Bipartite subgraphs and the smallest eigenvalue

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Abstract

Two results dealing with the relation between the smallest eigenvalue of a graph and its bipartite subgraphs are obtained. The first result is that the smallest eigenvalue μ of any non-bipartite graph on n vertices with diameter D and maximum degree Δ satisfies $\mu \geq -\Delta + \frac{1}{(D+1)n}$. This improves previous estimates and is tight up to a constant factor. The second result is the determination of the precise approximation guarantee of the Max Cut algorithm of Goemans and Williamson for graphs $G = (V, E)$ in which the size of the max-cut is at least $A|E|$, for all A between 0.845 and 1. This extends a result of Karloff.

1 Introduction

The smallest eigenvalue of (the adjacency matrix of) a graph G is closely related to properties of its bipartite subgraphs. In this paper we obtain two results based on this relation.

In [8], Problem 11.29 it is proved that if G is a d-regular, non-bipartite graph on n vertices with diameter D, then its smallest eigenvalue μ satisfies $\mu + d > \frac{1}{2dDn}$. Our first result here improves this bound as follows.

Theorem 1.1 *Let $G = (V, E)$ be a graph on n vertices with diameter D, maximum degree Δ and eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. If G is non-bipartite then*

$$\lambda_n \geq -\Delta + \frac{1}{(D+1)n}.$$

If $G = (V, E)$ is an undirected graph, and S is a nonempty proper subset of V, then $(S, V - S)$ denotes the cut consisting of all edges with one end in S and another one in $V - S$. The *size* of the

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cut is the number of edges in it. The MAX CUT problem is the problem of finding a cut of maximum size in G. This is a well known NP-hard problem (which is also MAX-SNP hard as shown in [9]- see also [6], [2]), and the best known approximation algorithm for it, due to Goemans and Williamson [5], is based on semidefinite programming and an appropriate (randomized) rounding technique. It is proved in [5] that the approximation guarantee of this algorithm is at least the minimum of the function $h(t)/t$ in $(0,1]$, where $h(t) = \frac{1}{\pi}\arccos(1-2t)$. This minimum is attained at $t_0 = 0.844..$ and is roughly 0.878. Karloff [7] showed that this minimum is indeed the correct approximation guarantee of the algorithm, by constructing appropriate graphs. The authors of [5] also proved that their algorithm has a better approximation guarantee for graphs with large cuts. If $A \geq t_0$, with t_0 as above, and the maximum cut of $G = (V,E)$ has at least $A|E|$ edges, then the expected size of the cut provided by the algorithm is at least $h(A)|E|$, showing that in this case the approximation guarantee is at least $h(A)/A$ (which approaches 1 as A tends to 1). Here we apply the relation between the smallest eigenvalue of G and the maximum size of a cut in it to show that this result is tight for every such A, that is, the precise approximation guarantee of the Goemans-Williamson algorithm for graphs in which the maximum cut is of size at least $A|E|$ is $h(A)/A$ for all A between t_0 and 1. This extends the result of Karloff, who proved the statement for $A = t_0$. The technical result needed for this purpose is the following.

Theorem 1.2 *For any rational η satisfying $-1 < \eta < 0$, there exists a graph $H = (V,E), V = \{1,\ldots,n\}$ and a set of unit vectors $w_1,\ldots,w_n$ in $R^k, 1 \leq k \leq n$, such that $w_i^t w_j = \eta$ for all $i \neq j$ with $\{i,j\} \in E$ and the size of a maximum cut in H is equal to*

$$\max_{\|v_i\|^2=1, v_i \in R^n} \sum_{\{i,j\}\in E} \frac{1 - v_i^t v_j}{2} = \sum_{\{i,j\}\in E} \frac{1 - w_i^t w_j}{2} = \sum_{\{i,j\}\in E} \frac{1-\eta}{2}.$$

The rest of this short paper is organized as follows. In Section 2 we prove Theorem 1.1 and present examples showing that its statement is optimal, up to a constant factor. In Section 3 we prove Theorem 1.2 by constructing appropriate graphs. Our construction resembles the one in [7], but is more general and its analysis is somewhat simpler. The main advantage of the new construction is that unlike the one in [7], it is a Cayley graph of an abelian group and therefore its eigenvalues have a simple expression, and can be compared with each other without too much efforts. The construction in [7] is an induced subgraph of one of our graphs. We also discuss in Section 3 the relevance of the construction to the study of the approximation guarantee of the algorithm of [5]. The final Section 4 contains some concluding remarks.

2 Non-bipartite graphs

In this section we prove Theorem 1.1 and show that its estimate is best possible up to a constant factor.

