Independent sets in tensor graph powers

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Abstract

The tensor product of two graphs, G and H, has a vertex set $V(G) \times V(H)$ and an edge between (u,v) and (u',v') iff both $uu' \in E(G)$ and $vv' \in E(H)$. Let $A(G)$ denote the limit of the independence ratios of tensor powers of G, $\lim \alpha(G^n)/|V(G^n)|$. This parameter was introduced in [5], where it was shown that $A(G)$ is lower bounded by the vertex expansion ratio of independent sets of G. In this note we study the relation between these parameters further, and ask whether they are in fact equal. We present several families of graphs where equality holds, and discuss the effect the above question has on various open problems related to tensor graph products.

1   Introduction

The *tensor* product (also dubbed as categorical or weak product) of two graphs, $G \times H$, is the graph whose vertex set is $V(G) \times V(H)$, where two vertices $(u,v),(u',v')$ are adjacent iff both $uu' \in E(G)$ and $vv' \in E(H)$, i.e., the vertices are adjacent in each of their coordinates. Clearly, this product is associative and commutative, thus G^n is well defined to be the tensor product of n copies of G.

The tensor product has attracted a considerable amount of attention ever since Hedetniemi conjectured in 1966 ([7]) that $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$ (where $\chi(G)$ denotes the chromatic number of G), a problem which remains open (see [17] for an extensive survey of this problem). For further work on colorings of tensor products of graphs, see [1], [9], [10], [13], [14], [16], [18].

It is easy to verify that Hedetniemi's conjecture is true when there is a homomorphism from G to H, and in particular when $G = H$, by examining a copy of G in $G \times H$, and it follows that

$\chi(G^n) = \chi(G)$ for every integer n. Furthermore, a similar argument shows that $\omega(G \times H)$ (the clique number of $G \times H$) equals $\min\{\omega(G), \omega(H)\}$ for every two graphs G and H, and in particular, $\omega(G^n) = \omega(G)$ for every integer n. However, the behavior of the independence ratios of the graphs G^n is far more interesting. Let $i(G) = \alpha(G)/|V(G)|$ denote the independence ratio of G. Notice that for every two graphs G and H, if I is an independent set of G, then the cartesian product $I \times V(H)$ is independent in $G \times H$, hence every two graphs G and H satisfy:

$$i(G \times H) \geq \max\{i(G), i(H)\} . \tag{1}$$

Therefore, the series $i(G^n)$ is monotone non-decreasing and bounded, hence its limit exists; we denote this limit, introduced in [5], where it is called the Ultimate Categorical Independence Ratio of G, by $A(G)$. In contrast to the clique numbers and chromatic numbers, $A(G)$ may indeed exceed its value at the first power of G, $i(G)$. The authors of [5] proved the following simple lower bound for $A(G)$: if I is an independent set of G, then $A(G) \geq \frac{|I|}{|I|+|N(I)|}$, where $N(I)$ denotes the vertex neighborhood of I. We thus have the following lower bound on $A(G)$: $A(G) \geq a(G)$, where

$$a(G) = \max_{I \text{ ind. set}} \frac{|I|}{|I| + |N(I)|} .$$

It easy to see that $a(G)$ resembles $i(G)$ in the sense that $a(G \times H) \geq \max\{a(G), a(H)\}$ (to see this, consider the cartesian product $I \times V(H)$, where I is an independent set of G which attains the ratio $a(G)$). However, as opposed to $i(G)$, it is not clear if there are any graphs G, H such that $a(G \times H) > \max\{a(G), a(H)\}$ and yet $a(G), a(H) \leq \frac{1}{2}$. This is further discussed later.

It is not difficult to see that if $A(G) > \frac{1}{2}$ then $A(G) = 1$, thus $A(G) \in (0, \frac{1}{2}] \cup \{1\}$, as proved in [5] (for the sake of completeness, we will provide short proofs for this fact and for the fact that $A(G) \geq a(G)$ in Section 2). Hence, we introduce the following variant of $a(G)$:

$$a^*(G) = \begin{cases} a(G) & \text{if } a(G) \leq \frac{1}{2} \\ 1 & \text{if } a(G) > \frac{1}{2} \end{cases} ,$$

and obtain that $A(G) \geq a^*(G)$ for every graph G. The following question seems crucial to the understanding of the behavior of independence ratios in tensor graph powers:

Question 1.1. *Does every graph G satisfy $A(G) = a^*(G)$?*

In other words, are non-expanding independent sets of G the only reason for an increase in the independence ratio of larger powers? If so, this would immediately settle several open problems related to $A(G)$ and to fractional colorings of tensor graph products. Otherwise, an example of a graph G satisfying $A(G) > a^*(G)$ would demonstrate a thus-far unknown way to increase $A(G)$. While it may seem unreasonable that the complicated parameter $A(G)$ translates into a relatively easy property of G, so far the intermediate results on several conjectures regarding $A(G)$ are consistent with the consequences of an equality between $A(G)$ and $a^*(G)$.

