Fair Partitions

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Abstract

A substantial number of results and conjectures deal with the existence of a set of prescribed type which contains a fair share from each member of a finite collection of objects in a space, or the existence of partitions in which this is the case for every part. Examples include the Ham Sandwich Theorem in Measure Theory, the Hobby-Rice Theorem in Approximation Theory, the Necklace Theorem and Ryser's Conjecture in Discrete Mathematics. The techniques in the study of these results combine combinatorial, topological, geometric, probabilistic and algebraic tools. This paper contains a brief description of the topic, focusing on several recent existence results and their algorithmic aspects. This is mainly a survey paper, but it also contains several novel results.

1 Introduction

The problem of the existence of a set with desired properties that has a fair share of each of a family of measures has been studied in several areas. The related notion of fair partitions has also received a considerable amount of attention. Although there have been several earlier results of this type it is common to view the Ham Sandwich Theorem as the initial statement in the area.

Theorem 1.1 (The Ham Sandwich Theorem) *For any collection of d probability measures in R^d, each absolutely continuous with respect to the Lebesgue measure, there is a hyperplane that bisects all measures.*

Thus, each of the two half-spaces determined by the separating hyperplane contains a fair share of each of the measures. This was conjectured by Steinhaus and proved by Banach, using the Borsuk-Ulam Theorem, a fundamental result in Topology which asserts that any continuous function from S^n to R^n maps two antipodal points to the same image.

The Ham Sandwich Theorem is first mentioned in [45], where Steinhaus attributes the proof to Banach (for $d = 3$, but the proof for general d is essentially identical).

There are numerous results and questions dealing with partitions of prescribed types of Euclidean spaces and the ways they can split measures. See [40] for a comprehensive recent survey of the subject. The formulation of most of these results is geometric, dealing with sets or measures in Euclidean spaces. There are, however, also purely combinatorial results and conjectures of the same flavor. Here we focus on questions of this type. The following examples of two results and two conjectures illustrate the diversity of the topic.

Theorem 1.2 (The Cycle and Triangles Theorem) *Let G be a cycle of length $3m$ and let P be an arbitrary partition of its set of vertices into pairwise disjoint sets $P_1, P_2, \ldots, P_m$, each of size 3. Then there is an independent set S of G that contains exactly one vertex of each set P_i. Moreover, all vertices of G can be partitioned into 3 independent sets S_1, S_2, S_3, each containing exactly one point of each P_i.*

This result (for one set S) was conjectured by Du, Hsu and Hwang in [16], the stronger conjecture is due to Erdős [18]. It was proved (in a strong form) by Fleischner and Steibitz in [21], using the algebraic technique of [8]. Additional proofs of the initial conjecture of [16] and of some variants appear in [42], [2] and [1].

Theorem 1.3 (The Necklace Theorem) *Let N be an open necklace with ka_i beads of type i, for $1 \leq i \leq t$. Then it is possible to cut N in at most $(k-1)t$ points and partition the resulting intervals into k collections, each containing exactly a_i beads of type i, for all $1 \leq i \leq t$.*

A continuous version of this result for $k = 2$ has been proved in [29], the discrete result for $k = 2$ is proved in [24], and a short derivation of it from the Borsuk-Ulam Theorem appears in [9]. The general result is proved in [3].

Conjecture 1.4 (Rota's Basis Conjecture) *Let $B_1, B_2, \ldots, B_n$ be n bases of a matroid of rank n. Then there is a partition of the elements in the (multi)set $B_1 \cup B_2 \cup \ldots \cup B_n$ into n pairwise disjoint bases $A_1, A_2, \ldots, A_n$ of the matroid, where each A_i contains exactly one element of each B_j.*

This was conjectured in [28]. It has been proved in several special cases. It is also known that there are always at least $n/2$ disjoint bases A_i satisfying the desired property [11] and that there are $(1 - o(1))n$ pairwise disjoint independent sets A_i, each of size $(1 - o(1))n$, and each containing at most one element from any A_i [35].

A *Latin Square of order* n is an n by n matrix in which each row and each column is a permutation of the n symbols $[n] = \{1, 2, \ldots, n\}$. A *Latin transversal* in such a square is a set of n entries containing one element in each row, one in each column and one copy of each symbol.

Conjecture 1.5 (Ryser's Conjecture, [41], [13]) *Every Latin Square of odd order contains a Latin transversal.*

An equivalent formulation of this conjecture is that for every proper edge coloring of the complete bipartite graph $K_{n,n}$ by n colors, where n is odd, there is a *rainbow* perfect matching, that is, a perfect matching in which no two edges have the same color. It is known that there is a rainbow matching of size at least $n - O(\log n / \log \log n)$, as proved in [31], improving an estimate of [30]. It is also known that for every n besides 3 there are examples of Latin Squares of order n that cannot be partitioned into Latin Transversals. Therefore while the existence of one fair matching is conjectured to always hold (for odd n) the corresponding partition result here fails. This was proved by Euler for all even n, by Mann for all $n \equiv 1 \bmod 4$ [33], and independently by Wanless and Webb and by Evans [46], [19] for the remaining cases.

In the rest of this paper we describe several recent variants and extensions of the examples above. The next section deals with the Necklace Theorem focusing on the investigation of random necklaces and on the algorithmic aspects of the problem. In Section 3 we consider problems dealing with subgraphs of prescribed type in edge colored graphs that contain a fair or nearly fair share of each color. The results in Subsection 3.2 here are new. The final Section 4 contains a discussion of open problems.

2 Necklaces

The bound $(k-1)t$ in the Necklace Theorem (Theorem 1.3) is tight for all admissible values of the parameters. One example demonstrating this is a necklace in which all the beads of each type appear contiguously. In this case at least $k-1$ cuts are needed somewhere in the interval of beads of type i for every i just in order to ensure that each of the collections contains a positive number of beads of each type. A possible interpretation of the Theorem is the following. Suppose that k mathematically oriented thieves want to distribute the necklace fairly among them. The statement ensures that if the number of beads of each of the t types is divisible by k then they can always do it by opening the necklace at the clasp and making at most $(k-1)t$ cuts between beads. This raises two natural questions. The first is if the bound $(k-1)t$ can typically be improved. The second is the algorithmic problem of finding the cuts and the partition into k fair collections efficiently. In this section we briefly describe recent results about both problems.