Proof of Theorem 1.1. Let A be the adjacency matrix of $G = (V, E)$ and let $V = \{1, \dots, n\}$. Denote by d_i the degree of the vertex i in G. Let $\mathbf{x} = (x_1, \dots, x_n)$ be an eigenvector, satisfying $\|\mathbf{x}\| = 1$, corresponding to the smallest eigenvalue λ_n of A. Then

$$\lambda_n = \lambda_n \|\mathbf{x}\|^2 = \mathbf{x}^t A \mathbf{x} = \sum_{i,j} a_{i,j} x_i x_j = 2 \sum_{\{i,j\} \in E} x_i x_j. \tag{1}$$

The fact that the maximum degree of G is Δ, together with the inequality $\Delta = \Delta \|\mathbf{x}\|^2 \geq \sum_i d_i x_i^2$ implies that

$$\Delta + \lambda_n \geq \sum_i d_i x_i^2 + 2 \sum_{\{i,j\} \in E} x_i x_j = \sum_{\{i,j\} \in E} (x_i + x_j)^2.$$

Partition the vertices of the graph into two parts by taking the first part to be the set of all vertices with negative coordinates and the second one to be the set of all remaining vertices. Although it is not difficult to show, using the Perron-Frobenius theorem, that both parts are nonempty, this fact is not needed in what follows. Since G is non-bipartite one of the parts must contain an edge. Therefore there exists an edge $\{i, j\}$ of G such that either both coordinates x_i, x_j are non-negative or both of them are negative.

Without loss of generality we can assume that x_1 is positive and has maximum absolute value among all entries of $\mathbf{x}$. First consider the case that $\{i, j\} \in E$ and $x_i \geq 0, x_j \geq 0$. Since the diameter of G is D, we can assume that $i \leq D + 1$, that $1, 2, \dots, i - 1, i$ is a shortest path from 1 to the set $\{i, j\}$ and that $j = i + 1$. Therefore, either the path $1, \dots, i$ or the path $1, \dots, i, i + 1$ has odd length and this length is bounded by $D + 1$. Let $1, \dots, k$ be such a path with $x_k \geq 0$, k even and $k \leq D + 2$. Then, by the Cauchy-Schwartz inequality

$$\Delta + \lambda_n \geq \sum_{\{i,j\} \in E} (x_i + x_j)^2 \geq \sum_{i=1}^{k-1} (x_i + x_{i+1})^2 \geq \frac{1}{k-1} \left(\sum_{i=1}^{k-1} |x_i + x_{i+1}| \right)^2$$

$$\geq \frac{1}{k-1} \Big((x_1 + x_2) + (-x_2 - x_3) + (x_3 + x_4) + (-x_4 - x_5) + \ldots + (x_{k-1} + x_k) \Big)^2 = \frac{1}{k-1} (x_1 + x_k)^2 \geq \frac{x_1^2}{D+1}.$$

Finally, since $\sum_i x_i^2 = 1$ and x_1 has maximum absolute value we conclude that

$$\Delta + \lambda_n \geq \frac{x_1^2}{D+1} \geq \frac{\sum_i x_i^2}{n(D+1)} = \frac{1}{n(D+1)}.$$

Next consider the case that both coordinates $x_i, x_j, \{i, j\} \in E$, are negative. Using the reasoning above it follows that there exists a path $1, \dots, k$ such that $x_k < 0$, k is odd and $k \leq D + 2$. Thus we have

$$\Delta + \lambda_n \geq \sum_{\{i,j\}\in E} (x_i + x_j)^2 \geq \sum_{i=1}^{k-1} (x_i + x_{i+1})^2 \geq \frac{1}{k-1} \left(\sum_{i=1}^{k-1} |x_i + x_{i+1}| \right)^2$$

$$\geq \frac{1}{k-1} \Big((x_1 + x_2) + (-x_2 - x_3) + (x_3 + x_4) + (-x_4 - x_5) + \ldots + (-x_{k-1} - x_k) \Big)^2 = \frac{1}{k-1} (x_1 - x_k)^2$$

$$\geq \frac{x_1^2}{D+1} \geq \frac{\sum_i x_i^2}{n(D+1)} = \frac{1}{n(D+1)}.$$

This completes the proof. □

As a corollary we obtain the following result for d-regular graphs.