As we show later, Question 1.1 has the following simple equivalent form:

Question 1.1'. *Does every graph G satisfy $a^*(G^2) = a^*(G)$?*

Conversely, is there a graph G which satisfies the following two properties:

1. Every independent set I of G has at least $|I|$ neighbors (or equivalently, $a(G) \leq \frac{1}{2}$).

2. There exists an independent set J of G^2 whose vertex-expansion ratio, $\frac{|N(J)|}{|J|}$, is strictly *smaller* than $\frac{|N(I)|}{|I|}$ for every independent set I of G.

In this note, we study the relation between $A(G)$ and $a^*(G)$, show families of graphs where equality holds, and discuss the effects of Question 1.1 on several conjectures regarding $A(G)$ and fractional colorings of tensor graph products. The rest of the paper is organized as follows:

In Section 2 we present several families of graphs where equality holds between $A(G)$ and $a^*(G)$. First, we extend some of the ideas of [5] and obtain a characterization of all graphs G which satisfy the property $A(G) = 1$, showing that for these graphs $a^*(G)$ and $A(G)$ coincide. In the process, we obtain a polynomial time algorithm for determining whether a graph G satisfies $A(G) = 1$. We conclude the section by observing that $A(G) = a(G)$ whenever G is vertex transitive, and when it is the disjoint union of certain vertex transitive graphs.

Section 3 discusses the parameters $i(G)$ and $a(G)$ when G is a tensor product of two graphs, G_1 and G_2. Taking $G_1 = G_2$, we show the equivalence between Questions 1.1 and 1.1'. Next, when G_1 and G_2 are both vertex transitive, the relation between $i(G)$ and $a(G)$ is related to a fractional version of Hedetniemi's conjecture, raised by Zhu in [16]. We show that for every two graphs G and H, $A(G + H) = A(G \times H)$, where $G + H$ is the disjoint union of G and H. This property links the above problems, along with Question 1.1, to the problem of determining $A(G + H)$, raised in [5] (where it is conjectured to be equal to $\max\{A(G), A(H)\}$). Namely, the equality $A(G + H) = A(G \times H)$ implies that if $A(H) = a^*(H)$ for $H = G_1 + G_2$, then:

$$ i(G_1 \times G_2) \leq a^*(G_1 + G_2) = \max\{a^*(G_1), a^*(G_2)\} \ . $$

This raises the following question, which is a weaker form of Question 1.1:

Question 1.2. *Does the inequality $i(G \times H) \leq \max\{a^*(G), a^*(H)\}$ hold for every two graphs G and H?*

We proceed to demonstrate that several families of graphs satisfy this inequality, and in the process, obtain several additional families of graphs G which satisfy $A(G) = a(G) = a^*(G)$.

Section 4 is devoted to concluding remarks and open problems. We list several additional interesting questions which are related to $a(G)$, as well as summarize the main problems which were discussed in the previous sections. Among the new mentioned problems are those of determining or estimating the value of $A(G)$ for the random graph models $\mathcal{G}_{n,d}$, $\mathcal{G}_{n,\frac{1}{2}}$ and for the random graph process.

2 Equality between $A(G)$ and $a^*(G)$

2.1 Graphs G which satisfy $A(G) = 1$

In this section we prove a characterization of graphs G satisfying $A(G) = 1$, showing that this is equivalent to the non-existence of a fractional perfect matching in G. A *fractional matching* in a graph $G = (V, E)$ is a function $f : E \to \mathbb{R}^+$ such that for every $v \in V$, $\sum_{v \in e} f(e) \leq 1$ (a matching is the special case of restricting the values of f to $\{0, 1\}$). The value of the fractional matching is defined as $f(E) = \sum_{e \in E} f(e)$ ($\leq \frac{|V|}{2}$). A *fractional perfect matching* is a fractional matching which achieves this maximum: $f(E) = \frac{|V|}{2}$.

Theorem 2.1. *For every graph G, $A(G) = 1$ iff $a^*(G) = 1$ iff G does not contain a fractional perfect matching.*

The proof of Theorem 2.1 relies on the results of [5] mentioned in the introduction; we recall these results and provide short proofs for them.

