Equilateral sets in the torus

Dedicated to the memory of Vera T. Sós

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Abstract

An equilateral set in a metric space X is a set in which all distances between two points are equal. We study the maximum possible cardinality of such a set in the n-dimensional torus with respect to several natural distance functions. For the ℓ_∞-distance (in the torus), we show that the maximum is exactly 3^n. For any finite real $p \geq 1$ we conjecture that for the ℓ_p-distance the maximum is linear in n, observe that $2n$ is a lower bound for every n, and prove that $c(p)n^{2+2/\lfloor p \rfloor}$ is an upper bound.

1 Introduction

Vera T. Sós has been a key figure in the development of Discrete Mathematics, Graph Theory and their connections to Number Theory, Discrepancy Theory and Analysis. I knew her for more than 30 years, participated in several conferences and workshops she organized, and took part in her 70th Birthday conference and in her 80th Birthday conference in Budapest. One of my very first papers addresses a problem she raised with Erdős and Simonovits [6]. Her seminal work on sequences obtained by irrational rotations on the torus and in particular her three-distance theorem reveals the interesting combinatorial behavior of these sequences.

The present note deals with the study of equidistant points on the n-dimensional torus. It is motivated by the investigation of point configurations on the torus that minimize interaction energies, which arise naturally in the context of discrepancy theory and quasi-Monte Carlo integration. The relevant arguments combine combinatorial tools with techniques from approximation theory and a simple construction from additive number theory. The question, its motivation, and the proof techniques are closely related to some of the research areas of Vera. I believe she could have enjoyed reading this note.

An *equilateral set* in a metric space X is a set A, so that all distances between pairs of distinct members of A are the same. The equilateral number $e(X)$ of the space is the

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maximum possible cardinality of such a set (infinity if there are such sets of arbitrarily large size).

The problem of determining or estimating these numbers, mainly for normed spaces, received a considerable amount of attention since the early 70s. It is easy and well known that the equilateral number of ℓ_2^n, that is, R^n with the usual ℓ_2-metric, is $n+1$. Somewhat surprisingly, the situation is far more complicated for the other ℓ_p norms. For a finite $p \geq 1$, the ℓ_p-distance between two points $a = (a_1, \ldots a_n)$ and $b = (b_1, \ldots, b_n)$ in R^n is $\|a-b\|_p = (\sum_{i=1}^n |a_i - b_i|^p)^{1/p}$. The ℓ_∞-distance between a and b is $\|a-b\|_\infty = \max_{1 \leq i \leq n} |a_i - b_i|$. For all $p \in [1, \infty]$, ℓ_p^n denotes the space R^n with the ℓ_p-distance. Thus $e(\ell_p^n)$ denotes the maximum possible cardinality of an equilateral set in ℓ_p^n.

Petty [14] proved that the maximum possible cardinality of an equilateral set in ℓ_p^n for any p is at most 2^n, and that equality holds only in l_∞^n, as shown by the set of all 2^n vectors with $\{0, 1\}$ coordinates. The set of standard basis vectors together with an appropriate multiple of the all 1 vector shows that $e(\ell_p^n) \geq n+1$ for all $1 \leq p < \infty$, and the set of standard basis vectors and their negatives shows that $e(\ell_1^n) \geq 2n$.

Kusner (see [8]) conjectured that $e(\ell_1^n) = 2n$ and that for any finite $p > 1$, $e(\ell_p^n) = n+1$. The best known upper bound for $e(\ell_1^n)$, proved in [2], is $O(n \log n)$ (for $n > 1$), see also [9] for an alternative proof that also shows that for every $p \in [1, 2)$ $e(\ell_p^n) \leq C(p) n \log n$. Swanepoel [16] proved that $e(\ell_4^n) = n+1$ and that for every $1 < p < 2$, $e(\ell_p^n) > (1+\epsilon_p)n$ for some positive $\epsilon_p > 0$. Thus Kusner's conjecture for $1 < p < 2$ is incorrect. He also proved that for any even integer p, $e(\ell_p^n) \leq 1 + pn/2$ if $p \equiv 0 \bmod 4$ and $e(\ell_p^n) \leq 1 + pn/2 + n$ if $p \equiv 2 \bmod 4$. For any odd integer p, an upper bound of $C(p) n \log n$ is proved in [2]. For real $p > 2$ it is proved in [2] that $e(\ell_p^n) \leq C(p) n^{\frac{2p+2}{2p-1}}$ improving a $C(p) n^{\frac{p+1}{p-1}}$ earlier estimate of Smyth [15].