2.1 Random Necklaces

As mentioned above the bound $(k-1)t$ in the Necklace Theorem is tight. Is the typical minimum number of required cuts smaller? This is studied in a recent joint work in progress with Dor Elboim, János Pach and Gábor Tardos, [6]. The random model considered is a necklace of total length $n = ktm$ consisting of exactly km beads of type i for each $1 \leq i \leq t$, chosen uniformly among all intervals of n beads as above. Call a set of cuts of such a necklace *fair*, if it is possible to split the resulting intervals into k collections, each containing exactly m beads of each type. For a necklace N, let $X = X(N)$ be the minimum number of cuts in a fair collection. When N is chosen randomly as above, X is a random variable which we denote by $X(k,t,m)$. By Theorem 1.3 we have $X(k,t,m) \leq (k-1)t$ with probability 1. In [6] we study the typical behavior of the random variable $X = X(k,t,m)$. The results are asymptotic, where at least one of the three variables k,t,m tends to infinity. As usual, we say that a result holds *with high probability* (*whp*, for short), if the probability that it holds tends to 1 when the relevant parameter(s) tend to infinity.

The problem of determining the asymptotic behavior of $X(k,t,m)$ turns out to be connected to several seemingly unrelated topics, including matchings in nearly regular hypergraphs with small codegrees and random walks in Euclidean spaces.

The first observation in [6] is the following.

Proposition 2.1 *For every fixed k and t, as m tends to infinity, $X = X(k,t,m) \geq \left\lceil \frac{(k-1)(t+1)}{2} \right\rceil$ whp.*

The proof is a simple first moment argument, whose details are omitted.

The main result describes the asymptotic behavior of $X = X(k,t,m)$ for two thieves ($k=2$) and any fixed number of types t, as m tends to infinity.

Theorem 2.2 *Let t be a fixed positive integer and $m \to \infty$.*

1. *For all $1 \leq s < \frac{t+1}{2}$,*

$$\mathbb{P}\big(X(2,t,m) = s\big) = \Theta\big(m^{s-\frac{t+1}{2}}\big). \tag{2.1}$$

2. When t is odd and $s = \frac{t+1}{2}$,

$$\mathbb{P}\big(X(2,t,m) = s\big) = \Theta\Big(\frac{1}{\log m}\Big). \tag{2.2}$$

3. For all $\frac{t+1}{2} < s \le t$,

$$\mathbb{P}\big(X(2,t,m) = s\big) = \Theta(1). \tag{2.3}$$

Two additional results deal with the case $m = 1$, in which every thief should get a single bead of each type.

Theorem 2.3 *For t and $k/\log t$ tending to infinity, the random variable $X = X(k,t,1)$ is $o(kt)$ whp.*

Theorem 2.4 *The random variable $X = X(2,t,1)$ is at least $2H^{-1}(1/2)t - o(t) = 0.220...t - o(t)$ whp, where $H^{-1}(x)$ is the inverse of the binary entropy function $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ taking values in the interval $[0,1/2]$.*

On the other hand, $X \le 0.4t + o(t)$ holds whp.

The upper bound above was obtained jointly with Alweiss, Defant and Kravitz, c.f. [6].

The proof of Theorem 2.2 applies the first and second moment and is rather lengthy and technical. A brief outline of the argument for the special case $t = 3$ follows. For $t = 3$ the probability that for the random necklace N, $X(N) = 1$ is easily seen to be $\Theta(1/m)$. By Theorem 1.3 for every N, we have $X(N) \le 3$. Thus, it remains to show that the probability that $X(N) \le 2$ is $\Theta(1/\log m)$. In order to estimate this probability, note that two cuts suffice if and only if there is a balanced partition of N into two cyclic intervals that is fair. There are exactly $3m$ balanced partitions into two cyclic intervals. For $0 \le i < 3m$, we denote by P_i the balanced partition into an interval starting at position $i+1$ and ending at position $i+3m$, and its complement.

Let $Y = Y(N)$ denote the random variable counting the number of fair partitions into cyclic intervals. Clearly, $X(N) \le 2$ if and only if Y is positive. This probability is lower bounded by the second moment method. It is not too difficult to check that the expectation of Y is $\Theta(1)$ and the expectation of Y^2 is $\Theta(\log m)$. Therefore, by the Paley-Zygmund Inequality [38], [39] the probability that Y is positive is at least $\Omega(1/\log m)$.

The proof of the upper bound for the probability that Y is positive is more interesting. It is done by defining another random variable $Z = Z(N)$. It is then shown that Z is positive with probability $O(1/\log m)$, and that the probability that Y is positive but Z is not, is even lower. The crucial step in bounding the probability that Z is positive, is the analysis of the probability that an appropriate two-dimensional random walk does not return to the origin in a certain number of steps. For this one can apply a slightly modified version of a classical argument of Dvoretzky and Erdős [17]. The details will appear in [6].

The proof of Theorem 2.3 applies a hypergraph edge-coloring result of Pippenger and Spencer [36]. This result asserts that the edges of any hypergraph of constant uniformity and large maximum degree k in which every pair of vertices lie in at most

$o(k)$ common edges, can be partitioned into $(1 + o(1))k$ matchings. By cutting the necklace into intervals of large constant size it is possible to define an appropriate hypergraph and show that whp it satisfies the conditions of the theorem of [36], which provides the required result.

2.2 The algorithmic aspects

The proof of Theorem 1.3 is topological. It starts by converting the problem into a continuous one dealing with interval coloring. Let $I = [0, 1]$ be the unit interval. An *interval t-coloring* is a coloring of the points of I by t colors, such that for each $i, 1 \leq i \leq t$, the set of points colored i is (Lebesgue) measurable. Given such a coloring, a *k-splitting of size r* is a sequence of numbers $0 = y_0 \leq y_1 \leq \ldots \leq y_r \leq y_{r+1} = 1$ and a partition of the family of $r + 1$ intervals $F = \{[y_i, y_{i+1}) : 0 \leq i \leq r\}$ into k pairwise disjoint subfamilies $F_1, \ldots, F_k$ whose union is F, such that for each $1 \leq j \leq k$ the union of the intervals in F_j captures precisely $1/k$ of the total measure of each of the t colors. The continuous version of the theorem is then the following.

