The limit points of the top and bottom eigenvalues of regular graphs

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Abstract

We prove that for each $d \geq 3$ the set of all limit points of the second largest eigenvalue of growing sequences of d-regular graphs is $[2\sqrt{d-1}, d]$. A similar argument shows that the set of all limit points of the smallest eigenvalue of growing sequences of d-regular graphs with growing (odd) girth is $[-d, -2\sqrt{d-1}]$. The more general question of identifying all vectors which are limit points of the vectors of the top k eigenvalues of sequences of d-regular graphs is considered as well. As a by product, in the study of discrete counterpart of the "scarring" phenomenon observed in the investigation of quantum ergodicity on manifolds, our technique provides a method to construct d-regular almost Ramanujan graphs with large girth and localized eigenvectors corresponding to eigenvalues larger than $2\sqrt{d-1}$, extending a result of Alon, Ganguly, and Srivastava [3].

1 Introduction

For a graph G on n vertices, let $\lambda_1(G) \geq \lambda_2(G) \ldots \geq \lambda_n(G)$ denote the ordered set of its adjacency matrix eigenvalues. If G is d-regular then $\lambda_1(G) = d$ and the Alon-Boppana bound ([1], [31], [32], see also [12]) asserts that $\lambda_2(G) \geq 2\sqrt{d-1}(1 - O(1/\log^2 n)) = (1 - o(1))2\sqrt{d-1}$. Therefore, any limit point of the values of $\lambda_2(G_i)$ for an infinite sequence G_i of d-regular graphs is at least $2\sqrt{d-1}$, and the existence of near-Ramanujan graphs for every degree d proved in [15] (see also [7], [26]) implies that $2\sqrt{d-1}$ is such a limit point. Are all other points in the interval $[2\sqrt{d-1}, d]$ also obtained as such limit

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points? This question was suggested to us by Peter Sarnak. It can be viewed as a variation of the inverse spectral problem, whose analogue for hyperbolic surfaces is studied in [23]. This analogue for all hyperbolic surfaces (with genus tending to infinity) is a consequence of the recent work of Hide and Magee [18].

Our first result in this paper is a short proof that this is indeed the case.

Theorem 1.1. *Let $A_2(d)$ denote the set of all limit points of sequences $\lambda_2(G_i)$ where G_i is an infinite sequence of d-regular graphs, then for every $d \geq 3$, $A_2(d) = [2\sqrt{d-1}, d]$.*

Note that trivially not every value in $A_2(d)$ can be achieved as $\lambda_2(G)$ for some finite d-regular graph G, as such a value needs to be a totally real algebraic integer (and also as the number of finite graphs is only countable). Indeed, most values in $A_2(d)$ can only be achieved as limit points of sequences $\lambda_2(G_i)$ as in the theorem. Note also that since the set $A_2(d)$ is an interval, the statement of the theorem is equivalent to the assertion that each point λ in this interval is obtained as a limit point of $\lambda_2(G_i)$ for a sequence of graphs G_i, where $\lambda_2(G_i) \neq \lambda$ for each i.

Our method implies a similar result for the set of all limit points of the smallest eigenvalue of growing sequences of d-regular graphs, provided the (odd) girth of these graphs tends to infinity.

Theorem 1.2. *Let $A_s(d)$ denote the set of all limit points of sequences of the last (smallest) eigenvalue of growing sequences of d-regular graphs in which the length of the shortest odd cycle tends to infinity. For every $d \geq 2$, $A_s(d) = [-d, -2\sqrt{d-1}]$.*

Without the assumption about the growing odd girth the set of limit points of the smallest eigenvalue is more complicated, contains isolated points, and is far from being fully understood. See [34] for the values of the first few largest points of this set.

We conjecture that the assertion of Theorem 1.1 can be extended to determine the limit points of the vectors in $\mathbb{R}^k$ of the top k eigenvalues, for any fixed k.

Conjecture 1.3. *For any $d \geq 3$ and any fixed k, the set of all limit points of the vectors*

$$(\lambda_1(G_i), \lambda_2(G_i), \ldots, \lambda_k(G_i))$$

for an infinite sequence G_i of d-regular graphs is exactly the set

$$B(d,k) = \{(\mu_1, \mu_2, \ldots, \mu_k) : \; d = \mu_1 \geq \mu_2 \geq \mu_3 \geq \ldots \geq \mu_k \geq 2\sqrt{d-1}\}.$$

The fact that the above set of limit points is contained in $B(d,k)$ follows from the known result, observed by several researchers, c.f., e.g., [11, 32], that for any fixed d

and k, any sufficiently large d-regular graph has at least k eigenvalues which are at least $2\sqrt{d-1}-o(1)$. However, we have not been able to decide whether or not the set of these limit points contains every point of $B(d,k)$ even for $k=3$. On the other hand, we can prove the following, showing that every point of $B(d,k)$ can be obtained as a limit if we relax the regularity condition.

Proposition 1.4. *For every $d \geq 3$ and every k, every point of $B(d,k)$ is a limit point of a sequence of vectors $(\lambda_1(G_i), \lambda_2(G_i), \ldots, \lambda_k(G_i))$ for an infinite sequence G_i of graphs with maximum degree at most d.*

Note that, of course, for sequences of graphs with maximum degree d there are also limit points as above that lie outside the set $B(d,k)$, so this result should be viewed mostly as a warmup for the next one that deals with regular graphs.

For d-regular graphs we can prove that Conjecture 1.3 is almost true, in the sense that every point of $B(d,k) \cap [2\sqrt{d-1}+o_d(1), d]^k$ can be obtained as a limit point. To be more precise, we have the following theorem.

Theorem 1.5. *For every $d \geq 3$ and every k, every point of*

$$\{(\mu_1, \mu_2, \ldots, \mu_k) :\ d = \mu_1 \geq \mu_2 \geq \mu_3 \geq \ldots \geq \mu_k \geq 2\sqrt{d-1} + \frac{1}{\sqrt{d-1}}\}.$$

is a limit point of a sequence of vectors $(\lambda_1(G_i), \lambda_2(G_i), \ldots, \lambda_k(G_i))$ for an infinite sequence G_i of d-regular graphs.

A byproduct of the technique is the following theorem, which constructs d-regular almost Ramanujan graphs G with large girth, while ensuring the presence of a localized eigenvector corresponding to an eigenvalue strictly greater than $2\sqrt{d-1}$. (An eigenvector is localized if a significant portion of its ℓ_2-mass is concentrated in a small number of vertices.) This in some way strengthens a result of Alon, Ganguly and Srivastava [3], who showed the existence of large girth d-regular graphs G with $\lambda_2(G) \leq 2.121\sqrt{d}$ and localized eigenvectors with eigenvalues in $(-2\sqrt{d}, 2\sqrt{d})$. Note, however that the localized eigenvectors in [3] are completely supported on a small set, unlike the eigenvectors constructed here, and that the construction here applies only to eigenvectors corresponding to eigenvectors that exceed $2\sqrt{d-1}$. Earlier research on constructing d-regular graphs with localized eigenvectors can be found in [17], but the graphs produced there are not expanding.

This line of research involving constructions of high girth (and expanding) graphs that exhibit localized eigenvectors can be viewed as a discrete version of the "scarring" phenomenon observed in the study of quantum ergodicity on manifolds. More about this topic can be found in [9], [6], [10], [17], and the references therein.

Theorem 1.6. *For any real numbers $\beta > 0$ and $C \geq 4$, and any positive integer $d = p+1$ where $p \equiv 1 \mod 4$ is a prime, there are infinitely many d-regular graphs G satisfying the following properties simultaneously:*

1. *The second largest eigenvalue is at most $2\sqrt{d-1} + \beta$,*
2. *The girth is at least $\frac{1}{5C} \log_d |V(G)|$, and*
3. *There is a vertex set S with $|S| \leq |V(G)|^{2/\sqrt{C}}$ and an eigenvalue strictly larger than $2\sqrt{d-1}$ whose eigenvector $\mathbf{v}$ satisfies $\sum_{u \in S} \mathbf{v}(u)^2 \geq (1-\beta)\|\mathbf{v}\|_2^2$.*

Our construction of these graphs is explicit. Note that we do not make any serious attempt to optimize the constants in the bounds for the girth and for the exponent of $|V(G)|$ in the bound for $|S|$. With a slightly more careful analysis, it is possible to prove $\sum_{u \in S} \mathbf{v}(u)^2 \geq (1 - o(1))\|\mathbf{v}\|_2^2$ where the $o(1)$ term here tends to 0 as $|V(G)|$ tends to infinity (where β, C, d are fixed). It is also worth noting that a similar result also holds for any $d \geq 3$, with the bound on the girth being $\Omega(\log\log|V(G)|)$. The construction can again be made explicit by a deterministic poly($|V(G)|$)-time algorithm, combining our arguments with the construction in [30].

