Refining the Graph Density Condition for the Existence of Almost K-factors

Noga Alon * Eldar Fischer †

Abstract

Alon and Yuster [4] have proven that if a fixed graph K on g vertices is $(h+1)$-colorable, then any graph G with n vertices and minimum degree at least $\frac{h}{h+1}n$ contains at least $(1-\epsilon)\frac{n}{g}$ vertex disjoint copies of K, provided $n > N(\epsilon)$. It is shown here that the required minimum degree of G for this result to follow is closer to $\frac{h-1}{h}n$, provided K has a proper $(h+1)$-coloring in which some of the colors occur rarely. A conjecture regarding the best possible result of this type is suggested.

1 Introduction

For an infinite family of graphs $\mathcal{F}$ and a fixed graph K with g vertices, we say that the graphs in $\mathcal{F}$ contain an almost K-factor if for any $\epsilon > 0$ there exists an $N = N(\epsilon)$ such that if G is a graph from $\mathcal{F}$ with $n > N$ vertices, then G contains at least $(1-\epsilon)\frac{n}{g}$ vertex disjoint copies of K.

When K is characterized only by its chromatic number, say $h+1$, it is sufficient to consider a complete $(h+1)$-partite graph, $K_{a_0,\ldots,a_h}$. Moreover, it is sufficient to consider a complete $(h+1)$-partite graph with equal color classes, since we can first find copies of the complete $(h+1)$-partite graph with all color classes of size $\sum_{i=0}^{h} a_i$, and from them extract the required copies of $K_{a_0,\ldots,a_h}$. When $\mathcal{F}$ is characterized by the minimum degree of its members, the following asymptoticaly tight result was proven by Alon and Yuster.

Theorem 1.1 ([4]) *For any natural a, h and any $\epsilon > 0$ there exists an $N = N(a, h, \epsilon)$ such that if G is a graph with $n > N$ vertices and minimum degree at least $\frac{h}{h+1}n$, then G contains at least $(1-\epsilon)\frac{n}{a(h+1)}$ vertex disjoint copies of the complete $(h+1)$-partite graph with color classes of size a each.*

*Department of Mathematics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, Israel. Research supported in part by a USA Israeli BSF grant.

†Department of Mathematics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel-Aviv University, Tel-Aviv, Israel.

Is it possible to relax the minimum degree condition on the members of $\mathcal{F}$ if more information is given about a particular $(h+1)$-coloring of K? In particular, is this possible if some of the color classes in this coloring are known to be smaller than others?

In the following we answer this question in the affirmative. We prove several results, leading to the following conjecture.

Conjecture 1.2 *For any sequence $a_0 \le a_1 \le \ldots \le a_h$ of natural numbers and any $\epsilon > 0$ there exists an $N = N(a_0, \ldots, a_h; \epsilon)$ such that if G is a graph with $n > N$ vertices and minimum degree at least $\frac{1}{h}\left(h-1+a_0/(\sum_{i=0}^{h} a_i)\right)n$, then G contains at least $(1-\epsilon)n/(\sum_{i=0}^{h} a_i)$ vertex disjoint copies of $K = K_{a_0,\ldots,a_h}$, the complete $(h+1)$-partite graph with color classes of sizes $a_0, \ldots, a_h$.*

The general conjecture remains open. Here we prove it for all bipartite K, (that is, for $h = 1$), and obtain some bounds for several other cases.

It suffices to prove the conjecture for the case in which $a_1, \ldots, a_h$ are all equal, since then the assertion for general $a_0, \ldots, a_h$ will follow by first finding copies of $K_{b_0,\ldots,b_h}$ with $b_0 = ha_0$ and $b_j = \sum_{i=1}^{h} a_i$ for $1 \le j \le h$, and then splitting each copy of $K_{b_0,\ldots,b_h}$ into copies of K. Also, this conjecture, if true, is best possible, as shown by considering a G which is the complete $(h+1)$-partite graph with one color class of size ak and h color classes of size bk, $a < b$. G has $n = (a+hb)k$ vertices and minimum degree $(a+(h-1)b)k = \frac{1}{h}(h-1+\frac{a}{a+hb})n$, and yet it has no almost $K = K_{a_0,\ldots,a_h}$-factor for $a_0 = a$, $\sum_{i=1}^{h} a_i < hb$. This is since each color class of a K copy must be contained in a color class of G, and hence by considering the smallest color class of G there are no more than k vertex disjoint copies of K in G. Note that in case all color classes of K are of equal size, the conjecture reduces to Theorem 1.1.

The rest of the paper is organized as follows. Some general lemmas needed for dealing with almost K-factors are stated in Section 2. Section 3 presents a proof of the conjecture for the bipartite case. Section 4 contains various results about the general case. We end this paper with some concluding remarks, mostly about the relation between the questions considered here and other known results and conjectures about the case of equal color classes.