Theorem 2.5 *Every interval t-coloring has a k-splitting of size $(k-1) \cdot t$.*

It is not difficult to show that this implies the Necklace Theorem. Indeed, the necklace can be converted to an interval coloring by replacing each bead by a small interval of the corresponding color. If the splitting ensured by the last theorem contains cuts that lie inside intervals corresponding to beads, it can be shown that these can be shifted to produce a splitting of the discrete necklace. The proof of Theorem 2.5 proceeds by first showing, by a simple combinatorial argument, that its validity for (t, k_1) and for (t, k_2) implies its validity for (t, k_1k_2). The main step is a proof that the assertion of the theorem holds for any prime k. This is done by applying a fixed point theorem of Bárány, Shlosman and Szücs [14], which can be viewed as an extension of the Borsuk-Ulam Theorem. Indeed, the case $k = 2$ of Theorem 2.5 admits a short proof using the Borsuk-Ulam Theorem, as shown in [9]. It can also be derived quickly from the Ham Sandwich Theorem, applying it to the measures obtained by placing the interval along the moments curve in R^t. The assertion of the theorem actually holds for general continuous probability measures, and not only for ones corresponding to interval colorings. Indeed, for $k = 2$ this extension is the Hobby-Rice Theorem [29], and the general case is proved in [3]. It is worth noting that a classical result of Liapounoff [32] implies that for any collection of t continuous probability measures μ_i on $[0, 1]$ and any $0 \leq \alpha \leq 1$ there is a subset A of $[0, 1]$ with μ_i measure α for each $1 \leq i \leq t$. The assertion of Theorem 2.5 for general continuous measures shows that for $\alpha = 1/k$ the interval can be partitioned into k such sets A_i, each being a union of a relatively small number of intervals.

The topological proof of the main step in the derivation of Theorem 1.3 is non-constructive, and does not supply any efficient algorithm for finding the required $(k-1)t$ cuts that provide a fair partition for a given input necklace. For $k = 2$ this algorithmic problem, raised in [4], is called *the Necklace Halving Problem.* A recent result of Filos-Ratsikad and Goldberg [20] shows that this is a hard problem.

PPA and PPAD are two complexity classes introduced by Papadimitriou, [34]. Although this is not our focus here, we include a very brief paragraph about the relevance of these classes to some of the problems discussed here. Both PPA and

PPAD are contained in the class TFNP, which is the complexity class of total search problems, consisting of all problems in NP where a solution exists for every instance. A problem is PPA-complete if and only if it is polynomially equivalent to the canonical problem LEAF, described in [34]. Similarly, a problem is PPAD-complete if and only if it is polynomially equivalent to the problem END-OF-THE-LINE. A problem is PPA-hard or PPAD-hard if the respective canonical problem is polynomially reducible to it. A number of important problems, such as several versions of Nash Equilibrium, have been proved to be PPAD-complete. It is known that PPAD $\subseteq$ PPA. Hence, PPA-hardness implies PPAD-hardness, and if a PPA-hard problem admits an efficient algorithm, so do all problems in PPA (and hence also in PPAD). Filos-Ratsikas and Goldberg [20] showed that the Necklace Halving problem, which is the problem of finding a collection of t cuts that provide a fair partition of a given input necklace with beads of t types and an even number of beads of each type, is PPA-hard [20]. This suggests the problem of finding an efficient algorithm for obtaining a fair partition using a somewhat larger number of cuts. An early result in this direction appears in [12], but it only provides a partition in which the number of beads of each type in the two collections are close to each other, and the number of cuts is exponential in the number of types. A recent improved algorithm is given in [7]. Its performance is described in the next result.

Theorem 2.6 *There is a polynomial time algorithm that given an input necklace with beads of t types, in which the number of beads of each type is an even number that does not exceed m, produces a collection of at most $t(\log m + O(1))$ cuts and a partition of the resulting intervals into two collections, each containing exactly half of the beads of each type.*

The algorithm proceeds by first converting the problem to the continuous interval coloring problem described above. The continuous problem is tackled using a linear algebra procedure based on Carathéodory's Theorem for cones. Its solution can then be rounded to produce a solution of the discrete problem. The details are sketched below.

Proof of Theorem 2.6 (sketch):

Given a necklace with m_i beads of color i for $1 \leq i \leq t$, where $m = \max m_i$, replace each bead of color i by an interval of μ_i-measure $1/m_i$ and μ_j-measure 0 for all $j \neq i$. These intervals are placed next to each other according to the order in the necklace, and their lengths are chosen so that altogether they cover $[0,1]$. We first describe a procedure that splits the interval into two collections so that for every i the difference between the μ_i-measures of the two collections is at most ε, where $\varepsilon = \frac{1}{2m}$. The number of cuts used here is at most $t(\log m + O(1))$. It is then not too difficult to round the cuts and get a solution of the discrete problem without increasing the number of cuts.

Let μ_i be the t measures defined above. Our objective is to describe an efficient algorithm that cuts the interval in at most $t(2 + \lceil \log_2 \frac{1}{\varepsilon} \rceil)$ places and splits the resulting intervals into two collections C_0, C_1 so that $\mu_i(C_j) \in [\frac{1}{2} - \frac{\varepsilon}{2}, \frac{1}{2} + \frac{\varepsilon}{2}]$ for all $i \in [t] = \{1, 2, \ldots, t\}, 0 \leq j \leq 1$.

For each interval $I \subset [0,1]$ denote $\mu(I) = \mu_1(I) + \ldots + \mu_t(I)$. Thus $\mu([0,1]) = t$. Using $2t - 1$ cuts split $[0,1]$ into $2t$ intervals $I_1, I_2, \ldots, I_{2t}$ so that $\mu(I_r) = 1/2$ for all

r. Note that it is easy to find these cuts efficiently, since each measure μ_i is uniform on its support. For each interval I_r let v_r denote the t-dimensional vector

$$(\mu_1(I_r), \mu_2(I_r), \ldots, \mu_t(I_r)).$$

By a simple linear algebra argument, which is a standard fact about the properties of basic solutions for Linear Programming problems, one can write the vector $(1/2, 1/2, \ldots, 1/2)$ as a linear combination of the vectors v_r with coefficients in $[0, 1]$, where at most t of them are not in $\{0, 1\}$. This follows from Carathédory's Theorem for cones. Here is the simple proof, which also shows that one can find coefficients as above efficiently. Start with all coefficients being $1/2$. Call a coefficient which is not in $\{0, 1\}$ *floating* and one in $\{0, 1\}$ *fixed*. Thus at the beginning all $2t$ coefficients are floating. As long as there are more than t floating coefficients, find a nontrivial linear dependence among the corresponding vectors and subtract a scalar multiple of it which keeps all floating coefficients in the closed interval $[0, 1]$ shifting at least one of them to the boundary $\{0, 1\}$, thus fixing it.