The rest of this paper is organized as follows. In the next section we describe the proof of the basic results: Theorem 1.1 and Theorem 1.2. Proposition 1.4 and several extensions are proved in Section 3. In Section 4 we describe the proof of Theorem 1.5. Near Ramanujan graphs with localized eigenvectors are constructed in Section 5 and the final Section 6 contains some concluding remarks.

2 Proofs of the basic results

2.1 The second eigenvalue

We start with the proof of Theorem 1.1 in the following stronger form.

Theorem 2.1. *For every $d \geq 2$, every even integer $n > d$ and every real $\lambda \in [2\sqrt{d-1}, d]$ there is a d-regular graph G on n vertices satisfying $|\lambda_2(G) - \lambda| \leq \frac{c}{\log\log n}$, where $c = c(d) > 0$.*

Proof. The proof is based on the fact that eigenvectors of high-girth d-regular graphs are nonlocalized. It is worth noting that stronger delocalization results are known for random

d-regular graphs (see [20]), but as our proof has to maintain this delocalization during the process described next it is better to rely on high girth, which can be maintained during this process. The delocalization enables us to start from a near Ramanujan d-regular graph of high girth and apply to it local changes (swaps), transforming it to a graph with a very small bisection width which has second eigenvalue close to d. This is done while maintaining the high girth. The nonlocalized nature of the eigenvectors is used to show that in each swap the second eigenvalue can change only by a small amount. A more precise description follows.

We prove that there are constants $c_1, c_2, c_3, c_4 > 0$ (depending on d) and a (finite) sequence of d-regular graphs $G_0, G_1, \ldots, G_t$, each being a graph on the set $V = \{v_1, v_2, \ldots, v_n\}$ of n labeled vertices, that satisfy the following properties.

1. $\lambda_2(G_0) \leq 2\sqrt{d-1} + c_1(\frac{\log\log n}{\log n})^2$.
2. $\lambda_2(G_t) \geq d - \frac{c_2}{\sqrt{n}}$.
3. The girth of each G_i is at least $c_3 \log\log n$.
4. For every $0 \leq i < t$, if either $\lambda_2(G_i)$ or $\lambda_2(G_{i+1})$ exceeds $2\sqrt{d-1}$ then
$$|\lambda_2(G_{i+1}) - \lambda_2(G_i)| \leq \frac{c_4}{\log\log n}.$$

This clearly implies the assertion of Theorem 2.1. The existence of G_0 satisfying properties (1) and (3) follows from the result of Friedman [15] and the known facts about the girth of random regular graphs. Indeed, Friedman proved that for every $0 < a < 1$ there is some $c_1 = c_1(a,d)$ so that with probability at least $1 - n^{-a}$ a random d-regular graph G_0 on n vertices satisfies (1). By the known results about the distribution of short cycles in random d-regular graphs (see [29]) the girth of G_0 exceeds some $c_3 \log\log n$ with probability exceeding $1/\sqrt{n}$. Taking $a = 1/2$ in Friedman's result it follows that with positive probability G_0 satisfies (1) and (3). Fix such a graph G_0.

For every $i \geq 0$, the graph G_{i+1} will be constructed from G_i by a single *swap*, that is, by deleting two vertex disjoint edges v_1v_2 and u_1u_2 and by adding the edges v_1u_1 and v_2u_2 instead, keeping the graph d-regular. We need the following result.

Claim 2.2. *Let $G = (V, E)$ be a graph with maximum degree d and with girth at least $2r$, and let x be an eigenvector of G with ℓ_2 norm $\|x\|_2 = 1$ corresponding to an eigenvalue μ with absolute value at least $2\sqrt{d-1}$. Then $\|x\|_\infty \leq 1/\sqrt{r}$.*

This is an immediate consequence of Lemma 3.2 of [2] where it is shown that if uv is any edge in such a graph, and N_i is the set of all vertices of distance exactly i from $N_0 = \{u, v\}$, $(0 \le i \le r-1)$, then for every $0 < i \le r-1$, $\sum_{v \in N_i} x_v^2 \ge \sum_{u \in N_{i-1}} x_u^2$. Thus $x_u^2 + x_v^2 \le 1/r$. A slight extension of this lemma for bounded degree (not necessarily regular) graphs is proved as Lemma 3.4 below. Note that a similar statement for regular graphs with a slightly worse quantitative estimate can be derived from the results in [17], even without any assumption on the eigenvalue μ. There are, however, simple examples showing that this more general statement does not hold for general bounded degree graphs.

Corollary 2.3. *If G and H are two d-regular graphs, each having girth at least $2r$, and one of them is obtained from the other by a swap, and if either $\lambda_2(G)$ or $\lambda_2(H)$ exceeds $2\sqrt{d-1}$, then $|\lambda_2(G) - \lambda_2(H)| \le 8/r$.*

Indeed, without loss of generality $\lambda_2(G) \ge \lambda_2(H)$ and $\lambda_2(G) \ge 2\sqrt{d-1}$. Let $\mathbf{v}$ be a normalized eigenvector of $\lambda_2(G)$. Then it is orthogonal to the constant vector, and $\mathbf{v}^T A_G \mathbf{v} = \lambda_2(G)$, where A_G is the adjacency matrix of G. Since H is obtained from G by a single swap, $\mathbf{v}^T A_G \mathbf{v} - \mathbf{v}^T A_H \mathbf{v}$ is a sum and difference of at most 8 terms of the form $\mathbf{v}(u)\mathbf{v}(v)$. Here A_H is the adjacency matrix of H. Since $\|\mathbf{v}\|_\infty \le 1/\sqrt{r}$ each such term has absolute value at most $1/r$. It follows that $\mathbf{v}^T A_H \mathbf{v} \ge \lambda_2(G) - 8/r$ and by the variational definition of $\lambda_2(H)$ this implies that $\lambda_2(H) \ge \lambda_2(G) - 8/r$, as needed.

Starting with G_0 satisfying (1) and (3) split its vertex set arbitrarily into two sets of equal cardinality B and C. Suppose we have already defined $G_0, G_1, \ldots, G_i$ so that (3) and (4) hold, and suppose that G_i still has more than $\sqrt{n}$ crossing edges, that is, edges with endpoints in B and in C. We show how to define G_{i+1} and decrease the number of these crossing edges by 2. Let v_1v_2 be an arbitrary crossing edge of G_i, with $v_1 \in B, v_2 \in C$. The number of edges of G_i whose distance from the edge v_1v_2 is at most $2r-1$ is at most $1 + 2(d-1) + 2(d-1)^2 + \cdots + 2(d-1)^{2r-1} < 2d^{2r}$. If r is smaller than $(1/4)\log(n/4)/\log d$ this number is smaller than $\sqrt{n}$, and hence there is at least one additional crossing edge u_1u_2 of G_i with $u_1 \in B, u_2 \in C$. Let G_{i+1} be the graph obtained from G_i by the swap that removes the edges v_1v_2, u_1u_2 and adds the edges u_1v_1, u_2v_2. Since any cycle of G_{i+1} that is not a cycle of G_i must contain at least one path in $G_i - \{v_1v_2, u_1u_2\}$ that connects two distinct vertices among these four, its length is at least the minimum between $1 + (2r-1)$ and the girth of G_i (in fact, twice this girth plus 2, but this is not crucial here). Therefore G_{i+1} also satisfies (3). By Corollary 2.3 condition (4) also holds for i. Since each graph G_{i+1} in the process has less crossing edges than the previous graph G_i the process must terminate with a graph G_t in which the number of crossing edges is at most $\sqrt{n}$. Let $\mathbf{v}'$

be the vector assigning $1/\sqrt{n}$ to each vertex of B and $-1/\sqrt{n}$ to each vertex of C. Then $\mathbf{v}'^T A_{G_t} \mathbf{v}' \geq (dn - 4\sqrt{n})/n = d - 4/\sqrt{n}$. Since $\|\mathbf{v}'\|_2^2 = 1$ and its sum of coordinates in 0, this implies that $\lambda_2(G_t) \geq d - 4/\sqrt{n}$, showing that condition (2) holds. This completes the proof. □

2.2 The smallest eigenvalue

Proof of Theorem 1.2. The proof is very similar to that of Theorem 1.1, we thus only include a brief description. The fact that $A_s(d)$ is contained in the closed interval $[-d, -2\sqrt{d-1}]$ follows from the result of Li [24] (see also [32]) that asserts that the smallest eigenvalue of any d-regular graph in which the length of the shortest odd cycle is r, is at most $-2\sqrt{d-1}(1 - O(1/r^2))$. In order to show that every point of this interval indeed lies in $A_s(d)$ we construct, for every even integer n, a sequence $G_0, G_1, \ldots, G_t$ of graphs on a set V of n vertices that satisfy the following, where c_1, c_2, c_3, c_4 are positive constants depending only on d.