Petty [14] conjectured that for any n-dimensional metric space X, $e(X) \geq n+1$. This is proved in [14] for dimensions $d \leq 3$, and proved for $d = 4$ in [13]. It is also known for norms that are sufficiently close to the n-dimensional Euclidean norm ([4], [5]), or more generally to the n-dimensional ℓ_p-norm ℓ_p^n for any $1 < p \leq \infty$ ([17]). Combining this with a result of [1] asserting that every n-dimensional normed space contains a subspace of dimension $r = e^{\Omega(\sqrt{\log n})}$ which is either close to ℓ_2^r or to ℓ_∞^r, it is proved by Swanepoel and Villa in [17] that for any n-dimensional normed space X, $e(X)$ is at least $e^{b\sqrt{\log n}}$ for some absolute constant $b > 0$.

In this note we consider the problem for the metric spaces $\ell_p(T_n)$ defined on the n-dimensional torus. This problem was suggested to me by Noah Kravitz and Stefan Steinerberger [10], motivated by a construction given by Bilyk, Nagel and Ruohoniemi in [3]. As explained in [3] this construction appears in the study of point configurations on the torus that minimize interaction energies with tensor product structure which arise in the context of discrepancy theory and quasi-Monte Carlo integration.

The set of points of the n-torus is $[0, 1]^n$ and the ℓ_p-distance between two points a, b

in it is $\sum_{i=1}^{n} \min\{|(a_i - b_i)|, 1 - |a_i - b_i|\}^p)^{1/p}$. The ℓ_∞ distance between them is defined in a similar way as $\max_i \min\{|(a_i - b_i)|, 1 - |a_i - b_i|\}$.

Let $e_p(T_n) = e(\ell_p(T_n))$ denote the equilateral number of the n-torus with the ℓ_p-distance. We also let T'_n denote the torus with the metric

$$d(a,b) = (\sum_{i=1}^{n} \sin^2[\pi(a_i - b_i)])^{1/2}. \tag{1}$$

Note that this last metric is the one obtained by considering the torus as a product of n pairwise orthogonal cycles with the usual Euclidean distance.

We prove the following results.

Theorem 1.1. *The equilateral number $e_\infty(T_n)$ is exactly 3^n for every n.*

Theorem 1.2. *For every fixed $p \geq 1$ there exists a constant $c(p)$ so that for every n*

$$2n \leq e_p(T_n) \leq c(p) n^{2+2/\lfloor p \rfloor}.$$

If $2n+1$ is a prime then $e_p(T_n) \geq 2n+1$.

Theorem 1.3. *The equilateral number $e(T'_n)$ of the n-torus with respect to the metric (1) is at most $2n+1$. If $2n+1$ is a prime this is tight.*

The proofs are described in the next section. The final section contains some brief concluding remarks and open problems.

2 Proofs

2.1 The lower bounds

The proofs of the lower bounds in all three theorems are rather simple. The set $A_{n,\infty} = \{0, 1/3, 2/3\}^n$ shows that $e_\infty(T_n) \geq 3^n$. For any finite real $p \geq 1$, the set $A_{n,p} = \{\pm \alpha_p e_i\}$, where $\alpha_p = 1/(2 + 2^{1/p})$ and e_i is the i-th coordinate basis vector shows that $e_p(T_n) \geq 2n$. A construction when $2n+1$ is a prime that works for all the distance functions considered here is a bit more complicated and (essentially) appears in [3]. Let A'_n be the set of points $x^{(j)}$ for $0 \leq j \leq 2n$ where

$$x^{(j)} = (\frac{j}{2n+1}, \frac{2j}{2n+1}, \frac{3j}{2n+1} \ldots, \frac{nj}{2n+1}),$$