This process clearly ends with at most t floating coefficients. The intervals with fixed coefficients with value 1 are now assigned to the collection C_1 and those with coefficient 0 to C_0. The rest of the intervals remain. Split each of the remaining intervals into two intervals, each with μ-value $1/4$. We get a collection $J_1, J_2, \ldots, J_m$ of $m \leq 2t$ intervals, each of them has the coefficient it inherits from its original interval. Each such interval defines a t-vector as before, and the sum of these vectors with the corresponding coefficients (in $(0, 1)$) is exactly what the collection C_1 should still get to have its total vector of measures being $(1/2, \ldots, 1/2)$.

As before, we can shift the coefficients until at most t of them are floating, assign the intervals with $\{0, 1\}$ coefficients to the collections C_0, C_1 and keep at most t intervals with floating coefficients. Split each of those into two intervals of μ-value $1/8$ each and proceed as before, until we get at most t intervals with floating coefficients, where the μ-value of each of them is at most $\varepsilon/2$. This happens after at most $\lceil \log_2(1/\varepsilon) \rceil$ rounds. In the first one, we have made $2t - 1$ cuts and in each additional round at most t cuts. Thus the total number of cuts is at most $t(2 + \lceil \log_2(1/\varepsilon) \rceil) - 1$.

From now on we do not increase the number of cuts, and show how to shift them and allocate the remaining intervals to C_0, C_1. Let $\mathcal{I}$ denote the collection of intervals with floating coefficients. Then $|\mathcal{I}| \leq t$ and $\mu(I) \leq \varepsilon/2$ for each $I \in \mathcal{I}$. This means that

$$\sum_{i=1}^{t} \sum_{I \in \mathcal{I}} \mu_i(I) \leq t\varepsilon/2.$$

It follows that there is at least one measure μ_i so that

$$\sum_{I \in \mathcal{I}} \mu_i(I) \leq \varepsilon/2.$$

Observe that for any assignment of the intervals $I \in \mathcal{I}$ to the two collections C_0, C_1, the total μ_i-measure of C_1 (and hence also of C_0) lies in $[1/2-\varepsilon/2, 1/2+\varepsilon/2]$, as this measure with the floating coefficients is exactly $1/2$ and any allocation of the intervals with the floating coefficients changes this value by at most $\varepsilon/2$. We can thus ignore this measure, for ease of notation assume it is measure number t, and replace each measure vector of the members in $\mathcal{I}$ by a vector of length $t-1$ corresponding to the other $t-1$ measures. If $|\mathcal{I}| > t-1$ (that is, if $|\mathcal{I}| = t$), then it is possible to shift the floating coefficients as before until at least one of them reaches the boundary, fix it assigning its interval to C_1 or C_0 as needed, and omit the corresponding interval from $\mathcal{I}$ ensuring its size is at most $t-1$. This means that for the modified $\mathcal{I}$ the sum

$$\sum_{i=1}^{t-1} \sum_{I \in \mathcal{I}} \mu_i(I) \leq (t-1)\varepsilon/2.$$

Hence there is again a measure μ_i, $1 \leq i \leq t-1$ so that

$$\sum_{I \in \mathcal{I}} \mu_i(I) \leq \varepsilon/2.$$

Again, we may assume that $i = t-1$, observe that measure number $t-1$ will stay in its desired range for any future allocation of the remaining intervals, and replace the measure vectors by ones of length $t-2$. This process ends with an allocation of all intervals to C_1 and C_0, ensuring that at the end $\mu_i(C_j) \in [1/2 - \varepsilon/2, 1/2 + \varepsilon/2]$ for all $1 \leq i \leq t$, $0 \leq j \leq 1$. These are the desired collections. It is clear that the procedure for generating them is efficient, requiring only basic linear algebra operations.

This completes the (sketch of the) proof. The full details can be found in [7]. □

3 Graphs

3.1 Fair representation

Theorem 1.2 and Conjecture 1.5 mentioned in Section 1 are two examples of fair representation problems dealing with graphs. There are quite a few additional results and conjectures of this type. We start this section by discussing several examples.

An *optimal proper edge coloring* of the complete graph K_{2n} on an even number of vertices is a coloring of the edges by $2n-1$ colors, each forming a perfect matching. Given such an edge coloring, the fair share of a spanning tree in each color class is exactly 1. Brualdi and Hollingsworth [10] conjectured that for each such edge coloring of $K = K_{2n}$ where $n > 4$ one can partition all edges of K into n pairwise edge disjoint *rainbow* spanning trees, that is, each tree containing exactly one edge of each color. Constantinos [15] conjectured that it is even possible to find such a partition in which all trees are isomorphic. This is proved for all sufficiently large n in a recent paper of Glock, Kühn, Montgomery and Osthus [23]. The proof is probabilistic, and is based on hypergraph matching results and the so-called absorption technique. This technique starts by removing an appropriate small part of the graph, finding an approximate partition of the rest, and then using the small part to complete it to a precise partition. The details, which require quite some work, can be found in [23]. Similar ideas are useful in the study of several related problems, as described in [23] and its references.

The Cycle and Triangles Theorem (Theorem 1.2) has been proved in [21] using the algebraic approach of [8]. This approach enables one to bound the chromatic number of a graph, and in fact even its so-called list chromatic number, by showing that a certain coefficient of an appropriate polynomial is nonzero. Subsequent proofs of the theorem (at least of the statement about the existence of a single independent set of the required form) apply topological ideas. The shortest proof is the one in [1] where the result is derived from Schrijver's Theorem on vertex critical subgraphs of the Kneser graph. This Theorem, which strengthens the result of Lovász about the chromatic number of the Kneser graph, is proved in [43] using the Borsuk-Ulam Theorem.

Theorem 3.1 (Schrijver [43]) *For $n > 2k$ the family of independent sets of size k in the cycle C_n cannot be partitioned into fewer than $n-2k+2$ intersecting families.*

Now let G be a cycle of length $3m$ and let P be a partition of its set of vertices into pairwise disjoint sets $P_1, P_2, \ldots, P_m$, each of size 3. Assuming the first assertion of Theorem 1.2 fails, there is no independent set of G that contains exactly one vertex of each set P_i. In this case each independent set of size m in the cycle G contains at least two vertices in some set P_i, and we can partition all these independent sets into m families, where a set S belongs to family number i iff i is the smallest index so that $|S \cap P_i| \geq 2$. Note that each such family is intersecting, as each member of it contains at least two vertices among the three vertices of P_i. But since $m < 3m - 2m + 2$ this contradicts Theorem 3.1 with $n = 3m$ and $k = m$, proving the existence of an independent set containing one vertex in each P_i.

The short proof above can be extended in several ways. In particular the following holds.