1. $\lambda_n(G_0) \geq -2\sqrt{d-1} - c_1(\frac{\log \log n}{\log n})^2$.
2. $\lambda_n(G_t) \leq -d + \frac{c_2}{\sqrt{n}}$.
3. The girth of each G_i is at least $c_3 \log \log n$.
4. For every $0 \leq i < t$, if the absolute value of either $\lambda_n(G_i)$ or $\lambda_n(G_{i+1})$ exceeds $2\sqrt{d-1}$ then $|\lambda_n(G_{i+1}) - \lambda_n(G_i)| \leq \frac{c_4}{\log \log n}$.

This clearly suffices to complete the proof of the theorem. As in the previous proof, the existence of G_0 satisfying properties (1) and (3) follows from the results of [15] and [29]. Assuming we have already constructed $G_0, G_1, \ldots G_i$ satisfying (3) and (4), we obtain G_{i+1} from G_i by a single swap. Split the set of vertices of G_0 into two sets B, C, each of size $n/2$. As long as the number of edges in the induced subgraph of G_i on B is at least, say, $\sqrt{n}$, so is the the number of edges in the induced subgraph of G_i on C. Indeed, these two numbers are equal since the sum of degrees in the induced subgraph on B is $|B|d - e(B, C)$ where $e(B, C)$ is the number of edges connecting B and C, and the sum of degrees in the induced subgraph on C is $|C|d - e(B, C)$. As in the previous proof, there are two edges v_1v_2 and u_1u_2 of G_i, where $v_1, v_2 \in B$, $u_1, u_2 \in C$ and the distance in G_i between these two edges is at least $\Omega(\log n/ \log d)$. Swapping these edges and replacing them by the two crossing edges v_1u_1 and v_2u_2 we obtain a graph G_{i+1} with girth satisfying condition (3). As the assertion of Corollary 2.3 clearly holds for the smallest eigenvalues too, condition (4) also holds for i. Since each graph G_{i+1} in the process has 2 more crossing edges than the previous graph G_i, the process must terminate with a graph G_t in which the number of non-crossing edges is at most $2\sqrt{n}$. As in the previous proof, let $\mathbf{v}'$ be the vector assigning $1/\sqrt{n}$ to each vertex of B and $-1/\sqrt{n}$ to each vertex of C. Then $\mathbf{v}'^T A_{G_t}\mathbf{v}' \leq (-dn + 8\sqrt{n})/n = -d + 8/\sqrt{n}$. Since $\|\mathbf{v}'\|_2^2 = 1$, this implies that $\lambda_n(G_t) \leq -d + 8/\sqrt{n}$, showing that condition (2) holds. This completes the proof of the theorem. □

3 The top eigenvalues of bounded degree graphs

3.1 Bounded degree graphs

In this subsection we prove Proposition 1.4. To do so we establish the following lemma.

Lemma 3.1. *For every $d \geq 2$ and every even integer n, and for every real $\lambda \in [2\sqrt{d-1}, d]$ there is a graph $G = G(n, \lambda)$ with maximum degree at most d, whose number of vertices is between $\sqrt{n}$ and n, satisfying*

1. $|\lambda_1(G) - \lambda| \leq 2\frac{d \log n}{\sqrt{n}}$
2. $\lambda_2(G) \leq 2\sqrt{d-1}$.

Proof. To simplify the presentation we omit all floor and ceiling signs whenever these are not crucial. We show that there is a sequence of graphs $G_0, G_1, \ldots, G_t$, where $t = n - \sqrt{n}$, G_i has exactly $n - i$ vertices, so that

1. $\lambda_1(G_0) = d$
2. $\lambda_1(G_t) \leq 2\sqrt{d-1} + \frac{d}{\sqrt{n}}$
3. For every $0 \leq i \leq t$, $\lambda_2(G_i) \leq 2\sqrt{d-1}$.
4. For every $0 \leq i < t$:

$$\lambda_1(G_i)\left(1 - 3\frac{\log(n-i)}{n-i}\right) \leq \lambda_1(G_{i+1}) \leq \lambda_1(G_i).$$

This clearly implies the assertion of the lemma. A d-regular graph G_0 on n vertices satisfying (1) and (3) exists by the result of Marcus, Spielman and Srivastava in [27]. Assuming we have already defined $G_0, G_1, \ldots, G_i$ satisfying (3) and (4), where G_j has $n - j$ vertices for all $j \leq i$ and where $i + 1 < t$, we define G_{i+1} as follows. Let ℓ be the largest even integer that does not exceed $(n-i)/2$ and consider all closed walks of length ℓ in G_i. By averaging, there is at least one vertex of G_i contained in at most half of these walks. Choose arbitrarily such a vertex v and let G_{i+1} be the graph obtained from G_i by removing the vertex v. Thus G_{i+1} has $n-i-1$ vertices. As it is an induced subgraph of G_i it satisfies condition (3) by eigenvalue interlacing (see, e.g., [14]). The same eigenvalue interlacing implies that $\lambda_1(G_{i+1}) \leq \lambda_1(G_i)$. In order to prove the other inequality in condition (4) we use the following simple fact.

Fact: Let H be a graph with q vertices, and let ℓ any even positive integer. Let $T = T(\ell)$ be the number of closed walks of length ℓ in H. Then $\sum_{i=1}^{q} \lambda_i^{\ell}(H) = T$ and hence $(T/q)^{1/\ell} \leq \lambda_1(H) \leq T^{1/\ell}$.

By the above fact applied to G_i, with ℓ being the largest even integer that does not exceed $(n-i)/2$ and T being the number of closed walks of length ℓ in G_i, it follows that $\lambda_1(G_i) \leq T^{1/\ell}$. Applying the fact to G_{i+1} with the same ℓ, and using the fact that the number of closed walks of length ℓ in it is at least $T/2$, we conclude that

$$\lambda_1(G_{i+1}) \geq \left(\frac{T}{2(n-i-1)}\right)^{1/\ell} \geq T^{1/\ell} \cdot \left(1 - \frac{3\log(n-i)}{n-i}\right) \geq \lambda_1(G_i) \cdot \left(1 - \frac{3\log(n-i)}{n-i}\right)$$

where here we used the fact that $n-i$ is large and that ℓ is close to half of it. This shows that condition (4) is maintained with G_{i+1}.

It remains to prove that condition (2) holds. This follows from the argument in the first part of the proof of Lemma 9.2.7 in [4]. For completeness we sketch the proof. Let f be an eigenvector corresponding to the largest eigenvalue of G_t. Let g be a vector defined on the vertex set of G_0, by letting $g(v) = f(v)$ for every vertex $v \in V(G_t)$ and by defining $g(u) = 0$ for all other vertices of G_0. Expressing this vector g as a linear combination of the all 1-vector (which is the top eigenvector of G_0) and an orthogonal vector h, and estimating $\lambda_1(G_t) = g^T A_{G_0} g$ using this expression and noting that $g^T 1 \leq \|g\|_2 n^{1/4}$ and $h^T A_{G_0} h \leq \lambda_2(G_0)\|h\|_2^2$, we get the required estimate in condition (2). This completes the proof of the lemma. $\square$

Proof of Proposition 1.4. Let $(\mu_1, \mu_2, \ldots, \mu_k)$ be a vector satisfying $d \geq \mu_1 \geq \mu_2 \ldots \geq \mu_k \geq 2\sqrt{d-1}$. By Lemma 3.1, for every i, $1 \leq i \leq k$ and for every even integer n there is a graph $G_j = G(n, \mu_j)$ with maximum degree at most d, whose number of vertices is between $\sqrt{n}$ and n, satisfying $|\lambda_1(G_j) - \mu_j| \leq 2\frac{d\log n}{\sqrt{n}}$ and $\lambda_2(G_j) \leq 2\sqrt{d-1}$. Let $G(n)$ be the vertex disjoint union of the graphs $G_1, G_2, \ldots, G_k$. Then $|\lambda_i(G(n)) - \mu_i| \leq \frac{2d\log n}{\sqrt{n}}$ for all $1 \leq i \leq k$. Any sequence of such graphs $G(n)$ for a growing sequence of values of n gives the required limit point $(\mu_1, \mu_2, \ldots, \mu_k)$. Note that if $\mu_1 = d$ then the maximum degree of each graph $G(n)$ is exactly d. $\square$

The graphs constructed in the proof of Proposition 1.4 are not connected. We can show, however, that the same set $B(k,d)$ of vectors is obtained by similar limits of the corresponding vectors of top eigenvalues of *connected* graphs with maximum degree at most d. This requires some additional ideas. The details follow.

Theorem 3.2. *For every $d \geq 2$ and every k, every point of $B(d,k)$ is a limit point of a sequence of vectors $(\lambda_1(G_i), \lambda_2(G_i), \ldots, \lambda_k(G_i))$ for an infinite sequence G_i of* connected *graphs G_i with maximum degree at most d.*

To establish this theorem we first prove the following variant of Lemma 3.1.