Claim 2.2 ([5]). *For every graph G, $A(G) \geq a(G)$.*

Proof. Let I be an independent set which attains the maximum of $a(G)$. Clearly, for every $k \in \mathbb{N}$, all vertices in G^k, which contain a member of $I \cup N(I)$ in one of their coordinates, and in addition, whose first coordinate out of $I \cup N(I)$ belongs to I, form an independent set. As k tends to infinity, almost every vertex has a member of $I \cup N(I)$ in at least one of its coordinates, and the second restriction implies that the fractional size of the set above tends to $\frac{|I|}{|I|+|N(I)|} = a(G)$. $\blacksquare$

Claim 2.3 ([5]). *If $A(G) > \frac{1}{2}$ then $A(G) = 1$.*

Proof. Assume, without loss of generality, that $i(G) > \frac{1}{2}$, and let I be a maximum independent set of G. For every power k, the set of all vertices of G^k, in which strictly more than $\frac{k}{2}$ of the coordinates belong to I, is independent. Clearly, since $\frac{|I|}{|G|} > \frac{1}{2}$, the size of this set tends to $|V(G)|^k$ as k tends to infinity (as the probability of more Heads than Tails in a sufficiently long sequence of tosses of a coin biased towards Heads is nearly 1), hence $A(G) = 1$. $\blacksquare$

Proof of Theorem 2.1. By Claims 2.2 and 2.3, if $a(G) > \frac{1}{2}$ (or equivalently, $a^*(G) = 1$) then $A(G) = 1$. Conversely, assuming that $a^*(G) = a(G) \leq \frac{1}{2}$, we must show that $A(G) < 1$. This will follow from the following simple lemma, proved by Tutte in 1953 (cf., e.g., [11] p. 216):

Lemma 2.4. *For a given set $S \subset V(G)$, let $N(S)$ denote that set of all vertices of G which have a neighbor in S; then every set $S \subset V(G)$ satisfies $|N(S)| \geq |S|$ iff every independent set $I \subset V(G)$ satisfies $|N(I)| \geq |I|$.*

Proof of lemma. One direction is obvious; for the other direction, take a subset S with $|N(S)| < |S|$. Define S' to be $\{v \in S \mid N(v) \cap S \neq \emptyset\}$, and examine $I = S \setminus S'$. Since $S' \subset N(S)$ and

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$|N(S)| < |S|$, I is nonempty, and is obviously independent. Therefore $|N(I)| \geq |I|$, however $|N(I)| \leq |N(S)| - |S'| < |S| - |S'| = |I|$, yielding a contradiction. ■

Returning to the proof of the theorem, observe that by our assumption that $a(G) \leq \frac{1}{2}$ and the lemma, Hall's criterion for a perfect matching applies to the bipartite graph $G \times K_2$ (where K_2 is the complete graph on two vertices). Therefore, G contains a factor $H \subset G$ of vertex disjoint cycles and edges (to see this, as long as the matching is nonempty, repeatedly traverse it until closing a cycle and omit these edges). Since removing edges from G may only increase $A(G)$, it is enough to show that $A(H) < 1$.

We claim that the subgraph H satisfies $A(H) \leq \frac{1}{2}$. To see this, argue as follows: direct H according to its cycles and edges (arbitrarily choosing clockwise or counter-clockwise orientations), and examine the mapping from each vertex to the following vertex in its cycle. This mapping is an invertible function $f : V \to V$, such that for all $v \in V$, $vf(v) \in E(H)$. Now let I be an independent set of H^k. Pick a random vertex $\underline{u} \in V(H^k)$, uniformly over all the vertices, and consider the pair $\{\underline{u}, \underline{v}\}$, where $\underline{v} = f(\underline{u})$ is the result of applying f on each coordinate of $\underline{u}$. Obviously $\underline{v}$ is uniformly distributed over H^k as-well, thus:

$$\mathbb{E}\,|I \cap \{\underline{u}, \underline{v}\}| \geq \frac{2}{|H^k|}|I| \ .$$

Choosing a vertex $\underline{u}$ for which $|I \cap \{\underline{u}, \underline{v}\}|$ is at least its expected value, and recalling that $\underline{u}$ and $\underline{v}$ are adjacent in H^k, we get:

$$\frac{2}{|H^k|}|I| \leq |I \cap \{\underline{u}, \underline{v}\}|) \leq 1 \ .$$

Hence, $i(H^k) \leq \frac{1}{2}$, and thus $A(H) \leq \frac{1}{2}$.

An immediate corollary from the above proof that $A(G) = 1$ iff $a^*(G) = 1$ is the equivalence between the property $A(G) \leq \frac{1}{2}$ and the existence of a fractional perfect matching in the graph G. It is well known (see for instance [11]) that for every graph G, the maximal fractional matching of G can be achieved using only the weights $\{0, \frac{1}{2}, 1\}$. Therefore, a fractional perfect matching is precisely a factor $H \subset G$, comprised of vertex disjoint cycles and edges, and we obtain another format for the condition $a(G) \leq \frac{1}{2}$: $A(G) \leq \frac{1}{2}$ iff G has a fractional perfect matching; otherwise, $A(G) = 1$.