Proposition 3.2 ([1]) *If $V = V_1 \cup V_2 \cup \cdots \cup V_m$ is a partition of the vertex set of a cycle C into m pairwise disjoint sets, and $|V_i|$ is odd for all i, then for any vertex v of C there is an independent set S of C so that $v \notin S$ and $|S \cap V_i| = (|V_i| - 1)/2$ for all i.*

In an attempt to strengthen Ryser's Conjectures (Conjecture 1.5), Stein [44] suggested the stronger conjecture that for any partition P of the edges of the complete bipartite graph $K_{n,n}$ into n pairwise disjoint color classes, each containing exactly n edges, there exists a rainbow matching of size $n-1$. This turned out to be too strong. A counterexample was found by Pokrovskiy and Sudakov in [37], where the authors describe a coloring as above so that every matching misses at least $\Omega(\log n)$ color classes. This shows that in some natural cases tight fair representations fail and suggests relaxed versions of questions of this type, as discussed in the following subsection.

3.2 Nearly fair representation

A special case of the approach described here was initiated in discussions with Eli Berger and Paul Seymour. Let $G = (V, E)$ be a graph and let P be an arbitrary partition of its set of edges into m pairwise disjoint subsets $E_1, E_2, \ldots, E_m$. The sets E_i are called the color classes of the partition. For any subgraph $H' = (V', E')$ of G, let $x(H', P)$ denote the vector $(x_1, x_2, \ldots, x_m)$, where $x_i = |E_i \cap E'|$ is the number of edges of H' that lie in E_i. Thus, in particular, $x(G,P) = (|E_1|, \ldots, |E_m|)$. In a completely fair representation of the sets E_i in H', each entry x_i of the vector $x(H',P)$ should be equal to $|E_i| \cdot \frac{|E'|}{|E|}$. Of course such equality can hold only if all these numbers are integers. But even when this is not the case the equality may hold up to a small additive error.

We are interested in results and conjectures asserting that when G is either the complete graph K_n or the complete bipartite graph $K_{n,n}$, then for certain graphs H and for any partition P of $E(G)$ into color classes $E_1, \ldots, E_m$, there is a subgraph H' of G which is isomorphic to H so that the vector $x(H',P)$ is close (or equal) to the vector $x(G,P)\frac{|E(H')|}{|E(G)|}$. As mentioned in the previous subsection, Stein [44] conjectured that if $G = K_{n,n}$ and P is any partition of the edges of G into n sets, each of size n, then there is always a rainbow matching of size $n-1$ in G. However, this turned out to be false as shown by a clever counter-example of Pokrovskiy and Sudakov [37].

In [1] it is conjectured that when $G = K_{n,n}$, P is arbitrary, and H is a matching of size n, then there is always a copy H' of H (that is, a perfect matching H' in G), so that

$$\|x(H',P) - \frac{1}{n}x(G,P)\|_\infty < 2.$$

This is proved in [1] (in a slightly stronger form) for partitions P with 2 or 3 color classes. Here we first prove the following, showing that when allowing a somewhat larger additive error (which grows with the number of colors m but is independent of n) a similar result holds for partitions with any fixed number of classes.

Theorem 3.3 *For any partition P of the edges of the complete bipartite graph $K_{n,n}$ into m color classes, there is a perfect matching M so that*

$$\|x(M,P) - \frac{1}{n}x(K_{n,n},P)\|_\infty \le \|x(M,P) - \frac{1}{n}x(K_{n,n},P)\|_2 < (m-1)2^{(3m-2)/2}.$$

It is worth noting that a random perfect matching M typically satisfies

$$\|x(M,P) - \frac{1}{n}x(K_{n,n},P)\|_\infty \le O_m(\sqrt{n}).$$

The main challenge addressed in the theorem is to get an upper bound independent of n.

Theorem 3.3 is a special case of a general result which we describe next, starting with the following definition.

Definition 3.4 Let G be a graph and let H be a subgraph of it. Call a family of graphs $\mathcal{H}$ (which may have repeated members) a *uniform cover of width s of the pair* (G,H) if the following four conditions hold.

- Every member H' of $\mathcal{H}$ is a subgraph of G which is isomorphic to H.
- The number of edges of each such H' which are not edges of H is at most s.
- Every edge of H belongs to the same number of members of $\mathcal{H}$.

- Every edge in $E(G) - E(H)$ belongs to the same positive number of members of $\mathcal{H}$.

An example of a uniform cover of width $s = 2$ for $G = K_{n,n}$ and H a perfect matching in it is the following. Let the n edges of H be $a_i b_i$ where $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ are the vertex classes of G. Let $\mathcal{H}$ be the family of all perfect matchings of G obtained from H by omitting a pair of edges $a_i b_i$ and $a_j b_j$ and by adding the edges $a_i b_j$ and $a_j b_i$. The width is 2, every edge of H belongs to exactly $\binom{n}{2} - (n-1)$ members of $\mathcal{H}$, and every edge in $E(G) - E(H)$ belongs to exactly 1 member of $\mathcal{H}$.

Theorem 3.5 *Let G be a graph with g edges, let F be a subgraph of it with f edges, and suppose there is a uniform cover of width s of the pair (G, F). Then for any partition P of the edges of G into m subsets, there is a copy H of F in G so that*

$$\|x(H,P) - \frac{f}{g}x(G,P)\|_\infty \leq \|x(H,P) - \frac{f}{g}x(G,P)\|_2 \leq (m-1)2^{(m-2)/2}s^m.$$

Theorem 3.3 is a simple consequence of Theorem 3.5. A similar simple consequence is the following.

Proposition 3.6 *For any partition P of the edges of the complete graph K_n into m color classes, there is a Hamilton cycle C so that*

$$\|x(C,P) - \frac{2}{n-1}x(K_n,P)\|_\infty \leq \|x(C,P) - \frac{2}{n-1}x(K_n,P)\|_2 < (m-1)2^{(3m-2)/2}.$$

Similar statements follow, by the same reasoning, for a Hamilton cycle in a complete bipartite graph, or for a perfect matching in a complete graph on an even number of vertices. We proceed to describe two more general applications.

For a fixed graph T whose number of vertices t divides n, a *T-factor* in K_n is the graph consisting of n/t pairwise vertex disjoint copies of T. In particular, when $T = K_2$ this is a perfect matching.

Theorem 3.7 *For any fixed graph T with t vertices and q edges and any m there is a constant $c = c(t,q,m) \leq (m-1)2^{(m-2)/2}(qt)^m$ so that for any n divisible by t and for any partition P of the edges of the complete graph K_n into m subsets, there is a T-factor H so that*

$$\|x(H,P) - \frac{2q}{(n-1)t}x(K_n,P)\|_\infty \leq \|x(H,P) - \frac{2q}{(n-1)t}x(K_n,P)\|_2 \leq c.$$

Another application, proved together with Sacheth Sathyanarayanan, is the following.