Lemma 3.3. *There are positive constants $c_1 = c_1(d), c_2 = c_2(d), c_3 = c_3(d)$ so that the following holds. For every $d \geq 2$ and every even integer n, and for every real $\lambda \in [2\sqrt{d-1}, d]$ there is a graph $G = G(n, \lambda)$ satisfying the following.*

1. *G is connected, it has at least $n/\log n$ and at most $n-1$ vertices, its girth is at least $c_1 \log\log n$, its maximum degree is at most d and it has at least 2 vertices of degree strictly smaller than d.*
2. *$|\lambda_1(G) - \lambda| \leq \frac{c_2}{\log n}$*
3. *$\lambda_2(G) \leq 2\sqrt{d-1} + c_3 \left(\frac{\log\log n}{\log n}\right)^2$.*

Proof. As in the proof of Lemma 3.1 we construct a sequence of graphs $G_0, G_1, \ldots, G_t$, where $t = n - n/\log n$, and G_i has maximum degree at most d and exactly $n-i$ vertices, so that

1. $\lambda_1(G_0) = d$
2. $\lambda_1(G_t) \leq 2\sqrt{d-1} + \frac{c_2}{\log n}$
3. For every $0 \leq i \leq t$, $\lambda_2(G_i) \leq 2\sqrt{d-1} + c_3(\frac{\log\log n}{\log n})^2$.
4. For every $0 \leq i < t$:
$$\lambda_1(G_i)\left(1 - 3\frac{\log^5 n}{n}\right) \leq \lambda_1(G_{i+1}) \leq \lambda_1(G_i).$$
5. Each G_i is connected, and has girth at least $c_1 \log\log n$. For $i \geq 1$ each G_i has at least two vertices of degree strictly smaller than d.

This easily implies the assertion of the lemma. A d-regular graph G_0 on n vertices satisfying the requirements in conditions (1), (3) and (5) exists by the work of Friedman [15] and the results in [29], as explained in the beginning of the proof of Theorem 2.1. All the other graphs G_i will be induced subgraphs of G_0, where each G_{i+1} is obtained from G_i by deleting a carefully chosen vertex. Note that trivially all graphs G_i will satisfy the girth condition in (5). Moreover, as the initial graph G_0 is a d-regular strong expander, it contains no cutpoints, and hence any nontrivial connected induced subgraph of it contains at least 2 vertices of degree strictly smaller than d. Since G_0 is a strong expander, its diameter is at most $D = O(\log n)$. Any BFS tree in it starting from an arbitrarily chosen root has at most $D+1$ levels. Fix such a tree T_0 in G_0. Assuming G_j and a spanning tree of it T_j of diameter at most $D+1$ have been defined already for all $j \leq i$, define G_{i+1} as follows. Let ℓ be the largest even number which does not exceed, say, $n/\log^4 n$. Let v be a leaf of T_i contained in the smallest number of closed walks of length ℓ in G_i. Define $G_{i+1} = G_i - \{v\}$, and $T_{i+1} = T_i - \{v\}$. Note that any spanning tree in a graph of m vertices which has at most $D+1$ levels has at least $m/(D+1)$ leaves, since all its vertices can be covered by all the root to leaf paths, and each such path contains at most $D+1$ vertices. Since $\ell = o(m/((D+1)\log n))$, by averaging in our process there would always be a leaf contained in at most a fraction of $O(1/\log n)$ of the closed walk. The estimate required in condition (4) thus follows, as in the proof of Lemma 3.1. Condition (2) also follows by the argument for establishing condition (2) in the proof of Lemma 3.1. This completes the proof. $\square$

We now prove the following useful lemma, whose special case is Lemma 3.2 in [2] (See also [3], [21] for related arguments.)

Lemma 3.4. *Let H be a graph with maximum degree $d \geq 2$ and let U be an independent set of vertices. Suppose, further, that the induced subgraph of H on the union of U with the $(l+1)$ neighborhood of U in some connected component of $H \setminus U$ is a collection of $|U|$ vertex disjoint trees, where the roots are the vertices of U. For each i satisfying $0 \leq i \leq l+1$, let X_i be the set of vertices of distance exactly i from U in those trees. Let $\mathbf{v}$ be a nonzero eigenvector of H with eigenvalue at least $2\sqrt{d-1}$. Then for every $1 \leq i \leq l-1$ and any $1 \leq j \leq \min(l-i, i)$,*

$$\sum_{u \in X_{i-j+1}} \mathbf{v}(u)^2 + \sum_{u \in X_{i+j}} \mathbf{v}(u)^2 \leq \sum_{u \in X_{i-j}} \mathbf{v}(u)^2 + \sum_{u \in X_{i+j+1}} \mathbf{v}(u)^2. \quad (1)$$

As a consequence, for any $1 \le i \le l-1$,

$$\sum_{u \in X_i \cup X_{i+1}} \mathbf{v}(u)^2 \le \frac{\|\mathbf{v}\|_2^2}{\min(i+1, l-i+1)}. \tag{2}$$

Proof. Let $u \in X_i$ where $1 \le i \le l$, and let u' be its unique neighbor in X_{i-1}. Then

$$\lambda \mathbf{v}(u) = \sum_{w \in X_{i+1}, w \sim u} \mathbf{v}(w) + \mathbf{v}(u'). \tag{3}$$

Thus by writing $\mathbf{v}(u') = (d(u')-1)\mathbf{v}(u')/(d(u')-1)$ and by Cauchy-Schwarz, we have

$$\sum_{w \in X_{i+1}, w \sim u} \mathbf{v}(w)^2 + \frac{\mathbf{v}(u')^2}{d(u')-1} \ge \frac{\lambda^2 \mathbf{v}(u)^2}{d(u)+d(u')-2} \ge 2\mathbf{v}(u)^2. \tag{4}$$

(Here it is convenient to denote, for $u' \in U$, by $d(u')-1$ the degree of u' as a root of the corresponding tree described in the lemma, even when the actual degree of u' in H may be larger than $d(u')$. This is convenient for the uniformity of the notation, and the only property needed in the proof is that $d(u') \le d$ with this notation too, which clearly holds).

Add up these inequalities for all $u \in X_i$. Noticing that each vertex w in X_{i+1} is adjacent to exactly one vertex in X_i while each vertex u' in X_{i-1} is adjacent to exactly $(d(u')-1)$ vertices in X_i, putting $S_i = \sum_{u \in X_i} \mathbf{v}(u)^2$, it follows that

$$2\sum\nolimits_{u \in X_i} \mathbf{v}(u)^2 \le \sum\nolimits_{w \in X_{i+1}} \mathbf{v}(w)^2 + \sum\nolimits_{u' \in X_{i-1}} \mathbf{v}(u')^2 \implies 2S_i \le S_{i+1} + S_{i-1}. \tag{5}$$

Thus if $1 \le i \le l-1$, by adding (5) for i and for $i+1$, we have

$$2S_i + 2S_{i+1} \le S_{i+1} + S_{i-1} + S_{i+2} + S_i \implies S_i + S_{i+1} \le S_{i-1} + S_{i+2}. \tag{6}$$

We now prove (1) by induction on j. The base case when $j = 1$ is (6). Suppose the claim holds up to $j-1$ where $j \ge 2$. Apply (5) to S_{i-j+1} and S_{i+j} to get that $2(S_{i-j+1} + S_{i+j}) \le S_{i-j} + S_{i-j+2} + S_{i+j-1} + S_{i+j+1} \le S_{i-j+1} + S_{i+j} + S_{i-j} + S_{i+j+1}$ by the inductive hypothesis. Therefore $S_{i-j+1} + S_{i+j} \le S_{i-j} + S_{i+j+1}$, as desired.

Finally, (2) is proved by applying (1) with $j = 1, \ldots, \min(l-i, i)$. $\square$

Remark: For eigenvalues λ bounded away from $2\sqrt{d-1}$ the estimate in (2) (and the resulting ones in the proofs where it is used) can be improved significantly. We make no attempt to optimize it, as any estimate which is $o(\|\mathbf{v}(u)\|_2^2)$ as i and $\ell - i$ grow to infinity suffices for our purpose here. Here is a sketch of the improved estimate. If $\lambda^2 \ge (1+\delta)(2\sqrt{d-1})^2$ then the right-hand-side in (4) can be improved to $2(1+\delta)\mathbf{v}(u)^2$.

This improves the statement in (5) to $(2+2\delta)S_i \le S_{i-1} + S_{i+1}$ and proceeding as in the proof above implies that the quantities $S_{i-j} + S_{i+j+1}$ grow exponentially as j increases. For fixed δ and ℓ logarithmic in the number of vertices n this provides in (2) an upper estimate of $\|\mathbf{v}(u)\|_2^2$ divided by a fractional power of n.