In order to simplify the presentation, we omit all floor and ceiling signs whenever the implicit assumption that a quantity is integral makes no essential difference.

2 The General Tools

For a graph G and disjoint sets A, B of vertices of G, the *density* of (A, B), denoted by $d(A, B)$, is the number of edges from A to B divided by $|A||B|$. The pair (A, B) is called γ-regular if for any $A' \subset A$,

$B' \subset B$ satisfying $|A'| \geq \gamma|A|$, $|B'| \geq \gamma|B|$, the two densities $d(A,B)$ and $d(A',B')$ differ by less than γ.

The main tool for obtaining such pairs is the well known Regularity Lemma of Szemerédi:

Lemma 2.1 ([13]) *For any $\gamma > 0$ and k there exists an $N = N_{2.1}(\gamma, k)$ such that the vertex set of any graph G with $n > N$ vertices can be partitioned into $C_0, \ldots, C_l$ where $k \leq l \leq N$, such that $|C_0| \leq \gamma n$, $C_1, \ldots, C_l$ are of the same size, and all but at most γl^2 of the pairs (C_i, C_j) are γ-regular.*

The relevance of this notion of regularity to graph packing problems is demonstrated in the following simple result.

Lemma 2.2 ([4]) *For any non decreasing sequence $a_0, \ldots, a_h$ and $\delta > 0$ there exist an $N = N_{2.2}(h, a_h; \delta)$ and $\gamma = \gamma_{2.2}(h, a_h; \delta) > 0$ such that every family of pairwise disjoint vertex sets $S_0, \ldots, S_h$ with more than N vertices each, in which all pairs are γ-regular and of density at least δ, contains a $K_{a_0,\ldots,a_h}$ with a_i vertices in S_i for all $0 \leq i \leq h$.*

When we need to find more than one copy of a given subgraph, the following simple and well known lemma will enable us to apply the previous one succesively and find the required copies one by one.

Lemma 2.3 *For every $\gamma \leq \delta$ and $\frac{1}{2} \geq \epsilon > 0$, if (A, B) is an $\epsilon\gamma$-regular pair of density at least δ, and $R \subset A$, $S \subset B$ satisfy $|R| \geq \epsilon|A|$, $|S| \geq \epsilon|B|$, then (R, S) is a γ-regular pair of density at least $\frac{1}{2}\delta$.*

The following lemma is the main tool in the proof of existence of almost K-factors, both in [4] and here. Similar statements have been proved and applied in various other existence proofs, such as in [9] and [10] (see also the "degree form" of the Regularity Lemma in [12]). The lemma is proved implicitly in [4]. Here we state and prove it explicitly.

Lemma 2.4 ([4], see also [12]) *For any $\epsilon > 0$ there exists a $\delta = \delta_{2.4}(\epsilon)$, such that for any k and $\gamma > 0$ there exists an $N = N_{2.4}(\epsilon, k, \gamma)$ satisfying for all α:*
If G is a graph with $n > N$ vertices and minimum degree at least αn, then there exists a partition $C_0, \ldots, C_l$ of G, $k \leq l \leq N$, and a graph H on a vertex set $v_1, \ldots, v_l$ with minimum degree at least $(\alpha - \epsilon)l$, such that $|C_0| < \epsilon n$ and if v_i, v_j are neighbours in H then (C_i, C_j) form a γ-regular pair in G with density at least δ.