and the numerators in all fractions are computed modulo the prime $2n+1$. For each $0 \leq i \neq j \leq 2n$, the difference $x^{(j)} - x^{(i)}$ is the vector

$$(\frac{a}{2n+1}, \frac{2a}{2n+1}, \frac{3a}{2n+1} \ldots, \frac{na}{2n+1}),$$

where $a = j - i \neq 0$ and here, too, the numerators are computed modulo $2n+1$. These numerators are nonzero, pairwise distinct, and for each pair of residues

$$t \text{ and } \quad -t \equiv 2n+1-t \bmod (2n+1)$$

include exactly one member of the pair. Therefore, for any function $f : [0,1] \mapsto R$ that satisfies $f(z) = f(1-z)$ for every z, the sum of the values of $f(z_i)$ over the n coordinates of any such difference vector is a constant. This shows that if $2n+1$ is a prime then for each of the distances considered here the equilateral number of the corresponding metric space on T_n is at least $2n+1$.

This establishes the lower bounds in all three theorems.

2.2 The value of $e_\infty(T_n)$

In this subsection we prove Theorem 1.1. We need the following simple lemma.

Lemma 2.1. *Let A be a finite set of points in the one-dimensional torus T_1 with the ℓ_∞-metric (which is identical in this space to the ℓ_p-metric for every p). Let d be the diameter of A, that is, the maximum distance between a pair of points of A. Let $G = G(A)$ be the graph on the vertex set A in which two points are adjacent iff the distance between them is d. Then the chromatic number $\chi(G)$ of G is at most 3.*

Proof. If $d \geq 1/3$ define $X_0 = [0, 1/3)$, $X_1 = [1/3, 2/3)$ and $X_2 = [2/3, 1)$. Note that X_0, X_1, X_2 form a partition of the set $[0,1)$ of all points of the 1-torus T_1. For each $i \in \{0,1,2\}$, $A \cap X_i$ is an independent set in G, since the distance between any two points of X_i is strictly smaller than $1/3 \leq d$. Therefore in this case $\chi(G) \leq 3$.

If $d < 1/3$ let $a \in A$ be an arbitrary point in A. Note that A is contained in the cyclic interval $[a-d, a+d]$, where both $a-d$ and $a+d$ are computed modulo 1, since the distance between a and any other point of the torus besides those in this cyclic interval exceeds d. In addition, it is impossible that both $a-d$ and $a+d$ lie in A, since the distance between these two points is $\min\{2d, 1-2d\} > d$. If $a-d$ is not in A then $A \cap (a-d, a]$ and $A \cap (a, a+d]$ are two independent sets of G that cover all vertices. This shows that in this case the graph G is 2-colorable. The same argument shows that this is also the case if $a+d$ is not in A. This completes the proof of the lemma. $\square$

Remark: As pointed out to me by Noah Kravitz [10], the assertion of the lemma holds for any d and not just for the diameter of A. Indeed, if d is rational, then the Cayley graph of T_1 with respect to $\pm d$ is a union of cycles, and its chromatic number is either 2 or 3 according to whether the denominator is even or odd. If d is irrational, then this Cayley graph is a union of bi-infinite paths, and has chromatic number 2.

Proof of Theorem 1.1. As proved in subsection 2.1, $e_\infty(T_n) \geq 3^n$. To prove the upper bound let $A \subset T_n$ be an equilateral set with respect to the ℓ_∞-distance on T_n. Let d denote its diameter and let G be the graph on A in which two points are adjacent iff the distance between them is d. Then G is a clique. Its set of edges can be naturally partitioned into n subsets E_i where an edge aa' for $a = (a_1, a_2, \ldots, a_n)$ and $a' = (a'_1, a'_2, \ldots, a'_n)$ belongs to E_i iff i is the smallest index of a coordinate j so that $\min\{|a_j - a'_j|, 1 - |a_j - a'_j|\} = d$. By Lemma 2.1, for each i the chromatic number of the graph (A, E_i) is at most 3. Therefore the chromatic number of G is at most 3^n, and as it is a clique, this shows that $|A| \le 3^n$, completing the proof. $\square$

2.3 Bounding $e_p(T_n)$

In this subsection we prove Theorem 1.2. The proof for $1 \le p < 2$ is similar to one of the proofs in [2] based on the approach in [15]. The proof of the improved bound for $p \ge 2$ is more complicated, and requires an additional idea. We first describe the proof for the simpler case.