Lemma 3.5. *Let F and H be two connected graphs, each having girth at least $2r+1$ and maximum degree at most d. Let G be the graph obtained from the vertex disjoint union of F and H by adding an arbitrary edge connecting them, keeping the maximum degree at most d. Let $\mu_1 \ge \mu_2 \ge \ldots \ge \mu_s$ be the s largest eigenvalues of the graph which is the disjoint union of F and H (that is, the s largest elements in the set of all eigenvalues of F and all eigenvalues of H, taken with multiplicities). Then*

$$|\mu_s - \lambda_s(G)| \le \frac{2s}{r+1}.$$

Proof. For each eigenvalue μ_i above which is an eigenvalue of H, let f_i be the corresponding normalized eigenvector viewed as a vector defined on $V(G) = V(H) \cup V(F)$, by extending its definition to be 0 on $V(F)$. Similarly, if μ_i is an eigenvalue of F let f_i be a corresponding eigenvector defined to be 0 on the vertices of H. These vectors are the normalized top s eigenvectors of $H \cup F$ and span a subspace of dimension s. Let $A' = A_{H\cup F}$ be the adjacency matrix of the disjoint union of H and F, then for any normalized vector $\mathbf{y}$ in this subspace $\mathbf{y}^T A' \mathbf{y} \ge \mu_s$. By Claim 2.2, the ℓ_∞ norm of each of the vectors f_i is at most $1/\sqrt{r+1}$ and hence by Cauchy-Schwarz the ℓ_∞ norm of each normalized vector y in this space is at most $\sqrt{s/(r+1)}$. Since the graph G is obtained from $H \cup F$ by the addition of a single edge, for each such $\mathbf{y}$, $\mathbf{y}^T A_G \mathbf{y}$ and $\mathbf{y}^T A' \mathbf{y}$ differ by only two terms of the form $\mathbf{y}(u)\mathbf{y}(v)$ and hence $\mathbf{y}^T A_G \mathbf{y} \ge \mu_s - 2s/(r+1)$. By the variational definition of $\lambda_s(G)$ this implies that $\lambda_s(G) \ge \mu_s - 2s/(r+1)$. To upper bound $\lambda_s(G)$ consider the subspace W spanned by the eigenvectors of the top s eigenvalues of G. This subspace contains a nonzero normalized vector $\mathbf{z}$ orthogonal to all the vectors f_i defined above for $1 \le i \le s-1$. In addition, its ℓ_∞-norm is at most $\sqrt{s/(r+1)}$. It is clear that $\mathbf{z}^T A' \mathbf{z} \le \mu_s$ and as before the fact that $\|\mathbf{z}\|_\infty \le \sqrt{s/(r+1)}$ implies that $\lambda_s(G) \le \mathbf{z}^T A_G \mathbf{z} \le \mu_s + 2s/(r+1)$. This supplies the desired upper bound for $\lambda_s(G)$, completing the proof. $\square$

Proof of Theorem 3.2. Let $(\mu_1, \ldots \mu_k)$ be a vector in $B(d,k)$. By Lemma 3.3 there are connected graphs $G_j = G(n, \mu_j)$ with the following properties.

1. Each G_j has maximum degree at most d and has at least two vertices of degree less than d.
2. $|\lambda_1(G_j) - \mu_j| \le \frac{c_2}{\log n}$

3. $\lambda_2(G_j) \leq 2\sqrt{d-1} + c_3(\frac{\log\log n}{\log n})^2$.

4. The girth of G_j is at least $c_3 \log\log n$.

Pick two vertices $u_{j,1}, u_{j,2}$ of degree less than d in each G_j for $2 \leq j \leq k-1$, one such vertex $u_{1,2}$ in G_1 and one such vertex $u_{k,1}$ in G_k. Adding the edges $u_{j,2}u_{j+1,1}$ for $1 \leq j < k$ to the vertex disjoint union of the graphs G_j we get a connected graph $G = G(n)$ with maximum degree at most d. Applying Lemma 3.5 $k-1$ times it follows that if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ are its top k eigenvalues then $|\lambda_i - \mu_i| \leq \frac{16k^2}{c_3 \log\log n}$ for all $1 \leq i \leq k$. Taking a growing sequence of values of n completes the proof of the theorem. $\square$

4 The top eigenvalues of regular graphs

We start with some notation. Let $H = (V, E)$ be a graph with maximum degree at most d. The *d-augmentation* $A_d(H)$ is the graph obtained from H as follows. For each vertex v of degree $d(v) < d$ of H, let U_v be a set of $d - d(v)$ new vertices, where all sets U_v are pairwise disjoint. Every point in U_v is adjacent to the vertex v. Thus, if H is d-regular then $A_d(H) = H$. In every other case all vertices of $A_d(H)$ are of degrees d or 1, and the number of leaves (vertices of degree 1) is exactly $d|V| - 2|E|$. Put $T^0(d)H = H$, $T^1(d)H = A_d(H)$ and $T^{i+1}(d)H = A_d(T^i(d)H)$ for all $i \geq 1$. Note that $T^r(d)H$ is obtained from the vertex disjoint union of H and $(d|V| - 2|E|)$ trees with r levels by joining the roots of these trees to the vertices of H of degree lower than d.

4.1 The top eigenvalues of regular graphs – a simple version

In this subsection we present the proof of the following Theorem 4.1, which is a less powerful version of Theorem 1.5, using a simpler analysis based on the same basic approach. A similar method with slightly more sophisticated arguments will be used later to establish Theorem 1.5.

Theorem 4.1. *For every $d \geq 3$ and every k, every point of*

$$C(d,k) = \{(\mu_1, \mu_2, \ldots, \mu_k) : \ d = \mu_1 \geq \mu_2 \geq \mu_3 \geq \ldots \geq \mu_k \geq 3\sqrt{d-1}\}.$$

is a limit point of a sequence of vectors $(\lambda_1(G_i), \lambda_2(G_i), \ldots, \lambda_k(G_i))$ for an infinite sequence G_i of d-regular graphs.

We need the following lemma.

Lemma 4.2. *Let $d \geq 3$ be an integer and let $\varepsilon > 0$ be a real number. Then for every real λ satisfying $2\sqrt{d-1} + 2\varepsilon \leq \lambda \leq d$ there exists a graph H with maximum degree at most d in which every connected component has at least one vertex of degree smaller than d satisfying the following properties.*

1. *The girth of H is at least $20d/\varepsilon$.*
2. *The number of leaves of $T^1(d)H$ is divisible by $2d$.*
3. *$\lambda_2(H) \leq 2\sqrt{d-1} + \varepsilon$.*
4. *For every $i \geq 4\sqrt{d-1}/\varepsilon$, $|\lambda_1(T^i(d)H) - \lambda| \leq 2\varepsilon$.*

Proof. As in the previous proofs we construct a family of graphs $H_0, H_1, \ldots$ and show that H can be chosen to be one of them. We start with a d-regular high-girth near Ramanujan graph G with a large even number m of vertices, and omit from it two non-adjacent vertices to get a graph H_0 satisfying (1) and (2). Each other graph H_i of the sequence will be obtained from the previous one by deleting two nonadjacent vertices and by adding, if needed, a set of at most d isolated edges to ensure that the number of leaves of $T^1(d)H_i$ is divisible by $2d$. Thus all these graphs satisfy (1) and (2). To prove (3) note that H_{i+1} is obtained from the vertex disjoint union of an induced subgraph of G and a collection of isolated edges and hence (3) follows by eigenvalue interlacing and the fact that G is near Ramanujan. It remains to analyze the largest eigenvalues of the graphs H_i and their augmentations. All of these graphs have largest degree at most d and hence the top eigenvalue of all is at most d. In addition, the top eigenvalue of H_0 is very close to d as its average degree is very close to d. Note also that eigenvalue interlacing implies that $\lambda_1(T^j(d)H_i)$ is an increasing function of j for every fixed i showing that $\lambda_1(T^j(d)H_0)$ is very close to d for all j. Consider some fixed j. Note that $T^j(d)H_{i+1}$ is obtained from $T^j(d)H_i$ by deleting two vertices, omitting several connected components each of which is a d-tree, and adding at most $3d$ such components, joining the root of some of them to the graph. By Claim 2.2 and the high girth the omission of two vertices does not change λ_1 by much. The subsequent removal and addition of the connected components which are trees does not change it at all, as long as the largest eigenvalue exceeds $2\sqrt{d-1}$. The addition of the edges also hardly changes it, by Lemma 3.5. Thus the difference between $\lambda_1(T^j(d)H_i)$ and $\lambda_1(T^j(d)H_{i+1})$ is smaller than ε. Since the graphs H_i end with one which is a union of disjoint trees for which $\lambda_1(T^j(d)H_i) \leq 2\sqrt{d-1}$, for every fixed j we can find some i so that $\lambda_1(T^j(d)H_i)$ is within ε of λ. It remains to show the following claim.