Proof: We may assume $\epsilon < 1$. Set $\delta = \frac{1}{2}\epsilon_0$ and $N = N_{2.1}(\min\{\gamma, \gamma_0, \epsilon_0\}, \frac{1-\epsilon_0}{1-2\epsilon_0}\max\{k, k_0\})$ with ϵ_0, γ_0, k_0 to be chosen later. Given G with $n > N$ vertices, partition it into $C'_0, \ldots, C'_{l'}$ using Lemma 2.1. Let G' be the graph obtained from G by removing all vertices in C_0 (decreasing the degrees of other vertices by no more than $\gamma_0 n$), all edges which are contained in a single C_i (decreasing the degrees by no more than $\frac{n}{k_0}$), all edges which are contained in a pair (C_i, C_j) which is not $\min\{\gamma, \gamma_0\}$-regular (this cannot decrease the degrees of more than $2\gamma_0^{\frac{1}{2}}n$ vertices by more than $\gamma_0^{\frac{1}{2}}n$), or in a pair (C_i, C_j) which is γ_0-regular but of density less than δ. This last step decreases the degrees of no more than $\gamma_0^{\frac{1}{2}}n$ vertices by more than $(\delta + \gamma_0 + \gamma_0^{\frac{1}{2}})n$, since for a given $i > 0$, no more than $\gamma_0|C_i|$ vertices in C_i have more than $(\delta + \gamma_0)|C_j|$ neighbours in C_j, for a given $j > 0$ such that (C_i, C_j) is γ_0-regular of density less than δ. Hence, no more than $\gamma_0^{\frac{1}{2}}|C_i|$ vertices of C_i (for each i) have more than $(\delta + \gamma_0)|C_j|$ neighbours in C_j for more than $\gamma_0^{\frac{1}{2}}l'$ possible values of j such that (C_i, C_j) are such pairs. As all other vertices of C_i are about to lose less than $(\delta + \gamma_0 + \gamma_0^{\frac{1}{2}})n$ neighbours in the last edge deletion step, this implies the required result. Define a graph H' on $w_1, \ldots, w_{l'}$ by declaring w_i, w_j to be adjacent iff there is at least one edge in G' from C'_i to C'_j. All vertices of G' but at most $3\gamma_0^{\frac{1}{2}}n$ of them were shown to be of degree at least $(\alpha - \gamma_0 - \frac{1}{k_0} - \gamma_0^{\frac{1}{2}} - (\delta + \gamma_0 + \gamma_0^{\frac{1}{2}}))n$. Remembering that $\delta = \frac{1}{2}\epsilon_0$, a proper choice (depending on ϵ_0) of γ_0 and k_0 will ensure that all vertices in G' but at most $\epsilon_0 n$ of them are of degree at least $(\alpha - \epsilon_0)n$ in G', which implies that all vertices in H' but at most $\frac{\epsilon_0}{(1-\epsilon_0)}l'$ of them are of degree at least $(\alpha - \epsilon_0)l'$. To obtain $C_0, \ldots, C_l$ and H, we delete from H' the vertices which are of degree less than $(\alpha - \epsilon_0)l'$, and group all the vertices in the appropriate C'_i together with C'_0 as the new C_0. H has at least k vertices, and a choice of $\epsilon_0 = \frac{\epsilon}{2+\epsilon}$ will ensure that $|C_0| \leq \epsilon n$ and that H has minimum degree at least $(\alpha - \epsilon_0 - \frac{\epsilon_0}{1-\epsilon_0})l' \geq (\alpha - \epsilon)l$ as required. Note that ϵ_0 and thereby $\delta = \frac{1}{2}\epsilon_0$ depend on ϵ only, as required. □

Thus, by proving in the first stage the existence of appropriate subgraphs of the graph H, to which we shall usualy refer as the *partition graph*, we can deduce by Lemma 2.2 and Lemma 2.3 the existence of certain subgraphs of G (in [12] H is refered to as the *reduced graph* of G with respect to the given partition). The following classical result of Hajnal and Szemerédi is often useful in finding subgraphs of the partition graph, and will also be used here.

Lemma 2.5 ([8]) *For every natural number h, any graph H with $k = hl$ vertices and minimum degree at least $\frac{h-1}{h}k = (h-1)l$ can be partitioned into l vertex disjoint copies of K_h, the complete graph on h vertices.*

Theorem 1.1 is proved in [4] by first applying Lemma 2.4 to G, then covering most of the partition graph H with vertex disjoint copies of K_h using Lemma 2.5, and finally extracting the required copies of $K_{a,\ldots,a}$ by repeated applications of Lemma 2.2, using Lemma 2.3 as well. The results here will be proven using a similar general approach, where in each case we find in the partition graph H an appropriate subgraph for the required purpose.

The following simple Lemma about the possibility of partitioning a graph (and the given packing problem with it) keeping high degrees, is useful in many packing problems, and will play a role in this paper too.

Lemma 2.6 ([2], see also [5]) *For any $\alpha > \beta > 0$ and $\eta > 0$ there exists an $N = N_{2.6}(\alpha, \beta, \eta)$ such that if G is a graph with $n > N$ vertices and minimum degree at least αn, then for any k, l satisfying $n = k + l$, $k \geq \eta n$, $l \geq \eta n$ the vertex set of G can be partitioned into sets A, B of sizes k, l respectively, such that all vertices of G have at least βk neighbours in A and at least βl neighbours in B.*

As a final remark, it should be mentioned that we could replace Lemma 2.2 and Lemma 2.3 by a much stronger result, the Blow Up Lemma of Komlós, Sárközy and Szemerédi [11]. However, since this heavy machinery will not simplify the proofs here considerably, we prefer to formulate them without it.

3 Almost Bipartite K-factors

In case $h = 1$ (bipartite graphs K), we can prove the precise statement of Conjecture 1.2, as stated in the following.

Theorem 3.1 *For any $a \leq b$ and $\epsilon > 0$ there exists an $N = N(a, b; \epsilon)$ such that any graph G with $n > N$ vertices and minimum degree at least $\frac{a}{a+b}n$ contains at least $(1-\epsilon)\frac{n}{a+b}$ vertex disjoint copies of $K_{a,b}$.*

The subgraph we need in the partition graph here is described in the following lemma. Note that the existence of any member of a rather large family of possible graphs suffices in this case.