We need the following well known, folklore lemma. A short proof can be found (among other places) in [2] .

Lemma 2.2. *For any real nonzero symmetric matrix M,*

$$rank(M) \ge \frac{(\sum_i M_{i,i})^2}{\sum_{i,j} M_{i,j}^2}.$$

We will also use the classical result of Jackson about the best approximation of Lipschitz functions by polynomials. It is convenient to use the version proved in [12], Theorem 6. Recall that a real function f on $[-1, 1]$ is M-Lipschitz iff for any $x, y \in [-1, 1]$, $|f(x) - f(y)| \le M|x - y|$.

Theorem 2.3 (Jackson's Theorem, [11], [12])**.** *If f is M-Lipschitz on $[-1, 1]$ then for every d there is a polynomial P of degree at most d so that for every $x \in [-1, 1]$,*

$$|f(x) - P(x)| \le \frac{M\pi^2}{2d + 4}.$$

Proof of Theorem 1.2 for $p \in [1, 2)$.

Let p be a fixed real in $[1, 2)$. As described in subsection 2.1, $e_p(T_n) \ge 2n$ and $e_p(T_n) \ge 2n+1$ if $2n+1$ is a prime. We proceed with a proof that $e_p(T_n) \le c(p)n^4$ for some constant $c(p)$. To simplify the presentation we omit, throughout the proof, all floor and ceiling signs whenever these are not crucial and assume that n is sufficiently large as a function of p whenever this is needed. We make no attempt to optimize the value of the constant $c(p)$ as even the power of n obtained here is unlikely to be tight.

Let $A \subset [0, 1]^n$ be an equilateral set in the n-torus with respect to the ℓ_p-metric. Put $|A| = m$. Therefore there exists a constant S so that for any two distinct points

$a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ in A,

$$\sum_{i=1}^{n} (\min\{|a_i - b_i|, 1 - |a_i - b_i|\})^p = S.$$

By shifting, if needed, we may assume, without loss of generality, that the point

$$(1/2, 1/2, \ldots, 1/2)$$

lies in A.

Note, first, that if $S < (1/3)^p$ then the distance between any two points of A is the usual ℓ_p-distance between them, considered as points in R^n. Indeed, for such an S every coordinate of any point of A lies in $(1/6, 5/6)$ and the difference and the absolute value $|a_i - b_i|$ of the difference between the i-th coordinates of any two points a, b of A is smaller than $2/3$. Therefore $1 - |a_i - b_i| > 1/3$ and by the assumption on S it follows that $\min\{|a_i - b_i|, 1 - |a_i - b_i|\} = |a_i - b_i|$.

This shows that for such a small S the distances between the points of A are the usual distances in ℓ_p^n and the desired upper estimate follows from the known results about the equilateral numbers of the normed space ℓ_p^n (with room to spare).

We thus may and will assume that $S \geq (1/3)^p$.

Let $g(z)$ be the real function on $[-1, 1]$ defined by $g(z) = [\min\{|z|, 1 - |z|\}]^p$. By the mean-value theorem it is easy to check that this function is, say, 2-Lipschitz. Therefore, by Theorem 2.3 there is a polynomial $P(z)$ of degree smaller than $10n\sqrt{m}$, so that for every $z \in [-1, 1]$, $|g(z) - P(z)| \leq \frac{1}{n\sqrt{m}}$. Since g is even we may replace $P(z)$ by $(P(z) + P(-z))/2$ without increasing the degree or the approximation error, hence we may and will assume that P is even.