Claim 4.3. *Let $J = \lceil 4\sqrt{d-1}/\varepsilon \rceil$. Assume the largest eigenvalue of $T^J(d)H_i$ is at least $2\sqrt{d-1}+\epsilon$. For any $R \geq J$, $|\lambda_1(T^J(d)H_i) - \lambda_1(T^R(d)H_i)| \leq \varepsilon$, namely, $\lambda_1(T^j(d)H_i)$ hardly grows after $j = 4\sqrt{d-1}/\varepsilon$.*

Proof. By the monotonicity of $\lambda_1(T^j(d)H_i)$ as j increases, it suffices to show that for arbitrarily large R there is some $j \leq 4\sqrt{d-1}/\varepsilon$ so that $|\lambda_1(T^j(d)H_i) - \lambda_1(T^R(d)H_i)| \leq \varepsilon$, namely, that $\lambda_1(T^j(d)H_i)$ hardly grows after $j = 4\sqrt{d-1}/\varepsilon$.

Let H' be a connected component of $T^R(d)H_i$ with the maximum eigenvalue of it, call it μ, and let A' be its adjacency matrix. By assumption, $\mu \geq 2\sqrt{d-1}+\varepsilon$. Let $\mathbf{v}$ be the normalized eigenvector of the maximum eigenvalue μ of H'.

For each $i \leq R$ let s_i denote the sum of squares of the entries of $\mathbf{v}$ on the vertices of H' that are of distance exactly i from the set of vertices U of the original graph H_i. (Call these vertices the vertices at level i). Thus $\sum_i s_i = 1$ and therefore there is some index $1 \leq i \leq 4\sqrt{d-1}/\varepsilon$ so that $s_i + s_{i+1} \leq \varepsilon/(\sqrt{d-1})$. Let $\mathbf{v}_s$ be the restriction of the vector v to the vertices of distance at most i from U, $\mathbf{v}_m$ the restriction of this vector to the two consecutive levels $i, i+1$, and $\mathbf{v}_l$ it's restriction to the levels at least $i+1$. With some abuse of notation consider each of these three vectors as one defined on all vertices of H', where the coordinates are set to 0 in the irrelevant levels. Note that since every level $j \geq 1$ is an independent set it follows that

$$\mu = \mathbf{v}^T A' \mathbf{v} = \mathbf{v}_s^T A' \mathbf{v}_s + \mathbf{v}_m^t A' \mathbf{v}_m + \mathbf{v}_l^T A' \mathbf{v}_l.$$

However $\mathbf{v}_m^T A' \mathbf{v}_m \leq \sqrt{d-1}\|\mathbf{v}_m\|_2^2$ since the induced subgraph on the two levels i and $i+1$ is a union of vertex disjoint stars, each with maximum degree at most $d-1$. Since $\|\mathbf{v}_m\|_2^2 \leq \varepsilon/\sqrt{d-1}$ this is at most ε. In addition $\mathbf{v}_l^T A' \mathbf{v}_l \leq 2\sqrt{d-1}\|\mathbf{v}_l\|_2^2$, as the induced subgraph on the vertices at levels exceeding i is a union of d-trees. It follows that if $\mu \geq 2\sqrt{d-1}+\varepsilon$ then $\mathbf{v}_s^T A' \mathbf{v}_s \geq (\mu - \varepsilon)\|\mathbf{v}_s\|_2^2$ since otherwise the sum of all three terms is smaller than

$$(\mu - \varepsilon)\|\mathbf{v}_s\|_2^2 + \varepsilon + 2\sqrt{d-1}(1 - \|\mathbf{v}_s\|_2^2) \leq (\mu - \varepsilon) + \varepsilon,$$

contradiction. □

On the other hand, if $\lambda_1(T^J(d)H_i) < 2\sqrt{d-1}+\varepsilon$, taking H to be a collection of d isolated edges satisfies all requirements. This completes the proof of the lemma. □

4.1.1 Proof of Theorem 4.1

We first prove the following patching lemma.

For each $1 \le i \le k-1$, let $F_i = (V_i, E_i)$ be a graph with maximum degree at most d. Let F_0 be a d-regular graph with girth at least $8R$. By an *R-patching* of $F_1, \dots, F_{k-1}$ to F_0, we mean a graph G constructed in the following way. Suppose $|V(F_0)|$ is large enough to ensure that F_0 contains a collection M of vertices such that $|M|d$ is the total number of leaves of all graphs $T^1(d)F_i$, $1 \le i \le k-1$, and the distance in F_0 between any two vertices in M exceeds $4R$. Suppose the total number of such leaves is divisible by $2d$. The graph G is obtained from the graphs $F_i = (V_i, E_i)$ as follows. Let G_0 be the induced subgraph of F_0 obtained by removing all vertices of M, where M is as above. This graph has exactly $|M|d$ vertices of degree $d-1$. On each set of vertices V_i for $1 \le i \le k-1$ we take a copy of F_i, and extend it to a copy of $T^1(d)F_i$ by identifying the leaves of $T^1(d)F_i$ with the required number of vertices of degree $d-1$ of F_0. Clearly G is a d-regular graph.

Lemma 4.4. *Let $d, R \ge 3$ be integers. For each $1 \le i \le k-1$, let F_i' be a graph with maximum degree at most d (which is not d-regular) and girth at least $4R$, and let $F_i = (V_i, E_i)$ be $T^{8R}(d)F_i'$. Let $F_0 = (V_0, E_0)$ be a d-regular graph with girth at least $8R$. For each $0 \le i \le k-1$ let μ_i be the largest eigenvalue of F_i, and let μ be the maximum of the second largest eigenvalue of F_i for $0 \le i \le k-1$. Suppose $\mu_1 \ge \mu_2 \cdots \ge \mu_{k-1} \ge \max(2\sqrt{d-1}, \mu)$. Let the graph G be an R-patching of $F_1, \dots, F_{k-1}$ to F_0 and let λ_i be the i-th largest eigenvalue of G. Then for each $1 \le i \le k$, $|\lambda_i - \mu_{i-1}| \le \max(\sqrt{d-1}/R, 2d^3 \frac{\sum_{i=1}^{k-1} |V_i|}{|V_0|})$.*

Proof. We first prove the following easier direction.

Claim 4.5. *The ordered k-tuple $(\lambda_1, \lambda_2, \dots, \lambda_k)$ is at least the k-tuple*

$$(d - 2d^3 \frac{\sum_{i=1}^{k-1} |V_i|}{|V_0|}, \mu_1, \dots, \mu_{k-1})$$

when arranged in descending order.

Proof. As G and F_0 are d-regular, $\lambda_1 = \mu_0 = d$. Let V_0' be the set of all vertices of G of distance at least two from $\bigcup_{i=1}^{k-1} V_i$. Clearly $V_0' \subset V_0$. Since F_0 is sufficiently large, $|V_0 \setminus V_0'|$ is tiny and thus the average degree of the induced subgraph $F_0[V_0']$ exceeds $(d|V_0| - 2\sum_{i=1}^{k-1} |V_i| d^3)/|V_0|$. Therefore the maximum eigenvalue of $F_0[V_0']$ exceeds $d - 2\sum_{i=1}^{k-1} |V_i| d^3/|V_0|$. In addition, for every $1 \le i \le k-1$, the induced subgraph of G on V_i is F_i, with maximum eigenvalue μ_i. In the induced subgraph of $G[V_0' \cup V_1 \cdots \cup V_{k-1}]$ there are no edges between different sets and thus all the eigenvalues of the k induced subgraphs of G on $V_0', V_1, \dots, V_{k-1}$ are also eigenvalues of this induced subgraph. The assertion of the claim follows by eigenvalue interlacing. $\square$

It remains to prove the upper bound.

Claim 4.6. *For every $1 \le i \le k$, $\lambda_i \le \mu_{i-1} + \sqrt{d-1}/R$.*

Proof. Since $\lambda_1 = \mu_0 = d$, fix $2 \le i \le k$. If $\lambda_i \le 2\sqrt{d-1}$, we are done by the assumption $\mu_{i-1} \ge 2\sqrt{d-1}$. Thus assume $\lambda_i > 2\sqrt{d-1}$. Let V_0' be $V(G) \setminus \bigcup_{j=1}^{k-1} V_j$. Let G' be the subgraph of G obtained by removing the edges between V_0' and $\bigcup_{j=1}^{k-1} V_j$. Thus G' is the disjoint union of k graphs $F_0[V_0'], F_1, \dots, F_{k-1}$. Let A, A' be the adjacency matrices of G and G' respectively. Let X be the set of vertices in G incident to the edges across V_0' and $\bigcup_{i=1}^{k-1} V_j$. Therefore G restricted to X is a union of vertex disjoint stars with maximum degree at most d. Let C be the matrix obtained from A by replacing all the entries of the rows and columns corresponding to vertices outside X by 0. Thus $A = A' + C$.