Notice that a fractional perfect matching f of G immediately induces a fractional perfect matching on G^k for every k (assign an edge of G^k a weight equaling the product of the weights of each of the edges in the corresponding coordinates). As it is easy to see that a fractional perfect matching implies that $i(G) \leq \frac{1}{2}$, this provides an alternative proof that if $a(G) \leq \frac{1}{2}$ then $A(G) \leq \frac{1}{2}$. ■

Since Lemma 2.4 also provides us with a polynomial algorithm for determining whether $a(G) > \frac{1}{2}$ (determine whether Hall's criterion applies to $G \times K_2$, using network flows), we obtain the following corollary:

Corollary 2.5. *Given an input graph G, determining whether $A(G) = 1$ or $A(G) \leq \frac{1}{2}$ can be done in polynomial time.*

2.2 Vertex transitive graphs

The observation that $A(G) = a(G)$ whenever G is vertex transitive (notice that $A(G) \leq \frac{1}{2}$ for every nontrivial regular graph G) is a direct corollary of the following result of [1] (the proof of this fact is by covering G^k uniformly by copies of G):

Proposition 2.6 ([1]). *If G is vertex transitive, then $A(G) = i(G)$.*

Clearly, for every graph G, $i(G) \leq a(G)$. Hence, for every vertex transitive graph G the following holds:

$$A(G) = i(G) \leq a(G) \leq A(G) \ ,$$

proving the following corollary:

Observation 2.7. *For every vertex transitive graph G, $A(G) = a^*(G) = a(G)$.*

We conclude this section by mentioning several families of vertex transitive graphs G and H whose disjoint union $G + H$ satisfies $A(G + H) = a(G + H) = \max\{A(G), A(H)\}$. These examples satisfy both the property of Question 1.1 and the disjoint union conjecture of [5].

The next two claims follow from the results of Section 3, as we later show. For the first claim, recall that a circular complete graph (defined in [16]), $K_{n/d}$, where $n \geq 2d$, has a vertex set $\{0, \ldots, n-1\}$ and an edge between i, j whenever $d \leq |i - j| \leq n - d$. A Kneser graph, $KN_{n,k}$, where $k \leq n$, has $\binom{n}{k}$ vertices corresponding to k-element subsets of $\{1, \ldots, n\}$, and two vertices are adjacent iff their corresponding subsets are disjoint.

Claim 2.8. *Let G and H be two vertex transitive graphs, where H is one of the following: a Kneser graph, a circular complete graph, a cycle or a complete bipartite graph. Then $G + H$ satisfies $A(G + H) = a(G + H) = \max\{A(G), A(H)\}$.*

Claim 2.9. *Let G and H be two vertex transitive graphs satisfying $\chi(G) = \omega(G) \leq \omega(H)$. Then $A(G + H) = a(G + H) = \max\{A(G), A(H)\}$.*

3 The tensor product of two graphs

3.1 The expansion properties of G^2

Question 1.1, which discusses the relation between the expansion of independent sets of G, and the limit of independence ratios of tensor powers of G, can be translated into a seemingly simpler question (stated as Question 1.1') comparing the vertex expansions of a graph and its square: can the minimal expansion ratio $|N(I)|/|I|$ of independent sets I decrease in the second power of G?

To see the equivalence between Questions 1.1 and 1.1', argue as follows: assuming the answer to Question 1.1 is positive, every graph G satisfies:

$$a^*(G) = A(G) = A(G^2) = a^*(G^2) \ ,$$

and hence $a^*(G^2) = a^*(G)$ (recall that every graph H satisfies $a(H^2) \geq a(H)$). Conversely, suppose that there exists a graph G such that $A(G) > a^*(G)$. By the simple fact that every graph H satisfies $i(H) \leq a^*(H)$ we conclude that there exists an integer k such that $a^*(G^{2^k}) \geq i(G^{2^k}) > a^*(G)$, and therefore there exists some integer $\ell \leq k$ for which $a(G^{2^\ell}) > a(G^{2^{\ell-1}})$.

3.2 The relation between the tensor product and disjoint unions

In this section we prove the following theorem, which links between the quantities $i(G_1 \times G_2)$, $a(G_1 \times G_2)$, $\chi_f(G_1 \times G_2)$ and $A(G_1 + G_2)$, where $\chi_f(G)$ denotes the fractional chromatic number of G:

Theorem 3.1. *For every two vertex transitive graphs G_1 and G_2, the following statements are equivalent:*

$$i(G_1 \times G_2) \quad \leq \quad \max\{a^*(G_1), a^*(G_2)\} \tag{2}$$

$$a^*(G_1 \times G_2) \quad \leq \quad \max\{a^*(G_1), a^*(G_2)\} \tag{3}$$

$$\chi_f(G_1 \times G_2) \quad = \quad \min\{\chi_f(G_1), \chi_f(G_2)\} \tag{4}$$

$$A(G_1 + G_2) \quad = \quad \max\{A(G_1), A(G_2)\} \tag{5}$$

Proof. The proof of Theorem 3.1 relies on the following proposition:

Proposition 3.2. *For every two graphs G and H, $A(G + H) = A(G \times H)$.*

We note that this generalizes a result of [5], which states that $A(G+H)$ is at least $\max\{A(G), A(H)\}$. Indeed, that result immediately follows from the fact that $A(G \times H)$ is always at least the maximum of $A(G)$ and $A(H)$ (by (1)).