Lemma 3.2 *For every natural numbers $a < b$, any graph H with k vertices and minimum degree at least $\frac{a}{a+b}k + a$ contains a spanning subgraph with minimum degree at least a and maximum degree at most b.*

Proof: Let L be a spanning subgraph of H with minimum degree at least a, which has a minimum number of edges incident with vertices whose degrees exceed b, and subject to this is minimal with respect to deleting edges. In particular, each edge of L is incident with at least one vertex of degree a. We claim that L contains no vertices of degree exceeding b, and hence is the required subgraph. Supposing otherwise, let v be a vertex with degree exceeding b, and w a neighbour of v in L, w being neccessarily of degree a in L. The number of vertices of degree at least b in L is less than $\frac{a}{a+b}k$. This is since, denoting their number by j, they have more than bj edges incident with them. As all their neighbours are of degree a in L, this means they have more than a total of $\frac{b}{a}j$ neighbours, and hence $j + \frac{bj}{a} < k$, implying that $j < \frac{a}{a+b}k$. Thus, w has more than a neighbours in H which are of degrees less than b in L. At least one of these neighbours is not a neighbour of w in L, and thus the edge leading to it from w can be added to L replacing the edge from w to v. This decreases the number of edges incident with vertices of degrees exceeding b, contradicting the choice of L and completing the proof. □

Corollary 3.3 *For any $a < b$ and $\epsilon > 0$ there exist $M = M_{3.3}(a, b, \epsilon)$ and $\gamma = \gamma_{3.3}(a, b, \epsilon)$, such that any graph H with $k > M$ vertices and minimum degree at least $(\frac{a}{a+b} - \gamma)k$ contains a spanning subgraph L with maximum degree at most b, all but at most ϵk of whose vertices are also of degree at least a.*

Proof: We add to H a complete graph on $2(\gamma k + a)$ new vertices, connecting them to all existing ones. Then we use Lemma 3.2, and remove the newly added vertices to obtain a spanning subgraph in which all but at most $2(\gamma k + a)b$ of the degrees are at least a. A proper choice of M and γ ensures that this last number is less than ϵk. $\square$

Proof of Theorem 3.1: We may assume $\epsilon < 1$. If $a = b$, this is Theorem 1.1. Otherwise, we set $N = \max\{N_1, N_2\}$, where $N_1 = N_{2.4}(\epsilon_0, M, \gamma_0)$, using $M = M_{3.3}(a, b, \frac{\epsilon}{4})$, $\epsilon_0 = \min\{\frac{\epsilon}{2}, \gamma_{3.3}(a, b, \frac{\epsilon}{4})\}$, and γ_0 and N_2 will be chosen later. Find according to Lemma 2.4 the appropriate partition $C_0, \ldots, C_l$ of G and the corresponding partition graph H. In H, find the appropriate L according to Corollary 3.3. Define $\gamma_0 = \frac{\epsilon}{8}\gamma_{2.2}(1, b; \frac{1}{2}\delta_{2.4}(\epsilon_0))$, and let N_2 be large enough to ensure for $1 \leq i \leq l$ that $\frac{\epsilon}{8}|C_i| > N_{2.2}(1, b; \frac{1}{2}\delta_{2.4}(\epsilon_0))$. We can use L to extract the required $K_{a,b}$ copies as follows.

Let $d_1, \ldots, d_l$ be the degrees of $v_1, \ldots, v_l$ in L. We deal with all edges in L which are incident with at least one vertex of degree at least a (and, of course, at most b). Pick such an edge, $v_i v_j$, with $d_i \geq d_j$, $d_i \geq a$. We now extract vertex disjoint copies of $K_{a,b}$ one by one using Lemma 2.2, occupying a total of $x = (1 - \frac{\epsilon}{8})\frac{|C_i|}{d_i}$ vertices from C_i and $y = (1 - \frac{\epsilon}{8})\frac{|C_j|}{\max\{a, d_j\}}$ vertices from C_j. Since $1 \geq \frac{x}{y} \geq \frac{a}{b}$, we can do it by picking $\frac{yb - xa}{b^2 - a^2}$ copies with a vertices in C_i and b vertices in C_j, and $\frac{xb - ya}{b^2 - a^2}$ copies with b vertices in C_i and a vertices in C_j. This process can be applied and completed for all of the above mentioned edges, since during the whole process at least $\frac{\epsilon}{8}|C_i|$ vertices remain in C_i for each i, thus ensuring by Lemma 2.3 that Lemma 2.2 is still applicable. Moreover, at the end of this process for each v_i with $d_i \geq a$, only $\frac{\epsilon}{8}|C_i|$ vertices from C_i will remain unused (since for each of the d_i edges incident with v_i, $(1 - \frac{\epsilon}{8})\frac{|C_i|}{d_i}$ vertices from C_i were taken for the $K_{a,b}$ copies corresponding to that edge).