Let $M = M(a, b)$ be the m by m matrix whose rows and columns are indexed by the points of A, where $M(a, b) = S - \sum_{i=1}^{n} P(a_i - b_i)$. By the triangle inequality, each diagonal entry of M is at least $S - 1/\sqrt{m}$ and at most $S + 1/\sqrt{m}$ whereas the absolute value of each non-diagonal entry is at most $1/\sqrt{m}$. By Lemma 2.2 the rank of M is at least

$$\frac{m^2(S - 1/\sqrt{m})^2}{m(S + 1/\sqrt{m})^2 + m^2(1/m)} \geq m/100$$

where here we used the fact that $S \geq 1/3^p \geq 1/9$ and that m $(\geq n)$ is sufficiently large.

Next we describe an upper bound for the rank of M. Note that this matrix is the sum of the rank-1 matrix in which all entries are S, and n additional matrices M_i, where $M_i(a, b) = -P(a_i - b_i)$. As the degree of the polynomial P is smaller than $10n\sqrt{m}$, the rank of each M_i is at most $10n\sqrt{m}$. Indeed, the row with index a of M_i is the vector of evaluations of the polynomial $P(a_i - y_i)$ in the m points b_i for $b \in A$. All these polynomials lie in the space of polynomials of y_i of degree less than $d = 10n\sqrt{m}$, and the dimension of this space is d as a basis for it is $\{1, y_i, y_i^2, \ldots, y_i^{d-1}\}$.

We thus conclude that the rank of M is at most 1 plus the sum of ranks of the matrices M_i (in fact a bit smaller as 1 is a common element of all bases and also spans the rows of the constant matrix S). Thus the rank of M is at most $n(10n\sqrt{m}) = 10n^2\sqrt{m}$. Therefore, $m/100 \leq 10n^2\sqrt{m}$, implying that $m \leq (1000)^2 n^4$. $\square$

Next we consider real numbers $p \geq 2$. We need the following two simple statements.

Lemma 2.4. *Let $s < p$ be two positive real numbers. Then for every real vector $(x_1, x_2, \ldots, x_n)$ with non-negative coordinates*

$$\sum_{i=1}^{n} x_i^s \leq (\sum_{i=1}^{n} x_i^p)^{s/p} n^{(p-s)/p}$$

Proof. Apply Hölder's Inequality to the two vectors $(x_1^s, \ldots, x_n^s)$ and $(1, 1, \ldots, 1)$. $\square$

Lemma 2.5. *Let $t_1, t_2 \ldots, t_n$ be non-negative reals and let $y_1, y_2, \ldots, y_n$ be reals satisfying $|t_i - y_i| \leq \varepsilon$ for all $1 \leq i \leq n$. Then for every integer $k \geq 2$*

$$|\sum_{i=1}^{n} t_i^k - \sum_{i=1}^{n} y_i^k| \leq 2^k(\varepsilon \sum_{i=1}^{n} t_i^{k-1} + n\varepsilon^k).$$

Proof. Since $t_i^k - y_i^k = (t_i - y_i)(t_i^{k-1} + t_i^{k-2}y_i + \ldots + y_i^{k-1})$ it follows that

$$|t_i^k - y_i^k| \leq \varepsilon(t_i^{k-1} + t_i^{k-2}|y_i| + \ldots + |y_i|^{k-1}).$$

Consider a sum $t_i^{k-1} + t_i^{k-2}|y_i| + \ldots + |y_i|^{k-1}$. If $t_i \leq \varepsilon$ then $|y_i| \leq 2\varepsilon$ and the sum is at most $2^k\varepsilon^{k-1}$. If $t_i > \varepsilon$ then $|y_i| \leq 2t_i$ and the sum is at most $2^k t_i^{k-1}$. By the triangle inequality

$$|\sum_{i=1}^{n} t_i^k - \sum_{i=1}^{n} y_i^k| \leq \sum_{i=1}^{n} |t_i^k - y_i^k|.$$

The total contribution of all terms with $t_i \leq \varepsilon$ to the last sum is at most $n2^k\varepsilon^k$ and the total contribution of all terms with $t_i > \varepsilon$ is at most $2^k\varepsilon \sum_i t_i^{k-1}$. $\square$

Proof of Theorem 1.2 for $p \geq 2$. The lower bound for $e_p(T_n)$ is discussed in subsection 2.1. To prove the upper bound let A be an equilateral set of points in $[0,1]^n$ with respect to the ℓ_p-metric. Put $|A| = m$ and define S to be the constant satisfying

$$\sum_{i=1}^{n} (\min\{|a_i - b_i|, 1 - |a_i - b_i|\})^p = S$$

for every two distinct points $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ in A. As in the discussion for $p \in [1, 2)$ we may and will assume that $S \geq (1/3)^p$ since otherwise the desired result follows from the known upper bound for $e(\ell_p^n)$. Note also that $S \leq n(1/2)^p \leq n/4$.