Theorem 3.8 *For any fixed d and m there is a constant $c = c(d,m)$ so that for any d-regular graph on n vertices H and for any partition P of the edges of the complete graph K_n into m subsets, there is a copy of H so that*

$$\|x(H,P) - \frac{d}{n-1}x(K_n,P)\|_\infty \leq \|x(H,P) - \frac{d}{n-1}x(K_n,P)\|_2 \leq c.$$

Note that Proposition 3.6 is a special case of the result above (with a specific value of $c(2,m)$).

We proceed with the proofs of the results above, starting with the proof of Theorem 3.5.

Proof of Theorem 3.5: Let P be a partition of the edges of G into m color classes E_i. Put

$$y = (y_1, y_2, \ldots, y_m) = \frac{f}{g} x(G,P).$$

Let H be a copy of F in G for which the quantity $\|y-x\|_2^2 = \sum_{j=1}^m (y_i - x_i)^2$ is minimum where $x = (x_1, x_2, \ldots, x_m) = x(H,P)$. Let $\mathcal{H}$ be a uniform cover of width s of the pair (G,H). Suppose each edge of H belongs to a members of $\mathcal{H}$ and each edge in $E(G)-E(H)$ belongs to $b>0$ such members. For each member H' of $\mathcal{H}$, let $v_{H'}$ denote the vector of length m defined as follows. For each $1 \leq i \leq m$, coordinate number i of $v_{H'}$ is the number of edges in $E(H')-E(H)$ colored i minus the number of edges in $E(H)-E(H')$ colored i. Note that the ℓ_1-norm of this vector is at most $2s$ and its sum of coordinates is 0. Therefore, its ℓ_2-norm is at most $\sqrt{2s^2}$. Note also that $x(H',P) = x(H,P) + v_{H'}$.

We claim that the sum S of all $|\mathcal{H}|$-vectors $v_{H'}$ for $H' \in \mathcal{H}$ is a positive multiple of the vector $(y-x)$. Indeed, each edge in $E(G)-E(H)$ is covered by b members of $\mathcal{H}$, and each edge of $E(H)$ is covered by a members of $\mathcal{H}$. In the sum S above this contributes to the coordinate corresponding to color number i, b times the number of edges of color i in $E(G)-E(H)$ minus $(|\mathcal{H}|-a)$ times the number of edges of color i in H. Equivalently, this is b times the number of all edges of G colored i minus $(|\mathcal{H}|+b-a)$ times the number of edges of H colored i. Since the sum of coordinates of each of the vectors $v_{H'}$ is zero, so is the sum of coordinates of S, implying that $bg = (|\mathcal{H}|+b-a)f$, that is, $|\mathcal{H}|+b-a = \frac{g}{f}b$. Since $\frac{g}{f}y = x(G,P)$ this implies that $S = \frac{bg}{f}(y-x)$, proving the claim.

Since the vector $y-x$ is a linear combination with positive coefficients of the vectors $v_{H'}$ it follows, by Carathéodory's Theorem for cones, that there exists a set L of linearly independent vectors $v_{H'}$ so that $y-x$ is a linear combination with positive coefficients of them. Indeed, starting with the original expression of $y-x$ mentioned above, as long as there is a linear dependence among the vectors $v_{H'}$ participating in the combination with nonzero (hence positive) coefficients, we can subtract an appropriate multiple of this dependence and ensure that at least one of the nonzero coefficients vanishes and all others stay non-negative (positive, after omitting all the ones with coefficient 0). As each vector $v_{H'}$ has m coordinates and their sum is 0, it follows that $|L| \leq m-1$.

We can now solve the system of linear equations $y - x = \sum z_{H'} v_{H'}$ with the variables $z_{H'}$ for $v_{H'} \in L$. Note that it is enough to consider any $|L| \leq m-1$ coordinates of $y-x$ and solve the system corresponding to these coordinates. By Cramer's rule applied to this system each $z_{H'}$ is a ratio of two determinants. The denominator is a determinant of a nonsingular matrix with integer coefficients, and its absolute value is thus at least 1. The numerator is also a determinant, and by Hadamard's Inequality its absolute value is at most the product of the ℓ_2-norms of the columns of the corresponding matrix. The norm of one column is at most $\|y-x\|_2$ (this can be slightly improved by selecting the $|L|$-coordinates with the smallest ℓ_2-norm, but we do not include this slight improvement here). Each other column has norm at most $(2s^2)^{1/2}$. Therefore each coefficient $z_{H'}$ satisfies $0 \leq z_{H'} \leq \|y-x\|_2(2s^2)^{(m-2)/2}$. By taking the inner product with $y-x$ we get

$$\|y-x\|_2^2 = \sum_{v_{H'}\in L} z_{H'}\langle y-x, v_{H'}\rangle$$

$$\leq \sum_{v_{H'}\in L, \langle y-x, v_{H'}\rangle > 0} z_{H'}\langle y-x, v_{H'}\rangle$$

$$\leq (m-1)\|y-x\|_2(2s^2)^{(m-2)/2}\max\langle y-x, v_{H'}\rangle.$$

Therefore, there is a $v_{H'}$ so that

$$\frac{\|y-x\|_2}{(m-1)(2s^2)^{(m-2)/2}} = \frac{\|y-x\|_2^2}{(m-1)(2s^2)^{(m-2)/2}\|y-x\|_2} \leq \langle y-x, v_{H'}\rangle,$$

that is,

$$\|y-x\|_2 \leq (m-1)(2s^2)^{(m-2)/2}\langle y-x, v_{H'}\rangle = (m-1)2^{(m-2)/2}s^{m-2}\langle y-x, v_{H'}\rangle. \quad (3.1)$$

By the minimality of $\|y-x\|_2^2$

$$\|x+v_{H'}-y\|_2^2 = \|x-y\|_2^2 - 2\langle y-x, v_{H'}\rangle + \|v_{H'}\|_2^2 \geq \|x-y\|_2^2,$$

implying that

$$2s^2 \geq \|v_{H'}\|_2^2 \geq 2\langle y-x, v_{H'}\rangle.$$

Plugging in (3.1) we get

$$\|y-x\|_2 \leq (m-1)2^{(m-2)/2}s^m,$$

and the desired results follows since $\|y-x\|_\infty \leq \|y-x\|_2$. □

The assertions of Theorem 3.3 and Proposition 3.6 follow easily from Theorem 3.5. Indeed, as described above there is a simple uniform cover of width $s=2$ for the pair $(K_{n,n}, M)$ where M is a perfect matching. There is also a similar uniform cover $\mathcal{H}$ of width $s=2$ for the pair (K_n, C) where C is a Hamilton cycle. The $n(n-3)/2$ members of $\mathcal{H}$ are all Hamilton cycles obtained from C by omitting two nonadjacent edges of it and by adding the two edges that connect the resulting pair of paths to a cycle.

