = A$, yielding $shift_1(\sigma) \sim shift_1(\tau)$. \square

At this point we need a connectivity result. This is best formulated in matrix language.

Lemma 6.6. *Let $A = (a_{ij})$ be an $n \times n$ 0-1 matrix and let $k \in [n-1]$. Let G be the graph whose vertices are the permutations $\sigma \in S_n$ satisfying $\sum_{i=1}^n a_{i\sigma(i)} \geq k$ and whose edges correspond to the \sim relation. If there exists $\rho \in S_n$ with $\sum_{i=1}^n a_{i\rho(i)} > k$, then G is connected.*

Proof. Without loss of generality $\rho = I$, meaning that $\sum_{i=1}^n a_{ii} > k$. We shall show that there is a path in G from ρ to σ for any $\sigma \in V(G) \setminus \{\rho\}$. We prove this claim by induction on $d(\sigma, \rho)$. Write $\ell = \sum_{i=1}^n a_{i\sigma(i)}$. Our aim is to find distinct $j \in [n]$ for which $\sigma(j) \neq j$ and $\sigma' = shift_j(\sigma) \in V(G)$. Then the induction hypothesis can be applied since $\sigma \sim \sigma'$ and σ' is closer to ρ than σ .

If $\ell \geq k + 2$ choose any $j \in [n]$ with $\sigma(j) \neq j$. Then we have $\sum_{i=1}^n a_{i\sigma'(i)} \geq \sum_{i=1}^n a_{i\sigma(i)} - 2 \geq k$, so $\sigma' \in V(G)$.

Suppose next that $\ell = k + 1$. By the assumption that $\sum_{i=1}^n a_{ii} > k$ we have $\sum_{i=1}^n a_{i\sigma(i)} \leq \sum_{i=1}^n a_{ii}$ and since $\sigma \neq \rho$ there must be some $j \in [n]$ for which $\sigma(j) \neq j$ and $a_{jj} \geq a_{j\sigma(j)}$. Taking $\sigma' = shift_j(\sigma) \in V(G)$ yields $\sum_{i=1}^n a_{i\sigma'(i)} \geq \sum_{i=1}^n a_{i\sigma(i)} - 1 = k$, so $\sigma' \in V(G)$.

Finally, if $\ell = k$ then $\sum_{i=1}^n a_{i\sigma(i)} < \sum_{i=1}^n a_{ii}$ and hence there must be some $j \in [n]$ for which $a_{jj} > a_{j\sigma(j)}$. Taking $\sigma' = shift_j(\sigma) \in V(G)$ we get $\sum_{i=1}^n a_{i\sigma'(i)} \geq \sum_{i=1}^n a_{i\sigma(i)} + 1 - 1 = k$, so $\sigma' \in V(G)$. \square

Corollary 6.7. *Let $A = (a_{ij})$ be an $n \times n$ 0-1 matrix and let $k \in [n]$. Let G be the graph whose vertices are the permutations $\sigma \in S_n$ with $\sum_{i=1}^n a_{i\sigma(i)} \geq k$ and whose edges correspond to the \sim relation. If $\sum_{i,j \leq n} a_{ij} \geq kn$ then G is connected.*

Proof. If there exists a permutation ρ with $\sum_{i=1}^n a_{i\rho(i)} > k$ then we are done by Lemma 6.6. If not, by König's theorem there exist sets $A, B \subseteq [n]$ with $|A| + |B| \leq k$ such that $a_{ij} = 0$ for $i \notin A$ and $j \notin B$. This is compatible with the condition $\sum_{i,j \leq n} a_{ij} \geq kn$ only if $|A| = 0$ and $|B| = k$ or $|B| = 0$ and $|A| = k$, and $a_{ij} = 1$ for all $(i, j) \in A \times [n] \cup [n] \times B$. In both cases $V(G) = S_n$, implying that the relation \sim is path connected since every permutation is reachable from every other permutation by a sequence of transpositions. \square

In the next two lemmas let $i \in [n]$ and $\sigma, \tau \in S_n$. We write $shift$ for $shift_i$.