Let $k = \lfloor p \rfloor$ be the integral part of p and define $r = p/k$. Thus $r \geq 1$ and $r < 3/2$. Let $g(z)$ be the function on $[-1, 1]$ defined by $g(z) = \min\{|z|, 1 - |z|\}^r$. By the mean-value theorem this function is 2-Lipschitz. Put $\varepsilon = \frac{S^{1/k}}{n^{1/k}\sqrt{m}}$ $(< 1/\sqrt{m})$. By Theorem 2.3 there exists a polynomial $P(z)$ of degree smaller than $10n^{1/k}\sqrt{m}/S^{1/k}$ so that $|g(z) - P(z)| \leq \varepsilon$ for all $z \in [-1, 1]$. As before, we may and will assume that $P(z)$ is even. Define $Q(z) = P(z)^k$. Then the degree of Q is smaller than $10kn^{1/k}\sqrt{m}/S^{1/k}$.

Let $M = M(a, b)$ be the m by m matrix with rows and columns indexed by the points of A, where for $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ in A,

$$M(a, b) = S - \sum_{i=1}^{n} Q(a_i - b_i).$$

Note that $\sum_{i=1}^n g(a_i - b_i)^k$ is exactly the the p-th power of the ℓ_p distance in T_n between a and b, which is 0 for $a = b$ and S for all $a \neq b \in A$. Put $z_i = (a_i - b_i)$, $t_i = g(z_i) = \min\{|z_i|, 1 - |z_i|\}^r$, and $y_i = P(z_i)$. Then $|t_i - y_i| \leq \varepsilon$ for all i and by the triangle inequality, Lemma 2.4 with $s = p(k-1)/k$ and Lemma 2.5

$$|(S - \sum_{i=1}^{n} \min\{|z_i|, 1 - |z_i|\}^p) - M(a, b)| = |\sum_{i=1}^{n} t_i^k - \sum_{i=1}^{n} y_i^k|$$

$$\leq 2^k \varepsilon \sum_{i=1}^{n} t_i^{k-1} + 2^k n \varepsilon^k \leq 2^k \varepsilon n^{1/k} S^{(k-1)/k} + 2^k n \varepsilon^k$$

$$= S2^k/\sqrt{m} + 2^k S/m^{k/2} \leq S2^{k+1}/\sqrt{m}.$$

It follows that each diagonal entry of M differs from S by at most $S2^{k+1}/\sqrt{m}$, and each non-diagonal entry is in absolute value at most $S2^{k+1}/\sqrt{m}$.

By Lemma 2.2 the rank of M is at least

$$\frac{m^2 S^2 (1 - 2^{k+1}/\sqrt{m})^2}{mS^2(1 + 2^{k+1}/\sqrt{m})^2 + m^2 S^2 4^{k+1}/m} > m/4^{k+2}$$

where in the last inequality we used the fact that m is sufficiently large as a function of k.

On the other hand, as in the proof for the case $p \in [1, 2)$, each row of M is a vector consisting of the sum of S with the sum of the evaluations of n polynomials $-Q(a_i - x_i)$ at m values of x_i (corresponding to the values of the coordinate number i of the points of A). As the degree of each $Q(x_i)$ is smaller than $10kn^{1/k}\sqrt{m}/S^{1/k}$ the rank is at most

$$10k\sqrt{m}n^{1+1/k}/S^{1/k} < 60k\sqrt{m}n^{1+1/k},$$

where here we used the fact that $S \geq (1/3)^p \geq (1/6)^k$.