Corollary 2.1 *Let $G = (V, E)$ be a d-regular graph on n vertices with diameter D and eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. If G is non-bipartite then*

$$\lambda_1 + \lambda_n = d + \lambda_n \geq \frac{1}{(D+1)n}.$$

Next we give examples of graphs which show that the estimate of Corollary 2.1 (and therefore also that of Theorem 1.1) is best possible up to a constant factor. In case $d = 2$ consider the cycle C_n of length $n = 2D + 1$. Clearly the diameter of C_{2D+1} is D and it is well known (see, e.g., [8]) that its maximum eigenvalue is $\lambda_1 = 2$ and the minimum one equals $\lambda_n = 2\cos(\frac{2D\pi}{2D+1})$. Thus

$$\lambda_1 + \lambda_n = 2 + 2\cos\left(\frac{2D\pi}{2D+1}\right) = 2\left(1 - \cos(\frac{\pi}{2D+1})\right) \leq \frac{\pi^2}{(2D+1)^2} \leq \frac{\pi^2}{2Dn}.$$

To generalize the previous example and show that our estimate is tight even if G has diameter D and more vertices and/or larger degrees, we construct for $d \geq 3$ a graph $H_{D,d}$ which is essentially a blow up of the cycle of odd length by complete bipartite graphs. Let $H_{D,d} = (V, E)$ be the graph on the set of vertices $V = \{v_{i,j}, u_{i,j} | 1 \leq i \leq 2D + 1, 1 \leq j \leq d\}$ whose set of edges includes all edges $\{v_{i,j}, u_{i,k}\}$ for all j, k except the cases $j = k = 1, j = k = d$ and the edges $\{v_{i,d}, v_{i+1,1}\}, \{u_{i,d}, u_{i+1,1}\}$ where $i + 1$ is reduced modulo $2D + 1$. Note that by definition, the graph $H_{D,d}$ is d-regular, has $n = (2D + 1)2d$ vertices and is non-bipartite, since it contains, for example, the odd cycle $v_{1,1}, u_{1,d}, u_{2,1}, v_{2,d}, \ldots, u_{1,1}, v_{1,2}, u_{1,2}, v_{1,1}$ of length $2(2D + 1) + 3$. Thus to show that the estimate of Theorem 1.1 is tight it is enough to prove the following proposition.

Proposition 2.2 *The graph $H_{D,d}$ has diameter $2D + 2$ and its largest and smallest eigenvalues λ_1, λ_n satisfy*

$$\lambda_1 + \lambda_n = O(\frac{1}{nD}).$$

Proof. The diameter is easily computed. To prove the estimate on $\lambda_1+\lambda_n$, let A_H be the adjacency matrix of $H_{D,d}$ and let $\lambda_1 = d \geq \ldots \geq \lambda_n$ be the eigenvalues of A_H. By the variational definition of the eigenvalues of A_H (see, e.g., [10], pp. 99–101) we have that

$$\lambda_n = \min_{\mathbf{x}\in R^n, \|\mathbf{x}\|=1} \mathbf{x}^t A_H \mathbf{x} = \min_{\mathbf{x}\in R^n, \|\mathbf{x}\|=1} 2 \sum_{\{u,v\}\in E(H)} x_u x_v. \tag{2}$$

Therefore, since $H_{D,d}$ is d regular, for any $\mathbf{x} \in R^n$, $\|\mathbf{x}\| = 1$ we obtain

$$\lambda_1 + \lambda_n = d + \lambda_n \leq d \sum_{v\in V(H)} x_v^2 + 2 \sum_{\{u,v\}\in E(H)} x_u x_v = \sum_{\{u,v\}\in E(H)} (x_u + x_v)^2. \tag{3}$$

We complete the proof by constructing a particular vector $\mathbf{x}$ for which the last sum is $O(\frac{1}{nD})$. Denote by $y = (y_1, \ldots, y_{2D+1})$ an eigenvector of the cycle C_{2D+1} corresponding to the smallest eigenvalue $2\cos(\frac{2D\pi}{2D+1})$ and satisfying $\sum_i y_i^2 = 1$. Then (1) implies that $2(y_1y_2+y_2y_3+\ldots y_{2D+1}y_1) = 2\cos(\frac{2D\pi}{2D+1})$ and hence

$$\sum_{i=1}^{2D+1} (y_i + y_{i+1})^2 + (y_1 + y_{2D+1})^2 = 2 + 2\cos\left(\frac{2D\pi}{2D+1}\right) \leq \frac{\pi^2}{(2D+1)^2}.$$

Let $\mathbf{x} = (x_v, v \in V(H))$ be the vector defined as follows

$$x_{v_{i,j}} = \frac{y_i}{\sqrt{2d}}, \qquad x_{u_{i,j}} = -\frac{y_i}{\sqrt{2d}} \qquad \forall i, j.$$