To prove Theorem 3.7 we need the following simple lemma.

Lemma 3.9 *Let T be a fixed graph with t vertices and q edges, suppose t divides n and let H be a T-factor in K_n. Then there is a uniform cover of width at most qt of the pair (K_n, H).*

Proof Let H be a fixed T-factor in K_n, it consists of $p = n/t$ (not necessarily connected) vertex disjoint copies of T which we denote by $T_1, T_2, \ldots, T_p$. Let $\mathcal{H}_1$ be the set of all copies H' of the T-factor obtained from H by replacing one the copies T_i by another copy of T on the same set of vertices, in all $t!$ possible ways. Note that if T has a nontrivial automorphism group some members of $\mathcal{H}_1$ are identical, and $\mathcal{H}_1$ is a multiset. By symmetry it is clear that each edge of H belongs to the same number of members of $\mathcal{H}_1$. Similarly, each edge connecting two vertices of the same T_i which does not belong to H lies in the same positive number of members of $\mathcal{H}_1$. Beside these two types of edges, no other edge of K_n is covered by any member of $\mathcal{H}_1$. Let $\mathcal{H}_2$ be the (multi)-set of all copies of the T-factor obtained from H by choosing, in all possible ways, t of the copies of T, say, $T_{i_1}, T_{i_2}, \ldots, T_{i_t}$, removing them, and replacing them by all possible placements of t vertex disjoint copies of T where each of the newly placed copies contains exactly one vertex of each T_{i_j}. Again by symmetry it is clear that each edge of H belongs to the same number of members of $\mathcal{H}_2$. In addition, each edge of K_n connecting vertices from distinct copies of T in H belongs to the same (positive) number of members of $\mathcal{H}_2$. No other edges of K_n are covered by any $H' \in \mathcal{H}_2$. It is now simple to see that there are two integers a, b, so that the multiset $\mathcal{H}$ consisting of a copies of each member of $\mathcal{H}_1$ and b copies of each member of $\mathcal{H}_2$ is a uniform cover of the pair (K_n, H). The width of this cover is clearly qt, as every member of $\mathcal{H}_2$ contains qt edges not in $E(H)$, and every member of $\mathcal{H}_1$ contains at most q edges not in $E(H)$. This completes the proof. □

The assertion of Theorem 3.7 clearly follows from the last lemma together with Theorem 3.5.

Proof of Theorem 3.8: By Theorem 3.5 it suffices to show that there is a uniform cover of bounded width of the pair (K_n, H). Fix a copy H' of H in K_n and let $\mathcal{H}$ be the (multi)-set of all $\binom{n}{2}$ graphs $H_{u,v}$ obtained from H' by swapping a pair of vertices u, v. Specifically, for every pair of distinct vertices u, v of H', let $H_{u,v}$ be the graph obtained from H' by removing all edges incident with u or with v, and by adding all edges connecting v to a neighbor of u and all edges connecting u to a neighbor of v. It is clear that each graph $H_{u,v}$ is isomorphic to H' (by the isomorphism swapping u and v). It is also clear that each $H_{u,v}$ contains at most $2d$ edges that do not lie in H'. Therefore the width of the collection is at most $2d$. It remains to check that this is a uniform cover.

Let xy be an edge of H'. This edge belongs to H_{uv} iff either $\{u,v\} \cap \{x,y\} = \emptyset$, or one of the members of $\{u,v\}$ is x and the other is a neighbor of y (or y itself), or, symmetrically, if one of the members of $\{u,v\}$ is y and the other is a neighbor of x. Altogether there are $\binom{n-2}{2} + 2d - 1$ such (unordered) pairs $\{u,v\}$. This is the same number for every edge xy, as needed for a uniform cover. Now let xy be a non-edge of H'. Then it belongs to $H_{u,v}$ iff either one of the members of $\{u,v\}$ is x and the other is a neighbor of y or one of the members of $\{u,v\}$ is y and the other is a neighbor of x. There are exactly $2d$ such pairs $H_{u,v}$, independently of the specific choice of the non-edge xy. This shows that $\mathcal{H}$ is indeed a uniform cover of (K_n, H), completing the proof. □

Remark

A similar argument shows that for every d-regular spanning subgraph H of the complete bipartite graph $K_{n,n}$ there is a uniform cover of width at most $2d$ of the pair $(K_{n,n}, H)$. Indeed, here the family of all $\binom{n}{2}$ copies of H obtained from a fixed one by swapping every pair of vertices in one of the two color classes is such a uniform cover. This and Theorem 3.5 implies a result about nearly fair representations of any regular spanning subgraph of $K_{n,n}$. Theorem 3.3 is a special case.

Additional remarks

- The statement of Theorem 3.7 holds for any graph H consisting of n/t (not necessarily connected) vertex disjoint components, each having t vertices and q edges. The proof applies with no need to assume that all these components are isomorphic.

- The proof of Theorem 3.5 is algorithmic in the sense that if the cover $\mathcal{H}$ is given then one can find, in time polynomial in n and $|\mathcal{H}|$, a copy H of F satisfying the conclusion. Indeed, the proof implies that as long as we have a copy H for which the conclusion does not hold, there is a member $H' \in \mathcal{H}$ for which $\|x(H',P) - \frac{f}{g}x(G,P)\|_2^2$ is strictly smaller than $\|x(H,P) - \frac{f}{g}x(G,P)\|_2^2$. By checking all members of $\mathcal{H}$ we can find an H' for which this holds. As both these quantities are non-negative rational numbers smaller than n^4 with denominator $g^2 < n^4$, this process terminates in a polynomial number of steps. We make no attempt to optimize the number of steps here.

- Theorem 3.5 can be extended to r-uniform hypergraphs by a straightforward modification of the proof.