Summing up, the $K_{a,b}$ copies found this way occupy at least $(1 - \frac{\epsilon}{2})(1 - \frac{\epsilon}{4})(1 - \frac{\epsilon}{8})n > (1 - \epsilon)n$ vertices, yielding the required result. $\square$

4 Almost $(h+1)$-partite K-factors

In case $h > 1$ we know less about the minimum degree required to ensure the existence of an almost K-factor. The following proposition supplies an upper bound when one of the color classes of K is known to be small.

Proposition 4.1 *For any natural h, g, a and any $\epsilon > 0$ there exists an $N = N(h, g, a; \epsilon)$ such that if G is a graph with $n > N$ vertices and minimum degree at least $\frac{1}{h}(h - 1 + \frac{1}{g})n$, then G contains at least $(1 - \epsilon)\frac{n}{(1+hg^2)a}$ vertex disjoint copies of K, the complete $(h + 1)$-partite graph with one color class of size a and h color classes of size g^2a.*

From this proposition it follows that graphs with n vertices and minimum degree at least $\frac{1}{h}(h-1+\frac{1}{g})n$ contain an almost K-factor for any fixed $K = K_{a_0,\ldots,a_h}$ satisfying $\sum_{i=1}^{h} a_i \geq hg^2a_0$, as copies of this K can be extracted from copies of a larger graph found using Proposition 4.1. Alternatively, the proof of the proposition itself can be slightly modified to include this more general case.

In case a second color class is known to be not too large, we can eliminate the square sign from our estimates. This is demonstrated in the following proposition, in which, for simplicity, we make no attempt to prove the most general result that can be obtained and merely illustrate the basic idea.

Proposition 4.2 *For any natural h, g, a, b with $a \leq b \leq 2ga$ and any $\epsilon > 0$ there exists an $N = N(h, g, a, b; \epsilon)$ such that if G is a graph with $n > N$ vertices and minimum degree at least $\frac{1}{h}(h - 1 + \frac{1}{g})n$, then G contains $(1 - \epsilon)\frac{n}{a+b+4(h-1)ga}$ vertex disjoint copies of K, the complete $(h + 1)$-partite graph with one color class of size a, one color class of size b and $h - 1$ color classes of size $4ga$.*

In order to prove Proposition 4.1 we define the following special graph, and then prove the existence of many vertex disjoint copies of such graphs in our partition graph H.

Definition: For any natural numbers g, h, define an *h-flower with g petals* to be the graph on $gh + 1$ vertices consisting of g vertex disjoint copies of K_h, the complete graph on h vertices, in which all vertices of these copies are adjacent to a single additional vertex.

Lemma 4.3 *For any $\epsilon > 0$ and natural h, g there exists an $M = M_{4.3}(h, g, \epsilon)$ such that if H is a graph with $k > M$ vertices and minimum degree at least $(\frac{1}{h}(h - 1 + \frac{1}{g}) + \epsilon)k$, and $gh + 1$ divides k, then H contains a spanning subgraph consisting of vertex disjoint h-flowers with up to g^2 petals each.*

Proof: Choose $M = N_{2.6}(\frac{1}{h}(h - 1 + \frac{1}{g}) + \epsilon, \frac{1}{h}(h - 1 + \frac{1}{g}), \frac{1}{gh+1})$. Given H, use Lemma 2.6 to partition its vertex set into two parts A, B of sizes $\frac{gh}{gh+1}k$, $\frac{1}{gh+1}k$, respectively, such that each vertex has at least $\frac{1}{h}(h - 1 + \frac{1}{g})\frac{gh}{gh+1}k > \frac{h-1}{h}\frac{gh}{gh+1}k$ neighbours in A and at least $\frac{1}{h}(h - 1 + \frac{1}{g})\frac{1}{gh+1}k$ neighbours in B. Use Lemma 2.5 to partition A into vertex disjoint copies of K_h. Simple counting shows that each of the above mentioned K_h copies has at least $\frac{k}{g(gh+1)}$ vertices in B to which all its vertices are adjacent. Similarly, each vertex in B is adjacent to all vertices of at least $\frac{k}{gh+1}$ of these K_h copies in A. Define a bipartite graph with color classes C, D as follows:

C consists of one vertex corresponding to each of the above mentioned K_h copies in A, D consists of g^2 vertices corresponding to each vertex of B. Two vertices $u \in C$, $w \in D$ are adjacent iff the vertex represented by w is adjacent in H to all vertices in the K_h represented by u. Let B' be an arbitrary subset of D in which each vertex of B has exactly one representative. Since each vertex in D has at least $|B'| = |B| = \frac{1}{gh+1}k$ neighbours in C, there is a matching from B' to C. Since each vertex of C has at least $|C| = \frac{g}{gh+1}k$ neighbours in D, the last matching can be extended to a matching from C to D, in which each vertex in B has a representative in D which is matched. The spanning subgraph of H corresponding to this matching consists of the required h-flowers. □