Proof of Proposition 3.2. Examine $(G + H)^n$, and observe that a vertex whose i-th coordinate is taken from G is disconnected from all vertices whose i-th coordinate is taken from H. Hence, we can break down the n-th power of the disjoint union $G + H$ to 2^n disjoint graphs, and obtain:

$$\alpha\left((G + H)^n\right) = \sum_{k=0}^{n} \binom{n}{k} \alpha\left(G^k H^{n-k}\right) . \tag{6}$$

To prove that $A(G + H) \geq A(G \times H)$, fix $\varepsilon > 0$, and let N denote a sufficiently large integer such that $i\left((G \times H)^N\right) \geq (1 - \varepsilon)A(G \times H)$. The following is true for every $n > 2N$ and $N \leq k \leq n - N$:

$$i(G^k H^{n-k}) = i\left((G \times H)^N G^{k-N} H^{n-k-N}\right) \geq i\left((G \times H)^N\right) \geq (1 - \varepsilon)A(G \times H) ,$$

where the first inequality is by (1). Using this inequality together with (6) yields:

$$i\left((G + H)^n\right) \quad \geq \quad \frac{1}{|(G + H)^n|} \sum_{k=N}^{n-N} \binom{n}{k} \alpha(G^k H^{n-k}) \geq$$

$$\geq \quad \frac{\sum_{k=N}^{n-N} \binom{n}{k} |G|^k |H|^{n-k}}{(|G| + |H|)^n} (1 - \varepsilon)A(G \times H) \xrightarrow[n \to \infty]{} (1 - \varepsilon)A(G \times H) .$$

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Therefore $A(G + H) \geq (1 - \varepsilon)A(G \times H)$ for any $\varepsilon > 0$, as required.

It remains to show that $A(G + H) \leq A(G \times H)$. First observe that (1) gives the following relation:

$$\forall\, k, l \geq 1, \quad i(G^k H^l) \leq i(G^k H^l \times G^l H^k) = i(G^{k+l} H^{k+l}) \leq A(G \times H) . \tag{7}$$

Using (6) again, we obtain:

$$
\begin{aligned}
i\left((G + H)^n\right) &= \sum_{k=0}^{n} \binom{n}{k} \frac{\alpha(G^k H^{n-k})}{(|G| + |H|)^n} = \sum_{k=0}^{n} \binom{n}{k} i(G^k H^{n-k}) \cdot \frac{|G|^k |H|^{n-k}}{(|G| + |H|)^n} \\
&\leq \frac{|G|^n}{(|G| + |H|)^n} i(G^n) + \frac{|H|^n}{(|G| + |H|)^n} i(H^n) + \left(1 - \frac{|G|^n + |H|^n}{(|G| + |H|)^n}\right) A(G \times H) \\
&\leq \frac{|G|^n}{(|G| + |H|)^n} A(G) + \frac{|H|^n}{(|G| + |H|)^n} A(H) + \left(1 - \frac{|G|^n + |H|^n}{(|G| + |H|)^n}\right) A(G \times H) \\
&\longrightarrow A(G \times H) ,
\end{aligned}
$$

where the first inequality is by (7), and the second is by definition of $A(G)$. ∎

Equipped with the last proposition, we can now prove that Question 1.2 is indeed a weaker form of Question 1.1, namely that if $A(G) = a^*(G)$ for every G, then $i(G_1 \times G_2) \leq \max\{a^*(G_1), a^*(G_2)\}$ for every two graphs G_1, G_2. Indeed, if $A(G_1 + G_2) = a^*(G_1 + G_2)$ then inequality (2) holds, as well as the stronger inequality (3):

$$a^*(G_1 \times G_2) \leq A(G_1 \times G_2) = A(G_1 + G_2) = a^*(G_1 + G_2) = \max\{a^*(G_1), a^*(G_2)\} ,$$

as required.

Having shown that a positive answer to Question 1.1 implies inequality (3) (and hence inequality (2) as well), we show the implications of inequality (2) when the two graphs are vertex transitive.