- There are graphs H for which no result like those proved above holds when G is either a complete or a complete bipartite graph even if the number of colors is small. A simple example is when $G = K_{2n}$, $H = K_{1,2n-1}$ and $m = 3$. The edges of K_{2n} can be partitioned into two vertex disjoint copies of K_n and a complete bipartite graph $K_{n,n}$. For this partition, every copy of the star H misses completely one of the color classes, although its fair share in it is roughly a quarter of its edges. More generally, let H be any graph with a vertex cover of size smaller than $m-1$ (that is, H contains a set of less than $m-1$ vertices touching all its edges). Consider a partition of the edges of the complete graph K_n into $m-1$ pairwise vertex disjoint copies of the complete graph on $\lfloor n/(m-1) \rfloor$ vertices, and an additional class containing all the remaining edges. Then any copy of H in this graph cannot contain edges of all those $m-1$ complete subgraphs, as the edges of the copy can be covered by less than $m-1$ stars. It is easy to see that similar examples exist for $G = K_{n,n}$ as well.

4 Open problems

Several open problems and conjectures dealing with the topic of this article are mentioned in the previous sections. In this final section we discuss several additional intriguing problems.

- Recall that the random variable $X(k,t,m)$ is the minimum number of cuts in a fair partition of a random necklace with km beads of each of t types into k parts. Theorem 2.2 determines the asymptotic behavior of this variable for $k = 2$ and fixed t, as m tends to infinity. It will be interesting to study the behavior of this random variable for all admissible values of the parameters.

- Theorem 2.4 provides upper and lower bounds for the ratio $X(2,t,1)/t$, implying that the liminf of this quantity is at least roughly 0.22 and the limsup is at most 0.4. Both these bounds can be slightly improved, as shown in [6], but there is still a large gap between the upper and lower bounds. While the problem of closing this gap appears to be difficult, it is not even known if the limit of this ratio as t tends to infinity exists. Naturally, we believe it does exist, but have not been able to prove it.

- The algorithmic aspects of the Necklace Theorem (Theorem 1.3) are discussed in Subsection 2.2. The computational aspects of some of the other results described here are also interesting. In particular, the algorithmic question corresponding to the Cycle and Triangles Theorem (Theorem 1.2) is challenging. The input of this problem is a cycle of length $3m$ and a partition P of its set of vertices into pairwise disjoint sets $P_1, P_2, \ldots, P_m$, each of size 3. The output, in the simpler problem, is an independent set containing exactly one vertex in each P_i, and in the harder problem it is a proper 3-coloring of the cycle in which each color class contains exactly one vertex of each P_i. There is no known efficient algorithm for solving this problem, and yet it is also not known to be PPA-hard (or PPAD hard). On the other hand, Haviv [25] proved that the more general algorithmic problem corresponding to Proposition 3.2 is PPA-hard. Specifically, he showed that the following problem is PPA-hard. The input is a partition $V = V_1 \cup V_2 \cup \cdots \cup V_m$ of the vertex set of a cycle C into m pairwise disjoint sets V_i, and the output is an independent set containing at least $|V_i|/2 - 1$ vertices of each V_i. The proof proceeds by reduction to the PPA-hardness results in [20], and the sets V_i have to be polynomially large (in m) for establishing hardness. Therefore this hardness result does not apply to the Cycle and Triangles Problem. It is worth noting that as Theorem 1.2 admits several very different proofs, including an algebraic one and a topological one, proving hardness for it may be difficult, as it would imply hardness for the algorithmic version of each of the corresponding techniques from which it can be deduced.

  Note that if we replace the sets P_i of size 3 in Theorem 1.2 by sets of size 4, then the corresponding algorithmic problem does admit a simple efficient solution. This follows from the proof in [5]. The invariant studied there is the so-called *strong chromatic number of a graph*. The strong chromatic number $s\chi(G)$ of a graph G with n vertices is the smallest number k such that after adding $k\lceil n/k \rceil - n$ isolated vertices to G, for any partition of the set of vertices of the resulting graph into disjoint subsets $P_1, P_2, \ldots, P_{\lceil n/k \rceil}$, each of size k, there is a proper vertex coloring of G by k colors so that each color class contains exactly one vertex of each P_i. The Cycle and Triangles Theorem asserts that the strong chromatic number of a cycle of length $3m$ is 3, and as shown in [5] it is easy to see that the strong chromatic number of any graph with maximum degree 2 is at most 4. For higher degrees, it is proved in [5] that the strong chromatic number of any graph with maximum degree d is at most $O(d)$. The hidden constant in this O notation has been vastly improved by Haxell [26], who showed that this maximum possible strong chromatic number is at most $3d - 1$. For large values d this is further improved in [27] to $(11/4 + \varepsilon)d$

for all $d > d_0(\varepsilon)$. It is believed, but not known, that the sharper bound $2d$ always holds. Regarding the algorithmic problem it is shown in [22] how to find efficiently one independent set containing a vertex of each set P_i in any partition of the vertex set of any given input graph G with maximum degree d into sets P_i, each of size at least $2d + 1$.

- The discussion in Subsection 3.2 suggests the following conjecture.

Conjecture 4.1 *For every d there exists a $c(d)$ so that for any graph H with at most n vertices and maximum degree at most d and for any partition P of the edges of K_n into m color classes, there is a copy H' of H in K_n so that*

$$\|x(H', P) - \frac{|E(H)|}{|E(K_n)|} x(K_n, P)\|_\infty \le c(d).$$

The analogous conjecture for bipartite bounded-degree graphs H with at most n vertices in each color class and for partitions of the edges of $K_{n,n}$ is also plausible. Note that the conjecture asserts that the same error term $c(d)$ should hold for any number of colors m. Note also that $c(d)$ must be at least $\Omega(d)$ as shown by the example of a star $H = K_{1,d}$ and the edge-coloring of K_{2n} with $m = 3$ colors described in Subsection 3.2.

- Another interesting question related to the results in Subsection 3.2 is whether or not for any d there is a constant $c(d)$ so that for any graph H on n vertices with maximum degree d there is a uniform cover of width at most $c(d)$ of the pair (K_n, H). Together with Sacheth Sathyanarayanan we proved that this is not the case for 3-uniform hypergraphs. Indeed, let $H_n^{(3)}$ be the tight 3-uniform cycle of length n in which the vertices are $v_0, v_1, v_2, \ldots v_{n-1}$ and the edges are all triples $\{v_i, v_{i+1}, v_{i+2}\}$ where the indices are reduced modulo n. This hypergraph is 3-regular, and it can be shown that if $K_n^{(3)}$ denotes the complete 3-uniform hypergraph on n vertices then any uniform cover of the pair $(K_n^{(3)}, H_n^{(3)})$ is of width $\Omega(n)$. The definition of a uniform cover of a pair of hypergraphs is defined just as it is defined for graphs in Definition 3.4.

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