Corollary 4.4 *For any $\epsilon > 0$ and natural $g > 1$, h there exist $M = M_{4.4}(h, g, \epsilon)$ and $\gamma = \gamma_{4.4}(h, g, \epsilon)$ such that if H is a graph with $k > M$ vertices and minimum degree at least $(\frac{1}{h}(h - 1 + \frac{1}{g}) - \gamma)k$, then H contains a subgraph with at least $(1 - \epsilon)k$ vertices consisting of vertex disjoint h-flowers with up to g^2 petals each.*

Proof: Set $M = \max\{M_{4.3}(h, g, \gamma), M_0\}$, M_0 as well as γ to be chosen later. If H is a graph with $k > M$ vertices and minimum degree at least $(\frac{1}{h}(h - 1 + \frac{1}{g}) - \gamma)k$, add to H a complete graph on l new vertices, where $4h\gamma k \leq l < 4h\gamma k + gh + 1$, connecting them to all existing ones, and ensuring that the number of vertices in the new graph is divisible by $gh + 1$. A calculation shows that Lemma 4.3 can be applied to this graph, and then the new vertices together with all flowers containing them can be removed, obtaining a subgraph consisting of vertex disjoint flowers with more than $k - (g^2h + 1)(4h\gamma k + gh + 1)$ vertices. An appropriate choice of M_0, γ will ensure that this is at least $(1 - \epsilon)k$. □

Proof of Proposition 4.1: We may assume $\epsilon < 1$ and $g > 1$. Set $N = \max\{N_1, N_2\}$, where $N_1 = N_{2.4}(\epsilon_0, M, \gamma_0)$, using $M = M_{4.4}(h, g, \frac{\epsilon}{4})$, $\epsilon_0 = \min\{\frac{\epsilon}{2}, \gamma_{4.4}(h, g, \frac{\epsilon}{4})\}$, and γ_0 and N_2 will be chosen later. Find according to Lemma 2.4 the appropriate partition $C_0, \ldots, C_l$ of G and the corresponding partition graph H. In H, find the h-flowers guarantied by Corollary 4.4. Define $\gamma_0 = \frac{\epsilon}{8}\gamma_{2.2}(h, g^2a; \frac{1}{2}\delta_{2.4}(\epsilon_0))$, and let N_2 be large enough to ensure for $1 \leq i \leq l$ that $\frac{\epsilon}{8}|C_i| > N_{2.2}(h, g^2a; \frac{1}{2}\delta_{2.4}(\epsilon_0))$. We can use the h-flowers to extract the required K copies as follows.

For each vertex w of H, denote the appropriate C_i by C_w. Consider an h-flower with t petals. Let v be the vertex adjacent to $1 \leq t \leq g^2$ copies of K_h. Consider one of the K_h copies in the h-flower, denoting its vertices by $u_1, \ldots, u_h$. We extract copies of K occupying a total of $(1 - \frac{\epsilon}{8})\frac{|C_v|}{t}$ vertices from C_v, and $(1 - \frac{\epsilon}{8})|C_{u_j}|$ vertices from C_{u_j}, $1 \leq j \leq h$. This process can be applied and completed for all the K_h copies in this flower and then for all the h-flowers, since during the whole process at least $\frac{\epsilon}{8}|C_i|$ vertices remain in C_i for each i, thus ensuring by Lemma 2.3 that Lemma 2.2 is still applicable. At the end of this process for each i such that v_i is in a flower, only $\frac{\epsilon}{8}|C_i|$ vertices from C_i will remain unused.

Summing up, the K copies found this way occupy at least $(1-\frac{\epsilon}{2})(1-\frac{\epsilon}{4})(1-\frac{\epsilon}{8})n > (1-\epsilon)n$ vertices, yielding the required result. □

For the proof of Proposition 4.2, we first prove the existence of a certain structure in H, which we define for this purpose.

Definition: For any natural h, g, define an (h,g)-*bush* to be a graph L consisting of a vertex set V, plus a list of vertex disjoint copies of K_h with a vertex set $U \subset V$ and an assignment of a vertex of V to each of the K_h copies satisfying:

- Each K_h-copy is a complete graph on h vertices in L.
- The vertex assigned to each copy of K_h does not belong to that copy and is adjacent in L to all the vertices of the copy.
- No vertex in V has more than g copies of K_h to whom it is assigned.
- No copy of K_h has more than a total of g copies of K_h to whom any of its vertices have been assigned.

The (h,g)-bush is said to *utilize* the vertex set U of all vertices in the above mentioned K_h copies.