Then $\|\mathbf{x}\|^2 = \sum_i 2d(\frac{y_i}{\sqrt{2d}})^2 = \sum_i y_i^2 = 1$. Substituting the vector $\mathbf{x}$ into (3) and noticing that the only edges $\{u,v\} \in E(H)$ for which $x_u + x_v \neq 0$ are the edges of the form $\{v_{i,d}, v_{i+1,1}\}$ or $\{u_{i,d}, u_{i+1,1}\}$ we obtain that

$$\lambda_1 + \lambda_n \leq \sum_{\{u,v\}\in E(H)} (x_u + x_v)^2 = 2\left(\sum_{i=1}^{2D+1} (\frac{y_i + y_{i+1}}{\sqrt{2d}})^2 + (\frac{y_1 + y_{2D+1}}{\sqrt{2d}})^2\right)$$

$$= \frac{1}{d}\left(\sum_{i=1}^{2D+1} (y_i + y_{i+1})^2 + (y_1 + y_{2D+1})^2\right) \leq \frac{\pi^2}{d(2D+1)^2} \leq \frac{\pi^2}{Dn}.$$

This completes the proof. $\square$

3 Max Cut

3.1 The Goemans-Williamson algorithm and its performance

We first describe the algorithm of Goemans and Williamson. For simplicity, we consider the unweighted case. More details appear in [5].

The MAX CUT problem is that of finding a cut $(S, V-S)$ of maximum size in a given input graph $G = (V, E)$, $V = \{1, \ldots, n\}$. By assigning a variable $x_i = +1$ to each vertex i in S and

$x_i = -1$ to each vertex i in $V - S$, it follows that this is equivalent to maximizing the value of $\sum_{\{i,j\} \in E} \frac{1 - x_i x_j}{2}$, over all $x_i \in \{-1, 1\}$. This problem is well known to be NP-hard, but one can relax it to the polynomially-solvable problem of finding the maximum

$$\max_{\|v_i\|^2 = 1} \sum_{\{i,j\} \in E} \frac{1 - v_i^t v_j}{2},$$

where each v_i ranges over all n-dimensional unit vectors. Note that all our vectors are considered as column vectors and hence $v^t u$ is simply the inner product of v and u. This is a semidefinite programming problem which can be solved (up to an exponentially small additive error) in polynomial time. The last expression is a relaxation of the max cut problem, since the vectors $v_i = (x_i, 0, \ldots, 0)$ form a feasible solution of the semidefinite program. Therefore, the optimal value z^* of this program is at least as large as the size of the max cut of G, which we denote by $OPT(G)$.

Given a solution $v_1, \ldots, v_n$ of the semidefinite program, Goemans and Williamson suggested the following rounding procedure. Choose a random unit vector r and define $S = \{i | r^t v_i \leq 0\}$ and $V - S = \{i | r^t v_i > 0\}$. This supplies a cut $(S, V - S)$ of the graph G. Let W denote the size of the random cut produced in this way and let $E[W]$ be its expectation. By linearity of expectation, the expected size is the sum, over all $\{i, j\} \in E$, of the probabilities that the vertices i and j lie in opposite sides of the cut. This last probability is precisely $\arccos(v_i^t v_j)/\pi$. Thus the expected value of the weight of the random cut is exactly

$$\sum_{\{i,j\} \in E} \frac{\arccos(v_i^t v_j)}{\pi}.$$

However the optimal value z^* of the semidefinite program is equal to

$$z^* = \sum_{\{i,j\} \in E} \frac{1 - v_i^t v_j}{2}.$$

Therefore the ratio between $E[W]$ and the optimal value z^* satisfies

$$\frac{E[W]}{z^*} = \frac{\sum_{\{i,j\} \in E} \arccos(v_i^t v_j)/\pi}{\sum_{\{i,j\} \in E} (1 - v_i^t v_j)/2} \geq \min_{\{i,j\} \in E} \frac{\arccos(v_i^t v_j)/\pi}{(1 - v_i^t v_j)/2}.$$

Denote $\alpha = \frac{2}{\pi} \min_{0 < \theta \leq \pi} \frac{\theta}{1 - \cos\theta}$. An easy computation gives that the minimum α is attained at $\theta = 2.3311..$, the nonzero root of $\cos\theta + \theta \sin\theta = 1$, and that $\alpha \in (0.87856, 0.87857)$. Thus, $E[W] \geq \alpha \cdot z^*$, and since the value of z^* is at least as large as the weight OPT of the maximum cut, we conclude that $E[W] \geq \alpha \cdot OPT$. It follows that the Goemans-Williamson algorithm supplies an α-approximation for MAX CUT. Moreover, by the above discussion, the expected size of the cut produced by the algorithm is not better than αOPT in case $OPT = z^*$ and for an optimal solution $v_1, \ldots, v_n$ for the semidefinite programming problem, $\frac{\arccos(v_i^t v_j)/\pi}{(1 - v_i^t v_j)/2} = \alpha$ for all $\{i, j\} \in E$.