Recall that for every two graphs G_1 and G_2, $i(G_1 \times G_2) \geq \max\{i(G_1), i(G_2)\}$, and consider the case when G_1, G_2 are both vertex transitive and have edges. In this case, $i(G_i) = a(G_i) = a^*(G_i)$ ($i = 1, 2$), hence inequalities (2) and (3) are equivalent, and are both translated into the form $i(G_1 \times G_2) = \max\{i(G_1), i(G_2)\}$. Next, recall that for every vertex transitive graph G, $i(G) = 1/\chi_f(G)$. Hence, inequality (2) (corresponding to Question 1.2), when restricted to vertex transitive graphs, coincides with (4). Furthermore, by Observation 2.7 and Proposition 3.2, for vertex transitive G_1 and G_2 we have:

$$i(G_1 \times G_2) = A(G_1 \times G_2) = A(G_1 + G_2) \geq \max\{A(G_1), A(G_2)\} = \max\{i(G_1), i(G_2)\} ,$$

hence in this case (4) also coincides with (5). Thus, all four statements are equivalent for vertex transitive graphs. ∎

By the last theorem, the following two conjectures, raised in [5] and [16], coincide for vertex transitive graphs:

Conjecture 3.3 ([5]). *For every two graphs G and H, $A(G + H) = \max\{A(G), A(H)\}$.*

Conjecture 3.4 ([16]). *For every two graphs G and H, $\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$.*

The study of Conjecture 3.4 is somewhat related to the famous and long studied Hedetniemi conjecture (stating that $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$), as for every two graphs G and H, $\omega(G \times H) = \min\{\omega(G), \omega(H)\}$, and furthermore $\omega(G) \leq \chi_f(G) \leq \chi(G)$.

It is easy to see that the inequality $\chi_f(G \times H) \leq \min\{\chi_f(G), \chi_f(H)\}$ is always true. It is shown in [13] that Conjecture 3.4 is not far from being true, by proving that for every graphs G and H, $\chi_f(G \times H) \geq \frac{1}{4}\min\{\chi_f(G), \chi_f(H)\}$.

So far, Conjecture 3.4 was verified (in [16]) for the cases in which one of the two graphs is either a Kneser graph or a circular-complete graph. This implies the cases of H belonging to these two families of graphs in Claim 2.8. Claim 2.9 is derived from the the following remark, which provides another family of graphs for which Conjecture 3.4 holds.

Remark 3.5. *Let G and H be graphs such that $\chi(G) = \omega(G) \leq \omega(H)$. It follows that $\omega(G \times H) = \omega(G) = \chi(G \times H)$, and thus $\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$, and $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$. In particular, this is true when G and H are perfect graphs.*

3.3 Graphs satisfying the property of Question 1.2

In this subsection we note that several families of graphs satisfy inequality (2) (and the property of Question 1.2). This appears in the following propositions:

Proposition 3.6. *For every graph G and integer ℓ, $i(G \times C_\ell) \leq \max\{a(G), a(C_\ell)\}$ (and hence, $i(G \times C_\ell) \leq \max\{a^*(G), a^*(C_\ell)\}$). This result can be extended to $G \times H$, where H is a disjoint union of cycles.*

Proof. We need the following lemma:

Lemma 3.7. *Let G and H be two graphs which satisfy at least one of the following conditions:*
  1. *$a(G) \geq \frac{1}{2}$, and every $S \subsetneq V(H)$ satisfies $|N(S)| > |S|$.*
  2. *$a(G) > \frac{1}{2}$ and every $S \subset V(H)$ satisfies $|N(S)| \geq |S|$.*
Then every maximum independent set $I \subset V(G \times H)$ contains at least one "full copy" of H, i.e., for each such I there is a vertex $v \in V(G)$, such that $\{(v, w) : w \in H\} \subset I$.

Proof of lemma. We begin with the case $a(G) \geq \frac{1}{2}$ and $|N(S)| > |S|$ for every $S \subsetneq V(H)$. Let J be a smallest (with respect to either size or containment) nonempty independent set in G such that $\frac{|J|}{|J| + |N(J)|} \geq \frac{1}{2}$. Clearly, $|N(J)| \leq |J|$. We claim that this inequality proves the existence of a one-one function $f : N(J) \to J$, such that $vf(v) \in E(G)$ (that is, there is a matching between $N(J)$ and J which saturates $N(J)$). To prove this fact, take any set $S \subset N(J)$ and assume $|N(S) \cap J| < |S|$; it is thus possible to delete $N(S) \cap J$ from J (and at least $|S|$ vertices from

$N(J)$) and since $|N(S) \cap J| < |S| \leq |N(J)| \leq |J|$ we are left with a nonempty $J' \subsetneq J$ satisfying $|N(J')| \leq |J'|$. This contradicts the minimality of J. Now we can apply Hall's Theorem to match a unique vertex in J for each vertex in $N(J)$.