The existence lemma we need here is the following:

Lemma 4.5 *If H is a graph with k vertices and minimum degree at least $\frac{1}{h}(h-1+\frac{1}{g})k$, and h divides k, then H contains an (h,g)-bush utilizing the whole vertex set of H.*

Proof: First we use Lemma 2.5 to partition H into vertex disjoint copies of K_h. We call a vertex and a (disjoint) copy of K_h adjacent iff the vertex is adjacent to all the vertices of the K_h. A simple counting argument shows that each of the above mentioned K_h copies is adjacent to at least $\frac{k}{g}$ vertices. We now define a bipartite graph with color classes A, B as follows:
The vertices of A consist of one vertex corresponding to each of the above mentioned K_h copies, while B consists of g vertices corresponding to each K_h. $u \in A$ and $w \in B$ are adjacent iff there is a vertex in the K_h copy corresponding to w which is adjacent to the K_h copy corresponding to u.

Since each vertex in A has at least $|A| = \frac{k}{h}$ neighbours in B, there is a matching from A to B. Assigning the vertices of H to the K_h copies according to this matching yields the required (h,g)-bush. □

Corollary 4.6 *For any $\epsilon > 0$ and natural $g > 1$, h there exist $M = M_{4.6}(h,g,\epsilon)$ and $\gamma = \gamma_{4.6}(h,g,\epsilon)$ such that if H is a graph with $k > M$ vertices and minimum degree at least $(\frac{1}{h}(h-1+\frac{1}{g})-\gamma)k$, then H contains an (h,g)-bush utilizing at least $(1-\epsilon)k$ vertices.*

Proof: Add to H a complete graph on l new vertices, where $2h\gamma k \le l < 2h\gamma k + h$, connecting them to all vertices of H, and ensuring that the number of vertices in the new graph is divisible by h. Apply now Lemma 4.5, and then remove the new vertices, discounting any copies of K_h (in the definition of the bush) which contain or are assigned any of the new vertices. Since less than $(g+1)(2h\gamma k+h)$ of the K_h copies were discounted, the remaining bush still utilizes more than $k - h(g+1)(2h\gamma k + h)$ vertices. An appropriate choice of M, γ will ensure that this is at least $(1-\epsilon)k$. $\square$

Proof of Proposition 4.2: We may assume $\epsilon < 1$ and $g > 1$. As in the proof of the previous proposition, set $N = \max\{N_1, N_2\}$, $N_1 = N_{2.4}(\epsilon_0, M, \gamma_0)$, $M = M_{4.6}(h, g, \frac{\epsilon}{4})$, $\epsilon_0 = \min\{\frac{\epsilon}{2}, \gamma_{4.6}(h, g, \frac{\epsilon}{4})\}$, where N_2 and γ_0 are to be chosen later. Find according to Lemma 2.4 the appropriate partition $C_0, \ldots, C_l$ of G and the corresponding partition graph H. In H, find the (h,g)-bush guarantied by Corollary 4.6. Define $\gamma_0 = \frac{\epsilon}{8}\gamma_{2.2}(h, 4ga; \frac{1}{2}\delta_{2.4}(\epsilon_0))$, and let N_2 be large enough to ensure for $1 \le i \le l$ that $\frac{\epsilon}{8}|C_i| > N_{2.2}(h, 4ga; \frac{1}{2}\delta_{2.4}(\epsilon_0))$. We can use this (h,g)-bush to extract the required K copies as follows.

Again for each vertex w of H we denote the appropriate C_i by C_w. Consider a K_h copy of the bush, denoting its vertices by $u_1, \ldots, u_h$, the vertex assigned to it by v, and the number of K_h copies to which $u_1, \ldots, u_h$ are assigned by $d_1, \ldots, d_h$, respectively. Extract one by one using Lemma 2.2 vertex disjoint copies of K, occupying a total of $(1-\frac{\epsilon}{8})\frac{|C_v|}{2g}$ vertices from C_v, and $(1-\frac{\epsilon}{8})\frac{(2g-d_j)|C_{u_j}|}{2g}$ vertices from each C_{u_j}, $1 \le j \le h$. The following is a brief explanation showing how it is done, for example, for the case $b = 2ga$ (the general case $a \le b \le 2ga$ is similar).