In case the value of the semidefinite program is a large fraction of the total number of edges of G, the above reasoning together with a simple convexity argument is used in [5] to show that the performance of the algorithm is better. Put $h(t) = \arccos(1-2t)/\pi$ and let t_0 be the value of t for which $h(t)/t$ attains its minimum in the interval $(0,1]$. Then t_0 is approximately 0.84458. Define $A = z^*/|E|$. If $A \geq t_0$ then, as shown in [5], $E[W] \geq \frac{h(A)}{A} z^* \geq \frac{h(A)}{A} OPT$. Here, as before, the actual expected size of the cut produced by the algorithm is not better than $\frac{h(A)}{A} OPT$ in case $OPT = z^*$ and for an optimal solution $v_1, \ldots, v_n$ for the semidefinite programming problem, $v_i^t v_j = 1-2A$ for all $\{i,j\} \in E$.

Karloff [7] proved that the approximation ratio of the algorithm is exactly α. To do so, he constructed, for any small $\delta > 0$, a graph G and vectors $v_1, \ldots, v_n$ which form an optimal solution of the semidefinite program whose value is $z^*(G)$, such that $OPT(G) = z^*(G)$ and $\frac{\arccos(v_i^t v_j)/\pi}{(1-v_i^t v_j)/2} < \alpha + \delta$ for all $\{i,j\} \in E$. Theorem 1.2 is a generalization of his result, and shows that the analysis of Goemans and Williamson is tight not only for the worst case, but also for graphs in which the size of the maximum cut is a larger fraction of the number of edges. Applying the above analysis to the graph H from Theorem 1.2 together with the vectors w_i as the solution of the semidefinite program we obtain that in this case $A = \frac{1-\eta}{2}$ and the approximation ratio is precisely

$$\frac{E[W]}{z^*(H)} = \frac{E[W]}{OPT(H)} = \min_{\{i,j\} \in E(H)} \frac{\arccos(w_i^t w_j)/\pi}{(1-w_i^t w_j)/2} = \frac{2}{\pi} \frac{\arccos \eta}{1-\eta} = \frac{h(A)}{A}.$$

3.2 The proof of Theorem 1.2

Our construction is based on the properties of graphs arising from the Hamming Association Scheme over the binary alphabet. Let $V = \{v_1, \ldots, v_n\}, n = 2^m$ be the set of all vectors of length m over the alphabet $\{-1,+1\}$. For any two vectors $x, y \in V$ denote by $d(x,y)$ their *Hamming distance*, that is, the number of coordinates in which they differ. The *Hamming graph* $H = H(m,2,b)$ is the graph whose vertex set is V in which two vertices $x, y \in V$ are adjacent if and only if $d(x,y) = b$. Here we consider only even values of b which are greater than $m/2$. We show that for any rational η there exists an appropriate Hamming graph which satisfies the assertion of Theorem 1.2. We may and will assume, whenever this is needed, that m is sufficiently large.

Note that by definition, $H(m,2,b)$ is a Cayley graph of the multiplicative group $Z_2^m = \{-1,+1\}^m$ with respect to the set U of all vectors with exactly b coordinates equal to -1. Therefore (see, e.g., [8], Problem 11.8 and the hint to its solution) the eigenvectors of $H(m,2,b)$ are the multiplicative characters χ_I of Z_2^m, where $\chi_I(x) = \prod_{i \in I} x_i$, I ranges over all subsets of $\{1, \ldots, m\}$. The eigenvalue corresponding to χ_I is $\sum_{x \in U} \chi_I(x)$. The eigenvalues of H are thus equal to the so called *binary*

Krawtchouk polynomials (see [3]),

$$P_b^m(k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{m-k}{b-j}, \; 0 \le k \le m. \tag{4}$$

The eigenvalue $P_b^m(k)$ corresponds to the characters χ_I with $|I| = k$ and thus has multiplicity $\binom{m}{k}$.

Consider any two adjacent vertices of $H(m,2,b)$, v_i and v_j. By the definition of H, the inner product $v_i^t v_j$ is $m-2b$. Choose m and b such that $b > m/2$ is even and $\frac{m-2b}{m} = \eta$. This is always possible since η is a rational number, $-1 < \eta < 0$. Let $w_i = \frac{1}{\sqrt{m}} v_i$ for all i; thus $\|w_i\|^2 = 1$ and $w_i^t w_j = \eta$ for any pair of adjacent vertices. We claim that for such choice of m and b the Hamming graph $H(m,2,b)$ together with the set of vectors $w_i, 1 \le i \le n$, satisfy the assertion of Theorem 1.2. To prove this we first need to establish a connection between the smallest eigenvalue of a graph and the semidefinite relaxation of the max cut problem.