Lemma 6.8. *If $d(\sigma, \tau) = 2$, then the 4-cycle $\sigma - \tau - shift(\tau) - shift(\sigma) - \sigma$ is null-homotopic in \mathcal{C} (i.e., it can be triangulated.)*

Proof. If either $\sigma \sim \text{shift}(\tau)$ or $\tau \sim \text{shift}(\sigma)$ then we are done. So, we may assume this does not happen and in particular $\sigma \neq \text{shift}(\sigma)$ and $\tau \neq \text{shift}(\tau)$. We may assume, without loss of generality, that $i = 1$, $\sigma = (12)$, $\tau = (12)(34)$, $\text{shift}(\sigma) = I$ and $\text{shift}(\tau) = (34)$. We can now fill the cycle as in Figure 11. \square

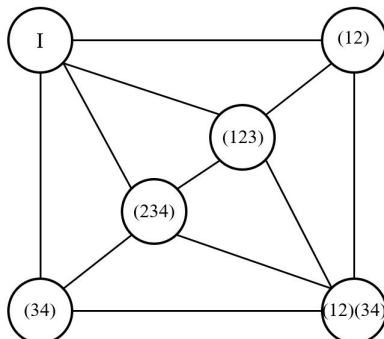


Figure 11

Lemma 6.9. *If $d(\sigma, \tau) = 3$ then the 4-cycle $\sigma - \tau - \text{shift}(\tau) - \text{shift}(\sigma) - \sigma$ is a null cycle in \mathcal{C} .*

Proof. Let $\rho \in S_n$ have distance 2 from both σ and τ . Denote $\sigma' = \text{shift}(\sigma)$, $\tau' = \text{shift}(\tau)$ and $\rho' = \text{shift}(\rho)$. We use the previous lemma to fill the cycle as in Figure 12. \square

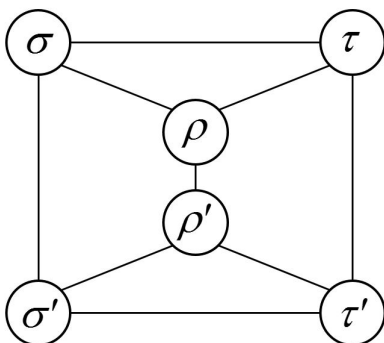


Figure 12

As a corollary from the above two lemmas we get

Corollary 6.10. *Let C be a cycle and let $f : C \rightarrow \mathcal{C}$ be a simplicial map, i.e., mapping each edge to an edge or a vertex. Let $\bar{f} : C \rightarrow \mathcal{C}$ be defined by $\bar{f}(v) = \text{shift}_i(f(v))$ for every $v \in V(C)$. Then \bar{f} is also simplicial and is homotopic to f .*

See Figure 13. As above, $f(\sigma)$ is denoted by σ' .

Lemma 6.11. *The simplicial complex \mathcal{C} is simply connected.*

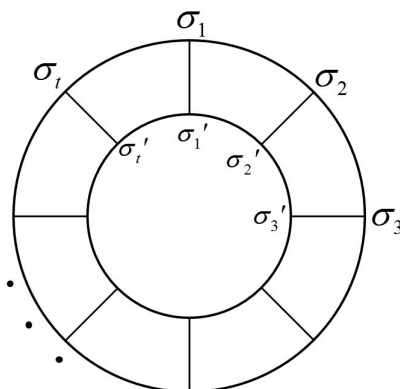


Figure 13

Proof. Let C be a cycle and let $f_0 : C \rightarrow C$ be a simplicial map. We need to show that f_0 is null-homotopic. For each $i \in [n]$, we define $f_i : C \rightarrow C$ by $f_i(v) = \text{shift}_i(f_{i-1}(v))$ for every $v \in V(C)$. Then by Corollary 6.10 f_0, \dots, f_n are all homotopic to each other. But $f_n(v) = I$ for every $v \in V(C)$. This means that f_0, \dots, f_n are all null-homotopic. \square

6.2 Associating a complex with the graph

Lemma 6.12. *Let the set E of edges of $K_{n,n}$ be partitioned to three sets $E = E_1 \dot{\cup} E_2 \dot{\cup} E_3$. Then there exists a perfect matching M with at least $\lfloor \frac{|E_1|}{n} \rfloor$ edges of E_1 and at most $\lfloor \frac{|E_3|}{n} \rfloor$ edges of E_3 .*