Let $\sigma_1 \ge \sigma_2 \ge \dots$ be the eigenvalues of G'. Thus $\sigma_1 \le d$. As $\mu_1 \ge \mu_2 \dots \ge \mu_{k-1} \ge \max(2\sqrt{d-1}, \mu)$, for each $2 \le j \le k$, $\sigma_j \le \mu_{j-1}$. We will use σ_i to upper bound λ_i.

For each $1 \le j \le k$, let $\mathbf{v}_j$ be the normal eigenvector of A corresponding to λ_j. Thus $\sqrt{N}\mathbf{v}_1 = \mathbf{1}$ where $N = |V(G)|$. By Lemma 3.4 where the set U is the set of vertices in $\bigcup_{i=1}^{k-1} V_i$ of distance exactly R from X, it follows that $\sum_{u \in X} \mathbf{v}_j(u)^2 \le 1/R$. As a consequence, as C corresponds to a disjoint union of stars of maximum degree at most d,

$$|\mathbf{v}_j^T C \mathbf{v}_j| \le \sqrt{d-1} \sum_{u \in X} \mathbf{v}_j(u)^2 \le \sqrt{d-1}/R. \tag{7}$$

Let W be the i-dimensional space spanned by $\mathbf{v}_1, \dots, \mathbf{v}_i$. By the min-max principal,

$$\sigma_i \ge \min\{\mathbf{v}^T A' \mathbf{v}, \text{ where } \mathbf{v} \in W, \ \|\mathbf{v}\|_2^2 = 1\}. \tag{8}$$

Write $\mathbf{v} = \sum_{j=1}^{i} c_j \mathbf{v}_j$, where $\sum_{j=1}^{i} c_j^2 = 1$. Then

$$\mathbf{v}^T A' \mathbf{v} = \sum_{j=1}^{i} c_j^2 \mathbf{v}_j^T A' \mathbf{v}_j = \sum_{j=1}^{i} c_j^2 \mathbf{v}_j^T A \mathbf{v}_j - \sum_{j=1}^{i} c_j^2 \mathbf{v}_j^T C \mathbf{v}_j \ge \sum_{j=1}^{i} c_j^2 \mathbf{v}_j^T A \mathbf{v}_j - \sqrt{d-1}/R$$

where the last inequality is by (7). Since $\mathbf{v}_j^T A \mathbf{v}_j = \lambda_j$, we thus have

$$\sigma_i \ge \min\{\sum_{j=1}^{i} c_j^2 \lambda_j - \sqrt{d-1}/R, \text{ where } \sum_{j=1}^{i} c_j^2 = 1\} = \lambda_i - \sqrt{d-1}/R.$$

Together with $\sigma_i \le \mu_{i-1}$ showed earlier, we have shown $\lambda_i \le \mu_{i-1} + \sqrt{d-1}/R$. □

□

We can now prove Theorem 4.1.

Proof of Theorem 4.1. For each sufficiently small ε, it suffices to prove the result for all $\mu_1 > \mu_2 > \dots > \mu_k > 3\sqrt{d-1}+\varepsilon$. Let $R = \lceil 4kd/\varepsilon \rceil$. For each $1 \le i \le k-1$, let F_i' be a graph satisfying the assertions of Lemma 4.2 for $\lambda = \mu_i$ and let $F_i = T^{8R}(d)F_i'$. Let F_0 be a sufficiently large near Ramanujan d-regular graph of girth at least $8R$. Theorem 4.1 will be proved by applying Lemma 4.4 to the R-patching of $F_1, \ldots, F_{k-1}$ to F_0. To check all the assumptions in Lemma 4.4 are satisfied, it is only needed to show that the second largest eigenvalues of $F_0, F_1, \ldots, F_{k-1}$ are at most $3\sqrt{d-1}+\varepsilon$. Since F_0 is near Ramanujan, $\lambda_2(F_0) \leq 2\sqrt{d-1}+\varepsilon < 3\sqrt{d-1}$. Fix $1 \leq i \leq k-1$. In F_i, let U_i be the vertex subset $V(F_i) \setminus V(F_i')$. The induced subgraph of F_i on U_i is a disjoint union of trees of maximum degree at most d. Let B be the adjacency matrix of the disjoint union of the graph F_i' and $F_i[U_i]$. Thus $\lambda_2(B) \leq 2\sqrt{d-1}+\varepsilon$. Let C be the adjacency matrix of F_i in which the only nonzero entries correspond to the cross-edges between U_i and $V(F_i')$. The matrix C corresponds to a disjoint union of stars of degree at most d. Thus $\lambda_1(C) \leq \sqrt{d-1}$. Since the adjacency matrix of F_i is $B+C$ and $\lambda_2(B+C) \leq \lambda_2(B)+\lambda_1(C)$, the second largest eigenvalue of F_i is at most $3\sqrt{d-1}+\varepsilon$, as desired. □

4.2 Proof of Theorem 1.5

Drawing inspiration from the proof of Theorem 4.1 as an application of Lemma 4.4, we seek to refine our estimation of the second largest eigenvalue of $T^\ell(F_i)$ for sufficiently large ℓ, given that $\lambda_2(F_i)$ is small. In the proof of Theorem 4.1, we present a short argument to establish that $\lambda_2(T^\ell(F_i)) \leq 3\sqrt{d-1}+\varepsilon$. The main ingredient in the proof of Theorem 1.5 is the following lemma, which provides an almost optimal upper bound for the second largest eigenvalue $\lambda_2(T^\ell(d)F_i)$.

Lemma 4.7. *Fix an integer $d \geq 3$. Let $\epsilon > 0$, and $R \geq 100d/\epsilon$. For any $z \in (2\sqrt{d-1}, d)$, there is a graph $\tilde{G}$ with maximum degree at most d and girth at least R satisfying the following properties simultaneously.*

1. *The largest eigenvalue of $T^\ell(d)\tilde{G}$ is within a range of ϵ to z for any $\ell \geq \lceil 10\sqrt{d-1}/\epsilon \rceil$.*
2. *The second largest eigenvalue of $T^\ell(d)\tilde{G}$ is at most $2\sqrt{d-1}+\frac{1}{\sqrt{d-1}}+\epsilon/2$ for any $\ell \geq 0$.*

We may also assume the number of vertices of degree one in $T^1(d)\tilde{G}$ is divisible by d.

Assuming Lemma 4.7 holds, Theorem 1.5 can be proved by using Lemma 4.4.

Proof of Theorem 1.5. Apply the patching lemma (Lemma 4.4) to assemble the augmentations of graphs that are guaranteed by Lemma 4.7 and a large near Ramanujan graph. Theorem 1.5 follows in a nearly identical way to the proof of Theorem 4.1, which also utilizes Lemma 4.4. Consequently, the details are omitted. □

4.2.1 Proof of Lemma 4.7

To simplify the analysis, we make the following definition similar to $T^{\ell}(d)F$. For any non-negative integers s, ℓ and a graph F, define the (d, s, ℓ)-*augmentation* $T^{\ell}_{s}(d)F$ to be the graph obtained from the vertex disjoint union of F and $s|V(F)|$ number of d-ary trees with ℓ levels [1] by joining each vertex of F to s of these trees by edges such that each tree joins exactly one vertex. Note that if F is d'-regular with $d' \leq d$, then $T^{\ell}(d)(F) = T^{\ell}_{d-d'}(d)F$. Our basic gadgets will be of the form $T^{\ell}_{s}(d)G'$ where G' is a regular graph. By interpolating between two basic gadgets, we will prove the existence of $\tilde{G}$ as desired in Lemma 4.7. We also sometimes write T^{ℓ}_{s} instead of $T^{\ell}_{s}(d)$ when there is no confusion.

The next lemma relates explicitly the top eigenvalues of a graph G' and $T^{\ell}_{s}(d)G'$. For each non-negative integer i, define an auxiliary function $a_i = a_i(\lambda)$ as

$$a_i = \frac{1}{\sqrt{\lambda^2 - 4(d-1)}}\left(\left(\frac{\lambda + \sqrt{\lambda^2 - 4(d-1)}}{2}\right)^{i+1} - \left(\frac{\lambda - \sqrt{\lambda^2 - 4(d-1)}}{2}\right)^{i+1}\right). \quad (9)$$

Lemma 4.8. *Let $d \geq 3, \ell \geq 2, s \geq 0$ be integers, and let G' be an arbitrary connected graph with the largest eigenvalue being μ_1. Then the following equation has at most one solution λ that is larger than $2\sqrt{d-1}$:*

$$\lambda - s a_{\ell-1}(\lambda)/a_{\ell}(\lambda) = \mu_1. \quad (10)$$

Furthermore, the top two eigenvalues of $T^{\ell}_{s}(d)G'$ satisfy the following statements.