Proposition 3.1 *Let $G = (V,E)$ be a graph on the set $V = \{1,2,\ldots,n\}$ of n vertices, with adjacency matrix $A = (a_{ij})$, and let $\lambda_1 \ge \ldots \ge \lambda_n$ be the eigenvalues of A. Then*

$$\sum_{i<j} a_{ij} \frac{1 - v_i^t v_j}{2} \le \frac{1}{2}|E| - \frac{1}{4}\lambda_n \cdot n,$$

for any $k > 0$ and any set $v_1,\ldots,v_n$ of unit vectors in R^k.

Proof. Let $B = (b_{ij})$ be the $n \times k$ matrix whose rows are the vectors $v_1^t,\ldots,v_n^t$. Denote by $u_1,\ldots,u_k$ the columns of B. By definition we have $\sum_{i=1}^k \|u_i\|^2 = \sum_{ij} b_{ij}^2 = \sum_{i=1}^n \|v_i\|^2 = n$. Therefore

$$\sum_{i<j} a_{ij} \frac{1 - v_i^t v_j}{2} = \frac{1}{2}|E| - \frac{1}{2}\sum_{i<j} a_{ij} v_i^t v_j = \frac{1}{2}|E| - \frac{1}{4}\sum_{i=1}^k u_i^t A u_i$$

By the variational definition of the eigenvalues of A (see equation (2)), for any vector $z \in R^n$, $z^t A z \ge \lambda_n \|z\|^2$ and equality holds if and only if $Az = \lambda_n z$. This implies that

$$\sum_{i<j} a_{ij} \frac{1 - v_i^t v_j}{2} \le \frac{1}{2}|E| - \frac{1}{4}\lambda_n \sum_{i=1}^k \|u_i\|^2 = \frac{1}{2}|E| - \frac{1}{4}\lambda_n \cdot n.$$

Note that in the last expression equality holds if and only if each u_i is an eigenvector of A with eigenvalue λ_n. $\square$

As we show later, the smallest eigenvalue of the adjacency matrix $A_H = (a_{ij})$ of the graph $H(m,2,b)$ is $P_b^m(1)$. By the above discussion it has multiplicity $\binom{m}{1} = m$ and eigenvectors $u_1,\ldots,u_m$ with ± 1 coordinates, where for each vertex $v_j = (v_{j1},\ldots,v_{jm})$, $u_i(v_j) = v_{ji}$. Therefore, the columns of the matrix, whose rows are the vectors w_i, are the eigenvectors $\frac{1}{\sqrt{m}}u_i$ of A_H corresponding to the eigenvalue $P_b^m(1)$. By the last sentence in the proof of Proposition 3.1 it follows that

$$\max_{\|v_l\|^2=1,\forall l}\sum_{i<j} a_{ij}\frac{1-v_i^t v_j}{2} = \frac{1}{2}|E(H)| - \frac{1}{4}P_b^m(1)\cdot n = \sum_{i<j} a_{ij}\frac{1-w_i^t w_j}{2}.$$

On the other hand, u_i is a vector with ± 1 coordinates. Thus the coordinates of u_i correspond to a cut in $H(m,2,b)$ of size equal to

$$\sum_{k<j} a_{kj}\frac{1-u_i(v_k)u_i(v_j)}{2} = \frac{1}{2}|E(H)| - \frac{1}{4}u_i^t A_H u_i = \frac{1}{2}|E(H)| - \frac{1}{4}P_b^m(1)\|u_i\|^2 = \frac{1}{2}|E(H)| - \frac{1}{4}P_b^m(1)\cdot n.$$

Thus the size of a maximum cut in $H(m,2,b)$ is equal to the optimal value of the semidefinite program (see Proposition 3.1). To complete the proof of Theorem 1.2 it remains to prove the following statement.