Proof. Let H be the graph with the edge set $E_1 \cup E_2$. König's edge coloring theorem, combined with an easy alternating paths argument, yields that H can be edge colored with n colors in a way that each color class is of size either $\lfloor \frac{|E(H)|}{n} \rfloor$ or $\lceil \frac{|E(H)|}{n} \rceil$. Clearly, at least one of these classes contains at least $\lfloor \frac{|E_1|}{n} \rfloor$ edges from E_1 . A matching with the desired property can be obtained by completing this color class in any way we please to a perfect matching of $K_{n,n}$. \square

In fact, a stronger property may hold:

Conjecture 6.13. *Let $G = (V, E)$ be a bipartite graph with maximal degree Δ and let $f : E \rightarrow \{1, 2, 3, \dots, k\}$ for some positive integer k . Then there exists a matching M in G such that every number $j \in \{1, 2, 3, \dots, k\}$ satisfies*

$$|\{e \in M \mid f(e) \leq j\}| \geq \left\lfloor \frac{|\{e \in E : f(e) \leq j\}|}{\Delta} \right\rfloor$$

Clearly, we only need to see to it that the condition holds for $j < k$.

In [14] this conjecture was proved for $G = K_{6,6}$.

We shall say that a perfect matching F has property $i^{(+)}$ if $|F \cap E_i| \geq k_i$, property $i^{(++)}$ if $|F \cap E_i| > k_i$, and property $i^{(-)}$ if $|F \cap E_i| \leq k_i$.

Lemma 6.14. *There exists a triangulation of the boundary of a hexagon, and an assignment of a perfect matching M_v and a color $i_v \in \{1, 2, 3\}$ to each vertex v of the triangulation, such that M_v has property $i_v^{(++)}$ and the coloring satisfies the conditions of Lemma 6.3.*

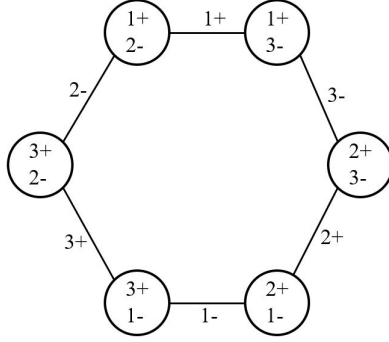


Figure 14

Proof. By Lemma 6.12 there exists a perfect matching M with properties $1^{(+)}$ and $3^{(-)}$. We assign it to one vertex of the hexagon. By permuting the roles of E_1, E_2, E_3 we can find six such perfect matchings and assign them to the six vertices of the hexagon as in Figure 14.

By Corollary 6.7, we can fill the path between the two permutations with property $i^{(-)}$ in a way that all perfect matchings in the path have property $i^{(-)}$. Similarly, we can fill the path between the two permutations with property $i^{(+)}$. For each vertex v we assign a color i_v such that M_v has property $i_v^{(++)}$. If Lemma 6.3 does not hold, then without loss of generality we have two perfect matchings $M_1 \sim M_2$, where M_1 has properties $3^{(+)}$ and $1^{(++)}$ and M_2 has properties $3^{(+)}$ and $2^{(++)}$. This yields Lemma 6.14. \square

Since \mathcal{C} is simply connected, we can extend the mapping we got in Lemma 6.14 to a triangulation of the hexagon. Applying Lemma 6.3 we obtain a triangle in the triangulation whose vertices are colored 1, 2 and 3. This means that there exist $\sigma_1, \sigma_2, \sigma_3 \in S_n$, pairwise \sim related and fairly representing E_1, E_2, E_3 respectively.

6.3 Proof of Theorem 6.1

We form a matrix $A = (a_{ij})_{i,j \leq n}$, where $a_{ij} = p$ ($p = 1, 2, 3$) if the edge ij belongs to E_p .

For each $\ell \in \{1, 2, 3\}$ and $\sigma \in S_n$ we write $d_\ell(\sigma) = |\{i : a_{i\sigma(i)} = \ell\}| - k_\ell$.

Lemma 6.15. *Suppose that the triple $\{\sigma_1, \sigma_2, \sigma_3\}$ is in \mathcal{C} , and that $d_\ell(\sigma_\ell) > 0$ for each $\ell \in \{1, 2, 3\}$. Then there exists $\sigma \in S_n$ with $|d_\ell(\sigma)| \leq 1$ for each $\ell \in \{1, 2, 3\}$.*

Since the existence of such $\sigma_1, \sigma_2, \sigma_3$ follows from Lemmas 6.3, 6.14 and 6.11, this will finish the proof of Theorem 6.1.