It is enough to find a particular solution of the following linear programing problem: Find $x_1, \ldots, x_h$, $y_1, \ldots, y_h$, which are all non-negative and satisfy $a\sum_{i=1}^h x_i + 2ga\sum_{i=1}^h y_i = (1-\frac{\epsilon}{8})\frac{|C_v|}{2g}$ and $2gax_j + 4ga\sum_{i\ne j} x_i + ay_j + 4ga\sum_{i\ne j} y_i = (1-\frac{\epsilon}{8})\frac{(2g-d_j)|C_{u_j}|}{2g}$ for $1 \le j \le h$. This is since each variable will then tell us how many copies of K in a certain configuration to extract. To find one such solution, we denote $x = \sum_{i=1}^h x_i$, $y = \sum_{i=1}^h y_i$, $s = (1-\frac{\epsilon}{8})\sum_{j=1}^h \frac{(2g-d_j)|C_{u_j}|}{2g}$, and solve $ax + 2gay = (1-\frac{\epsilon}{8})\frac{|C_v|}{2g}$, $(4hg-2g)ax + (4(h-1)g+1)ay = s$. Both x and y can be shown to be non-negative. Now, since $\sum_{i=1}^h d_i \le g$, we can find the $x_1, \ldots, x_h$, $y_1, \ldots, y_h$ which have sums x, y respectively and satisfy the following conditions for all $1 \le j \le h$, which in particular imply the conditions of the original problem: $2gax_j + 4ga\sum_{i\ne j} x_i = \frac{(4hg-2g)ax}{s}(1-\frac{\epsilon}{8})\frac{(2g-d_j)|C_{u_j}|}{2g}$, $ay_j + 4ga\sum_{i\ne j} y_i = \frac{(4(h-1)g+1)ay}{s}(1-\frac{\epsilon}{8})\frac{(2g-d_j)|C_{u_j}|}{2g}$. Although the solution is usualy not integral, replacing each x_i and y_i by the closest integer above or below as appropriate to the case still yields the desired result.

As usual, Lemma 2.3 guaranties that this process can be applied and completed for all the K_h copies of the bush, since during the whole process at least $\frac{\epsilon}{8}|C_i|$ vertices remain in C_i for each i. At the end, for each v_i which is utilized by the bush, only $\frac{\epsilon}{8}|C_i|$ vertices from C_i will remain unused, so the K copies found this way occupy at least $(1-\frac{\epsilon}{2})(1-\frac{\epsilon}{4})(1-\frac{\epsilon}{8})n > (1-\epsilon)n$ vertices, yielding the required result. □

5 Concluding Remarks

- As is the case with the results in [4], [5] and [2], the results here can be made algorithmic using the algorithmic version of the Regularity Lemma in [3]. The price will be an increase of the (already horrible) estimate of the lower bound N for the number of vertices in the graph as a function of ϵ.

- We can prove that for any natural $h > 1$,$a > 1$ there exists an $\eta = \eta(h,a) > 0$ such that any graph G with n vertices and minimum degree at least $(\frac{h}{h+1} - \eta)n$ contains an almost K-factor, K being the complete $(h+1)$-partite graph with one color class of size $a-1$ and h color classes of size a, so the required minimum degree can be somewhat reduced for any K with unequal color classes. For the proof we use Theorem 1.1 and methods similar to ones appearing in [2] and [5] to prove that the partition graph can be covered by vertex disjoint complete $(h+1)$-partite graphs which are with h color classes of size $ha-1$ and one color class of size either $ha-1$ or ha. We omit the full details.

- El-Zahar [7] conjectured that if $n = \sum_{i \in I} c_i$ where $c_i \geq 3$ are integers, exactly l of which are odd, then any graph on n vertices with minimum degree at least $\frac{n+l}{2}$ contains vertex disjoint cycles C_i, $(i \in I)$, where C_i is of length c_i. Methods similar to those used in [2] can be used to obtain some asymptotic version of this conjecture if Conjecture 1.2 holds for $h = 2$. See also [1] for some related results.

- In [5] it is proven from Theorem 1.1 that any graph G on $n = (h+1)ak$ vertices with minimum degree at least $(\frac{h}{h+1}+\epsilon)n$ contains k vertex disjoint copies of the complete $(h+1)$-partite graph with a vertices in each color class, provided $n > N(a,h,\epsilon)$. The question arises as to whether a similar refinment of the minimum degree condition can be achieved in this case, supposing that Conjecture 1.2 is proven. The answer is, however, negative: Taking any $K_{a_0,\ldots,a_h}$ such that $h+1$ divides a_i for all i, but does not divide $\sum_{i=0}^{h} \frac{a_i}{h+1}$, and G the complete $(h+1)$-partite graph with all color classes of size $(k(h+1)+1)\sum_{i=0}^{h} \frac{a_i}{h+1}$ (k any natural number), we see that G cannot be partitioned into copies of $K_{a_0,\ldots,a_h}$, although its minimum degree is $\frac{h}{h+1}n$, $n = (k(h+1)+1)\sum_{i=0}^{h} a_i$.

- We can formulate, in the spirit of [4], [5], the stronger conjecture that for any $a_0 \leq a_1 \leq \ldots \leq a_h$ there exists a constant $c = c(a_0,\ldots,a_h)$ such that any graph G with n vertices and minimum degree at least $\frac{1}{h}\left(h-1+a_0/(\sum_{i=0}^{h} a_i)\right) n$ contains at least $n/(\sum_{i=0}^{h} a_i)-c$ vertex disjoint copies of $K_{a_0,\ldots,a_h}$. This seems to be true.

- It would be interesting to see if the minimum degree condition for the existence of an almost K-factor (or a K-factor) can be refined in different ways, when more information about K is available.

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