We conclude that

$$m/4^{k+2} \leq 60k\sqrt{m}n^{1+1/k},$$

implying that

$$m \leq C(k)n^{2+2/k} = C(p)n^{2+2/\lfloor p \rfloor},$$

for an appropriately defined constant $C(p)$. This completes the proof. $\square$

2.4 The metric space T'_n

Proof of Theorem 1.3. To prove the upper bound consider the following function mapping the points of the torus T_n to the Euclidean space R^{2n}.

$$f((x_1, x_2, \dots x_n)) = \frac{1}{2}(\cos[2\pi x_1], \sin[2\pi x_1], \cos[2\pi x_2], \sin[2\pi x_2], \dots, \cos[2\pi x_n], \sin[2\pi x_n]).$$

It is easy to check that for every two points x, y in T'_n, $d(x, y) = \|f(x) - f(y)\|_2$, where the distance d is the distance defined in (1). Therefore f maps any equilateral set A in the torus T'_n to an equilateral set in ℓ_2^{2n} showing that $|A| \leq 2n + 1$. This is tight when $2n + 1$ is a prime, as proved in subsection 2.1. $\square$

3 Concluding remarks and open problems

- As mentioned in the introduction the problem of determining or estimating the equilateral numbers of normed spaces received a considerable amount of attention. The more general problem of studying the corresponding numbers for metric spaces is much less studied and the discussion here suggests that it may be interesting for some spaces.

- It seems plausible to conjecture that for every fixed (finite) $p \geq 1$, $e_p(T_n) \leq C(p)n$. It will be interesting to determine these numbers precisely or at least establish better bounds closing the gaps between the upper and lower bounds

- Theorem 1.3 and its proof imply an intriguing (even if admittedly not very natural) result in additive combinatorics, described in the next proposition. Let m, n be natural numbers and let v be a vector in Z_m^n. Let $T(v)$ denote the set of all vectors obtained from v by permuting the coordinates and by replacing some of the coordinates by their additive inverses in Z_m. Thus if v contains no two equal coordinates, no two coordinates of the form x and $-x$ mod (m), and no single coordinate equal to $m/2$, then $|T(v)| = 2^n n!$

  Proposition 3.1. *For any natural numbers m, n and for any vector $v \in Z_m^n$, the maximum possible cardinality of a subset A of Z_m^n satisfying $A - A \subset T(v) \cup 0$ is at most $2n + 1$. If $2n + 1$ is a prime that divides m then this is tight for some vectors v.*

  Proof. The upper bound follows from Theorem 1.3 applied to the vectors $\{u(a) : a \in A\}$, where for each $a = (a_1, \dots, a_n) \in Z_m^n$, $u(a) = (a_1/m, \dots, a_n/m)$. The lower bound in case $2n + 1$ is a prime dividing m is obtained from the construction described in subsection 2.1. An example for $n = 3$ and $m = 7$ is

$$\{(0,0,0), (1,2,3), (2,4,6), (3,6,2), (4,1,5), (5,3,1), (6,5,4)\}$$

where here $v = (1, 2, 3)$. $\square$

Note added in proof: Following our paper, Ge, Xu and Zhou [7] improved the bounds in Theorem 1.2 by replacing the application of Jackson's Theorem by a more sophisticated analytic argument. Their approach enables them to avoid the loss arising from the non-smoothness of the cyclic distance at the point 1/2, which is inevitable when using a global uniform polynomial approximation. The bounds proved in [7] are $e_p(T_n) \leq c(p)n\log(2n)$ for $1 \leq p \leq 2$ and $e_p(T_n) \leq c(p)n^{3/2-1/p}$ for $p > 2$. Moreover, their method provides new results in the study of Kusner's Conjecture, showing, in particular, that the conjecture is true for $2 \leq p \leq 4$. The specific results are $e(\ell_p^n) \leq (2k+1)n + 1$ for $p \in [4k+2, 4k+4]$ where $k \geq 0$ is an integer, and $e(\ell_p^n) \leq c(p)n\log(2n)$ for $p \in (4k, 4k+2), p \geq 1$ where $k \geq 0$ is an integer.

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