Proposition 3.2 *Let $P_b^m(k), 0 \le k \le m$ be the binary Krawtchouk polynomials and let b be an even integer satisfying $b = \frac{1-\eta}{2}m$ for some fixed $-1 < \eta < 0$. Then $P_b^m(1) \le P_b^m(k)$ for all $0 \le k \le m, m > m_0(\eta)$.*

Proof. We need the following well known properties of the Krawtchouk polynomials (see, e.g., [3]),

$$(m-k)P_b^m(k+1) = (m-2b)P_b^m(k) - kP_b^m(k-1), \tag{5}$$

$$P_b^m(k) = (-1)^b P_b^m(m-k), \qquad P_b^m(k) = \binom{m}{b}P_k^m(b)\binom{m}{k}^{-1}. \tag{6}$$

By (4) we have that $P_b^m(1) = \frac{m-2b}{m}\binom{m}{b} < 0 < P_b^m(0) = \binom{m}{b}$ and $P_b^m(k) = P_b^m(m-k)$ since b is even. Therefore it is enough to prove the statement of the proposition only for $1 \le k \le m/2$. First assume that k is at most $\frac{1+\eta}{2}m$. Then the equality (5) implies that

$$|P_b^m(k+1)| = \frac{1}{m-k}|(m-2b)P_b^m(k) - kP_b^m(k-1)| \le \frac{2b-m}{m-k}|P_b^m(k)| + \frac{k}{m-k}|P_b^m(k-1)|$$

$$\le \frac{2b-m+k}{m-k}\max(|P_b^m(k)|, |P_b^m(k-1)|).$$

Since b is equal to $\frac{1-\eta}{2}m$ and $k \le \frac{1+\eta}{2}m$, it follows that $\frac{2b-m+k}{m-k} = 2\frac{b}{m-k} - 1 \le 1$. Therefore, arguing by induction on k we obtain that $|P_b^m(k+1)| \le \max(|P_b^m(1)|, |P_b^m(2)|)$. By calculation from (4), $|P_b^m(2)| = |\frac{(m-2b)^2-m}{m(m-1)}\binom{m}{b}| < |P_b^m(1)|$. This proves that $|P_b^m(1)| \ge |P_b^m(k)|$ for all $k \le \frac{1+\eta}{2}m$.

From now on, till the end of the proof, $c_1, c_2, c_3, \ldots$ always denote positive constants depending only on η. Whenever needed we use the assumption that m is sufficiently large as a function of η.

To complete the proof we show, next, that for $\frac{1+\eta}{2}m \leq k \leq m/2$ the value of $|P_b^m(k)|$ is at most $c_1 m^{-1/3}\binom{m}{b}$. This would imply that $|P_b^m(1)| = \frac{2b-m}{m}\binom{m}{b} \geq |P_b^m(k)|$, since by our assumptions about b, $\frac{2b-m}{m} = -\eta$ is a constant, bounded away from zero. By the equality (6)

$$\frac{P_b^m(k)}{\binom{m}{b}} = \frac{P_k^m(b)}{\binom{m}{k}} = \sum_{j=0}^{b}(-1)^j \binom{b}{j}\binom{m-b}{k-j}\binom{m}{k}^{-1} = S_1 + S_2,$$

where

$$S_1 = \sum_{r\leq j\leq q}(-1)^j \binom{b}{j}\binom{m-b}{k-j}\binom{m}{k}^{-1},$$

S_2 contains all the remaining summands, and $r = (b/m - m^{-1/3})k + c_2$ as well as $q = (b/m + m^{-1/3})k + c_3$ are chosen so that S_1 contains an even number of terms. Note that S_1 is a sum of at most $c_4 m^{2/3}$ summands $t_j = (-1)^j\binom{m}{k}^{-1}(\binom{b}{j}\binom{m-b}{k-j} - \binom{b}{j+1}\binom{m-b}{k-j-1})$, where j runs over all integers equal to r modulo 2 in an appropriate interval. Therefore to bound the absolute value $|S_1|$ of S_1 it is enough to bound $|t_j|$ for $r \leq j \leq q$. A simple calculation shows that

$$|t_j| = \binom{m}{k}^{-1}\binom{b}{j+1}\binom{m-b}{k-j-1}\left|\frac{(jm-bk)+(m-b-k+2j+1)}{(b-j)(k-j)}\right|.$$

From the assumption about j we have that $jm - bk \leq c_5 m^{2/3}k \leq c_5 m^{5/3}$ and that $b - j > k - j \geq k(1 - b/m - m^{-1/3}) - c_6 \geq k/c_7 \geq m/c_8$. Thus $|t_j| \leq c_9 m^{-1/3}\binom{m}{k}^{-1}\binom{b}{j+1}\binom{m-b}{k-j-1}$. Hence we obtain the following inequality,

$$|S_1| \leq \sum_{j=r}^{q}|t_j| \leq c_9 m^{-\frac{1}{3}}\binom{m}{k}^{-1}\sum_{j=-\infty}^{+\infty}\binom{b}{j+1}\binom{m-b}{k-j-1} = c_9 m^{-\frac{1}{3}},$$

where here we used the fact that $\sum_{j=-\infty}^{+\infty}\binom{b}{j+1}\binom{m-b}{k-j-1} = \binom{m}{k}$. Next we obtain an upper bound on $|S_2|$. Let $t = 2m^{-1/3}$ and $p = b/m$; by definition

$$|S_2| \leq \sum_{j=1}^{(p-t)k}\binom{m}{k}^{-1}\binom{b}{j}\binom{m-b}{k-j} + \sum_{j=(p+t)k}^{k}\binom{m}{k}^{-1}\binom{b}{j}\binom{m-b}{k-j}.$$