Assume the lemma is false, and let I be a counterexample. Examine the intersection of I with a pair of copies of H, which are matched in the matching above between $N(J)$ and J. As we assumed that there are no full H copies in I, each set S of vertices in a copy of H has at least $|S|+1$ neighbors in an adjacent copy of H. Thus, each of the matched pairs of $N(J) \to J$ contains at most $|H|-1$ vertices of I. Define I' as the result of adding all missing vertices from the H copies of J to I, and removing all existing vertices from the copies of $N(J)$ (all other vertices remain unchanged). Then I' is independent, and we obtain a contradiction to the maximality of I.

The case of $a(G) > \frac{1}{2}$ and $|N(S)| \geq |S|$ for every $S \subset V(H)$ is essentially the same. The set J is now the smallest independent set of G for which $|N(J)| < |J|$, and again, this implies the existence of a matching from $N(J)$ to J, which saturates $N(J)$. By our assumption on H, each pair of copies of H in the matching contributes at most $|H|$ vertices to a maximum independent set I of $G \times H$, and by the assumption on I, the unmatched copies of H (recall $|J| > |N(J)|$) are incomplete. Therefore, we have strictly less than $|H||J|$ vertices of I in $(N(J) \cup J) \times H$, contradicting the maximality of I. ∎

Returning to the proof of the proposition, let I be a maximum independent set of $G \times C_\ell$. Remove all vertices, which belong to full copies of C_ℓ in I, if there are any, along with all their neighbors (note that these neighbors are also complete copies of C_ℓ, but this time empty ones). These vertices contribute a ratio of at most $a(G)$, since their copies form an independent set in G. Let G' denote the induced graph of G on all remaining copies. The set I', defined to be $I \cap (G' \times C_\ell)$, is a maximum independent set of $G' \times C_\ell$, because for any member of I we removed, we also removed all of its neighbors from the graph.

Notice that C_ℓ satisfies the expansion property required from H in Lemma 3.7: for every k, every set $S \subsetneq V(C_{2k+1})$ satisfies $|N(S)| > |S|$, and every set $S \subset V(C_{2k})$ satisfies $|N(S)| \geq |S|$. We note that, in fact, by the method used in the proof of Lemma 2.4 it is easy to show that every regular graph H satisfies $|N(S)| \geq |S|$ for every set $S \subset V(H)$, and if in addition H is non-bipartite and connected, then every $S \subsetneq V(H)$ satisfies $|N(S)| > |S|$.

We can therefore apply the lemma on $G' \times C_\ell$. By definition, there are no full copies of C_ℓ in I', hence, by the lemma, we obtain that $a(G') \leq \frac{1}{2}$ (and even $a(G') < \frac{1}{2}$ in case ℓ is odd). In particular, we can apply Hall's Theorem and obtain a factor of edges and cycles in G'. Each connected pair of non-full copies has an independence ratio of at most $i(C_\ell) = a(C_\ell)$ (by a similar argument to the one stated in the proof of the lemma), and double counting the contribution of the copies in the cycles we conclude that $\frac{|I'|}{|G'||C_\ell|} \leq a(C_\ell)$. Therefore $i(G \times C_\ell)$ is an average between values, each of which is at most $\max\{a(G), a(C_\ell)\}$, completing the proof. ∎

Proposition 3.8. *For every graph G and integer k: $i(G \times K_k) \leq \max\{a(G), a(K_k)\}$ (and hence, such graphs satisfy the inequality of Question 1.2). This result can be extended to $G \times H$, where H is a disjoint union of complete graphs.*

Proof. Let I denote a maximum independent set of $G \times K_k$, and examine all copies of K_k which contain at least two vertices of I. Such a copy of K_k in $G \times K_k$ forces its neighbor copies to be empty (since two vertices of K_k have the entire graph K_k as their neighborhood). Therefore, by the maximality of I, such copies must contain *all* vertices of K_k. Denote the vertices of G which represent these copies by $S \subset V(G)$; then S is an independent set of G, and the copies represented by $S \cup N(S)$ contribute an independence ratio of at most $a(G)$. Each of the remaining copies contains at most one vertex, giving an independence ratio of at most $\frac{1}{k} = a(K_k)$. Therefore, $i(G \times K_k)$ is an average between values which are at most $\max\{a(G), a(K_k)\}$, and the result follows. ∎

Corollary 3.9. *Let G be a graph satisfying $a(G) = \frac{1}{2}$; then for every graph H the following inequality holds: $i(G \times H) \leq \max\{a(G), a(H)\}$.*

Proof. By Theorem 2.1 we deduce that G contains a fractional perfect matching; let G' be a factor of G consisting of vertex disjoint cycles and edges. Since $a(G') \leq \frac{1}{2}$ as-well, it is enough to show that $i(G' \times H) \leq \max\{a(G'), a(H)\}$. Indeed, since G' is a disjoint union of the form $C_{\ell_1} + \ldots + C_{\ell_k} + K_2 + \ldots + K_2$, the result follows from Proposition 3.6 and Proposition 3.8. ∎