Proof. Define a 3×3 matrix $B = (b_{ij})$ by $b_{ij} = d_i(\sigma_j)$. We know that the diagonal entries in B are positive, the sum in each column is zero, and any two entries in the same row differ by at most 3. This means that the minimal possible entry in B is -2. We may assume each column has some entry not in $\{-1, 0, 1\}$.

Let us start with the case that all of the diagonal entries of B are at least 2. This implies that all off-diagonal entries are at least -1. Since each column must sum up to zero, we must have

$$B = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

This implies that the distance between any two of $\sigma_1, \sigma_2, \sigma_3$ is exactly 3, and without loss of generality $\sigma_1 = I, \sigma_2 = (123), \sigma_3 = (132)$, and the matrix A has the form

$$A = \begin{pmatrix} 1 & 2 & 3 & * & \dots & * \\ 3 & 1 & 2 & * & \dots & * \\ 2 & 3 & 1 & * & \dots & * \\ * & * & * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & \dots & * \end{pmatrix}$$

We can now take $\sigma = (12)$ and we are done.

We are left with the case that some diagonal entry of B is 1. Without loss of generality $b_{11} = 1$. We also assume without loss of generality that $b_{21} \leq b_{31}$. Since the first column must sum up to zero, we have $b_{21} + b_{31} = -1$, and thus $-0.5 = 0.5(b_{21} + b_{31}) \leq b_{31} = -1 - b_{21} \leq 1$. In other words, either $b_{21} = -1$ and $b_{31} = 0$ or $b_{21} = -2$ and $b_{31} = 1$. In the first case we can just take $\sigma = \sigma_1$ and we are done. Therefore we assume the second case.

$$B = \begin{pmatrix} 1 & * & * \\ -2 & * & * \\ 1 & * & * \end{pmatrix}$$

Since $d_3(\sigma_1) > 0$, we may assume $\sigma_3 = \sigma_1$, and due to the -2 entries in the second row, we must have $b_{22} = 1$. We now get

$$B = \begin{pmatrix} 1 & * & 1 \\ -2 & 1 & -2 \\ 1 & * & 1 \end{pmatrix}$$

Without loss of generality $b_{12} \leq b_{32}$ and by arguments similar to the above we can fill the second column

$$B = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}$$

The distance between σ_1 and σ_2 is exactly 3, so without loss of generality $\sigma_1 = I$ and $\sigma_2 = (123)$. In order to achieve the values of $b_{12} = -2, b_{11} = 1, b_{21} = -2, b_{22} = 1$ we must have $a_{ii} = 1$ and $a_{i\sigma_2(i)} = 2$ for each $i \in \{1, 2, 3\}$.

The only case in which none of the choices $\sigma = (12)$ or $\sigma = (23)$ or $\sigma = (13)$ works is if $a_{13} = a_{21} = a_{32} = 3$, so once again we get

$$A = \begin{pmatrix} 1 & 2 & 3 & * & \dots & * \\ 3 & 1 & 2 & * & \dots & * \\ 2 & 3 & 1 & * & \dots & * \\ * & * & * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & \dots & * \end{pmatrix}$$

We have $b_{31} = 1$ which means that 3 appears $k_3 + 1$ times on the diagonal. Without loss of generality $a_{44} = a_{55} = \dots = a_{k_3+4, k_3+4} = 3$. In any of the following cases one can easily find some $\sigma \in S_n$ with $|d_\ell(\sigma)| \leq 1$ for each $\ell \in \{1, 2, 3\}$:

- If either $a_{ij} \neq 3$ or $a_{ji} \neq 3$ for some $i \in \{4, \dots, k_3 + 4\}$ and $j \in \{1, 2, 3\}$.
- If $a_{ij} \neq 3$ for some $i, j \in \{4, \dots, k_3 + 4\}$
- If both $a_{ij} \neq 3$ and $a_{ji} \neq 3$ for some $i \in \{4, \dots, k_3 + 4\}$ and $j \in \{k_3 + 5, \dots, n\}$.

If none of the above occurs then

$$k_3 n = |\{(i, j) : a_{ij} = 3\}| \geq 2 \cdot 3 \cdot (1 + k_3) + (1 + k_3)^2 + \frac{1}{2} \cdot 2(k_3 + 1)(n - k_3 - 4)$$

which is a contradiction. □

Remark 6.16. After the above topological proof of Theorem 6.1 was found, a combinatorial proof was given in [14].

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