Note that the right hand side in the last inequality is exactly the probability that a hypergeometric distribution with parameters (m, b, k) deviates by tk from its expectation. By the result of [1], the probability of this event is bounded by $2e^{-2t^2k} \leq 2e^{-m^{1/3}/c_{10}}$. This implies that $|S_2| \leq c_{11}m^{-1/3}$. Therefore

$$\left|\frac{P_b^m(k)}{\binom{n}{b}}\right| \leq |S_1| + |S_2| \leq c_1 m^{-\frac{1}{3}} \leq \frac{2b-m}{m} = \left|\frac{P_b^m(1)}{\binom{m}{b}}\right|.$$

It follows that $|P_b^m(1)| \geq |P_b^m(k)|$ for all $1 \leq k \leq m/2$. Since the value of $P_b^m(1)$ is negative, $P_b^m(1) = -|P_b^m(1)| \leq -|P_b^m(k)| \leq P_b^m(k)$. This completes the proof of the proposition. □

4 Concluding remarks

1. The examples described in Section 2 can be easily extended to similar examples with other values of the degree d, the diameter D and the number of vertices n, by replacing the complete bipartite graphs substituted in these examples with other regular bipartite graphs.

2. In [8], Problem 11.29 it is shown that the difference between the first and second largest eigenvalues of d-regular graphs with n vertices and diameter D is always bigger than $\frac{1}{Dn}$. This is essentially tight, as the examples in Section 2 also show that there are d-regular graphs with n vertices and diameter D for which the difference between the largest and second largest eigenvalues is $O(\frac{1}{Dn})$. Indeed, consider the graph $H = H_{D,d}$. Denote by $y = (y_1, \ldots, y_{2D+1})$ an eigenvector of the cycle C_{2D+1} corresponding to the eigenvalue $2\cos(\frac{2\pi}{2D+1})$ and satisfying $\sum_i y_i^2 = 1$. Let $\mathbf{x} = (x_v, v \in V(H))$ be the vector such that $x_{v_{i,j}} = x_{u_{i,j}} = \frac{y_i}{\sqrt{2d}}$. By definition x satisfies $\sum_{v \in V(H)} x_v = \sqrt{2d} \sum_i y_i = 0, \|x\|^2 = 1$, and a computation similar to the one in Section 2 shows that the second eigenvalue of H satisfies $\lambda_2(H) \geq x^t A_H x \geq d - O(\frac{1}{Dn})$. This implies that $\lambda_1(H) - \lambda_2(H) = d - \lambda_2(H) = O(\frac{1}{Dn})$. It is worth noting that the hidden constant in the $O(\frac{1}{Dn})$ bound above can be improved by a factor of two in several ways, for example, by replacing the complete bipartite graphs with complete graphs in the construction of the graph H.

3. Let $a_{ij}, 1 \leq i < j \leq n$, and b be reals. We call a constraint
$$\sum_{i<j} a_{ij}(v_i^t v_j) \geq b$$
valid if it is satisfied whenever each v_i is an integer in $\{-1, 1\}$. Feige, Goemans and Williamson (see [4],[5]) proposed adding to the semidefinite program a family of valid constrains, in the hope of narrowing the gap between the optimal value of the semidefinite program and the weight of the max cut. It is easy to see that, as observed in [7], since the vectors $w_1, \ldots, w_n$ from Section 3 have all their coordinates equal to $\pm 1/\sqrt{m}$ they satisfy any valid constraint. Therefore the proof of Theorem 1.2 shows that the addition of any family of valid constraints cannot improve the performance ratio of the Goemans-Williamson algorithm even for graphs containing large cuts.

4. Very recently we determined, together with U. Zwick, the precise approximation guarantee of the Goemans Williamson algorithm for graphs $G = (V, E)$ in which the size of the max cut is at least $A|E|$ for all values of A ($\geq 1/2$). It turns out that this approximation guarantee is $h(t_0)/t_0$ for all A between $1/2$ and t_0, where t_0 is as in Section 1 (note that for A close to $1/2$

this gives a cut of size less than half the number of edges!). The examples demonstrating this fact are based on the ones constructed here, but require several additional ideas. The details will appear somewhere else.

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