4   Concluding remarks and open problems

We have seen that answering Question 1.1 is imperative to the understanding of the behavior of independent sets in tensor graph powers. While it is relatively simple to show that $A(G)$ equals $a(G)$ whenever G is vertex transitive, proving this equality for $G = G_1 + G_2$, the disjoint union of two vertex transitive graphs G_1 and G_2, seems difficult; it is equivalent to Conjecture 3.4, the fractional version of Hedetniemi's conjecture, for vertex transitive graphs. These two conjectures are consistent with a positive answer to Question 1.1, and are in fact direct corollaries in such a case.

The assertion of Conjecture 3.3 for several cases can be deduced from the spectral bound for $A(G)$ proved in [1]. For a regular graph G with n vertices and eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, denote $\Lambda(G) = \frac{-\lambda_n}{\lambda_1 - \lambda_n}$. As observed in [1], the usual known spectral upper bound for the independence number of a graph implies that for each regular G, $A(G) \leq \Lambda(G)$. It is not difficult to check that for regular G and H, $\Lambda(G \times H) = \max\{\Lambda(G), \Lambda(H)\}$. Therefore, by Proposition 3.2, if G and H are regular and satisfy $\Lambda(G) \leq \Lambda(H) = A(H)$, then the assertion of Conjecture 3.3 holds for G and H. Several examples of graphs H satisfying $\Lambda(H) = A(H)$ are mentioned in [1].

It is interesting to inspect the expected values of $A(G)$ for random graph models. First, consider $G^t \sim \mathcal{G}_{n,t}$, the random graph process on n vertices after t steps, where there are t edges chosen

uniformly out of all possible edges (for more information on the random graph process, see [4]). It is not difficult to show, as mentioned in [2], that $a(G^t)$ equals the minimal degree of G^t, $\delta(G^t)$, as long as $\delta(G^t)$ is fixed and $|G|$ is sufficiently large. When considering $A(G)$, the following is a direct corollary of the fractional perfect matching characterization for $A(G) = 1$ (Theorem 1.1), along with the fact that the property "G contains a fractional perfect matching" almost surely has the same hitting time as the property "$\delta(G) \geq 1$":

Remark 4.1. *With high probability, the hitting time of the property $A(G) < 1$ equals the hitting time of $\delta(G) \geq 1$. Furthermore, almost every graph process at that time satisfies $A(G) = \frac{1}{2}$.*

Question 4.2. *Does almost every graph process satisfy $A(G) = \frac{1}{\delta(G)+1}$ as long as $\delta(G)$ is fixed?*

Second, the expected value of $a(G)$ for a random regular graph $G \sim \mathcal{G}_{n,d}$ is easily shown to be $\Theta(\frac{\log d}{d})$, as the independence ratio of $\mathcal{G}_{n,d}$ is almost surely between $\frac{\log d}{d}$ and $2\frac{\log d}{d}$ as $n \to \infty$ (see [3], [15]). As for $A(G)$, the following is easy to prove, by the spectral upper bound $\Lambda(G)$ mentioned above, and by the eigenvalue estimations of [8]:

Remark 4.3. *Let G denote the random regular graph $\mathcal{G}_{n,d}$; almost surely: $\Omega(\frac{\log d}{d}) \leq A(G) \leq O(\frac{1}{\sqrt{d}})$ as $d \to \infty$.*

Question 4.4. *Is the expected value of $A(G)$ for the random regular graph $G \sim \mathcal{G}_{n,d}$ equal to $\Theta(\frac{\log d}{d})$?*

The last approach can be applied to the random graph $G \sim \mathcal{G}_{n,\frac{1}{2}}$ as well. To see this, consider a large regular factor (see [12]), and use the eigenvalue estimations of [6] to obtain that almost surely $\Omega(\frac{\log n}{n}) \leq A(G) \leq O(\sqrt{\frac{\log n}{n}})$, whereas $a(G)$ is almost surely $(2 + o(1))\frac{\log_2 n}{n}$.

Question 4.5. *Is the expected value of $A(G)$ for the random graph $G \sim \mathcal{G}_{n,\frac{1}{2}}$ equal to $\Theta(\frac{\log n}{n})$?*

We conclude with the question of the decidability of $A(G)$. Clearly, deciding if $a(G) > \beta$ for a given value β is in NP, and we can show that it is in fact NP-complete. It seems plausible that $A(G)$ can be calculated (though not necessarily by an efficient algorithm) up to an arbitrary precision:

Question 4.6. *Is the problem of deciding whether $A(G) > \beta$, for a given graph G and a given value β, decidable?*

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