1. *If (10) has a solution larger than $2\sqrt{d-1}$, then this solution is the largest eigenvalue of $T^{\ell}_{s}(d)G'$.*
2. *If (10) has no solution larger than $2\sqrt{d-1}$, then $T^{\ell}_{s}(d)G'$ has no eigenvalue larger than $2\sqrt{d-1}$.*
3. *For any $\epsilon \geq 0$, if the second largest eigenvalue of G' is strictly less than $2\sqrt{d-1+\epsilon} - \frac{s}{\sqrt{d-1+\epsilon}+\sqrt{\epsilon}}$, then the second largest eigenvalue of $T^{\ell}_{s}(d)G'$ is less than $2\sqrt{d-1+\epsilon}$.*

To prove Lemma 4.8, we need to establish two claims. The first claim demonstrates that in the graph $T^{\ell}_{s}(d)(G')$, an explicit relationship exists among the entries of an eigenvector on each of the d-ary trees. We refer to a vector that is supported on the vertices of a tree T as *radial* if the vertices at the same distance from the root in T have identical values in the vector.

[1] A *d-ary trees* is a tree T where the root has degree $d-1$ and each vertex except the root and leaves has $d-1$ children. A d-ary tree is said to have ℓ *levels* if the distance between each leaf and the root is $\ell - 1$.

Claim 4.9. *Let $d \geq 3$ be a fixed integer. Let $G = (V, E)$ be a graph and let $v \in V$ be a fixed vertex. Suppose some component in $G \setminus \{v\}$ is a d-ary tree T of $\ell + 1$ levels where $\ell \geq 0$ and in T the root is the only neighbor of v. If $\lambda > 2\sqrt{d-1}$ is an eigenvalue of G then the eigenvector $\mathbf{v}$ of λ is radial on T. Furthermore, if the entries of $\mathbf{v}$ corresponding to the leaves of T have the value x, then for each $0 \leq i \leq \ell$, the entries of $\mathbf{v}$ corresponding to vertices in T that are at a distance of $\ell - i$ from the root have the value $a_i x$, where $a_i = a_i(\lambda)$ is defined as in (9).*

Proof. We prove the claim by induction on ℓ. Suppose $\ell = 1$. Let $\mathbf{v}$ be an eigenvector of eigenvalue λ. Let $x_1, x_2, \dots, x_{d-1}$ be its entries of the leaves and let y be the entry of the root. Then $\lambda x_i = y$ for each $1 \leq i \leq d-1$. Since $\lambda \neq 0$, it forces $x_1 = \cdots = x_{d-1} = x = a_0 x$ for some x. Furthermore, $y = \lambda x = a_1 x$ as $a_1(\lambda) = \lambda$. The base case is proved.

Suppose the claim holds for trees of at most ℓ levels. We now show it holds for trees with $\ell + 1$ levels. Let z be the entry of the root v in the eigenvector, and $y_1, \dots, y_{d-1}$ be the entries of its children, which are vertices $u_1, \dots, u_{d-1}$ respectively. For $1 \leq i \leq d-1$, let T_i be the subtree of T rooted at u_i. Thus T_i is a d-ary tree with ℓ levels. By the inductive hypothesis, $\mathbf{v}$ is radial on T_i. Let x_i' be the entry of the leaves in T_i. By the inductive hypothesis, $y_i = a_{\ell-1} x_i'$. Furthermore, each of the $d-1$ children of u_i has entry $y_i' = a_{\ell-2} x_i'$. On the other hand, we also have

$$\lambda y_i = (d-1) y_i' + z \implies \lambda a_{\ell-1} x_i' = (d-1) a_{\ell-2} x_i' + z.$$

This implies $(\lambda a_{\ell-1} - (d-1) a_{\ell-2}) x_i' = z$, which is equivalent to $a_\ell x_i' = z$. Since $\lambda > 2\sqrt{d-1}$, we have $a_\ell \neq 0$. Thus we conclude $x_1' = x_2' = \cdots = x_{d-1}' = x$ for some x, and the root v has value $z = a_\ell x$. The entries on the other levels are proved by the inductive hypothesis and the fact $x_1' = x_2' = \cdots = x_{d-1}' = x$. □

The next claim shows an explicit relationship between the eigenvalues and eigenvectors of G' and those of $T_s^\ell(d)G'$.

Claim 4.10. *Let $d \geq 3, s, \ell \geq 2$ be positive integers and G' a fixed graph. Let G be $T_s^\ell(d)G'$. Then for any $\lambda > 2\sqrt{d-1}$, the following two statements are equivalent:*

1. *λ is an eigenvalue of G;*

2. *Let $a_i = a_i(\lambda)$ be defined in (9). The following value μ is an eigenvalue of G':*

$$\lambda - s a_{\ell-1}(\lambda)/a_\ell(\lambda) = \mu. \tag{11}$$

In addition, suppose $\mathbf{v}'$ is an eigenvector of G corresponding to λ. Then $\mathbf{v}'$ restricted to $V(G')$ is an eigenvector $\mathbf{v}$ of G'. For each vertex u that is on a d-ary tree joined to some vertex $v \in V(G')$ and is at a distance i from the leaf of that tree, the value of $\mathbf{v}'$ at u is $\mathbf{v}(v)a_i/a_\ell$.

Proof. Let v be a vertex in G', and it joins s trees $T_1, \ldots, T_s$ of ℓ levels each, where the roots are $u_1, \ldots, u_s$ respectively. To show Statement 1 implies 2, we can apply Claim 4.9 to each of T_i. Suppose $\mathbf{v}'$ restricted to the leaves of T_i have entries x_i (this is well-defined since the eigenvector is radial on T_i by Claim 4.9). Then each root u_i has value $a_{\ell-1}x_i$. Similarly, the values of the $d-1$ children of u_i in T_i are $a_{\ell-2}x_i$. Then since λ is an eigenvalue, we have

$$\lambda a_{\ell-1}x_i = (d-1)a_{\ell-2}x_i + \mathbf{v}'(v).$$

Therefore by a similar argument as in the proof of Claim 4.9 we have $\mathbf{v}'(v) = a_\ell x_i$. Since $a_\ell \neq 0$, we have $x_1 = x_2 = \cdots = x_s = x$ for some x. Thus $x = \mathbf{v}'(v)/a_\ell$.

We also have $\lambda \mathbf{v}'(v) = \sum_{u:(u,v)\in E(G')} \mathbf{v}'(u) + sa_{\ell-1}x$ where the summation is over all the vertices adjacent to v in G'. Since $x = \mathbf{v}'(v)/a_\ell$,

$$\lambda \mathbf{v}'(v) = \sum_{u:(u,v)\in E(G')} \mathbf{v}'(u) + sa_{\ell-1}\mathbf{v}'(v)/a_\ell \implies (\lambda - sa_{\ell-1}/a_\ell)\mathbf{v}'(v) = \sum_{u:(u,v)\in E(G')} \mathbf{v}'(u).$$

Therefore $\lambda - sa_{\ell-1}/a_\ell$ is an eigenvalue of G', with eigenvector $\mathbf{v}'$ restricted to $V(G')$. The fact that Statement 2 implies Statement 1 is by the same argument, constructing $\mathbf{v}'$ directly from $\mathbf{v}$ and noticing that λ is an eigenvalue corresponding to $\mathbf{v}'$. □

Proof of Lemma 4.8.

The case when $s = 0$ is trivial. Thus we assume $s \geq 1$. Proving that there is at most one solution to (10) that is larger than $2\sqrt{d-1}$ directly from the equation is challenging. Nevertheless, we can establish this fact by examining the eigenvector. As per Claim 4.10, the eigenvector of λ in G that is restricted to $V(G')$ is also the unique eigenvector $\mathbf{v}$ of G' for the top eigenvalue μ_1, with each entry being non-negative. For each vertex which is on the d-ary tree grew out from vertex $v \in V(G')$, its value on the eigenvector is $\mathbf{v}(v)a_i/a_\ell \geq 0$ for some $0 \leq i \leq \ell - 1$. Suppose λ and λ' are two distinct solutions to (10) that are larger than $2\sqrt{d-1}$. In that case, their eigenvectors are orthogonal but have all non-negative entries, a contradiction.

We now prove statements 1 and 2. Let λ_0 be the largest solution to (10). By Claim 4.10, it suffices to show that for any $\mu < \mu_1$, the following equation in terms of λ has no solution larger than $\max(2\sqrt{d-1}, \lambda_0)$.

$$h(\lambda) := \lambda - sa_{\ell-1}/a_\ell = \mu. \tag{12}$$

For the sake of contradiction, suppose $\lambda' > \max(2\sqrt{d-1}, \lambda_0)$ is a solution to (12) for some $\mu < \mu_1$, i.e., $h(\lambda') = \mu < \mu_1$. Note $a_\ell, a_{\ell-1}$ are also functions of λ. When $\lambda > 2\sqrt{d-1}$, $a_{\ell-1}/a_\ell < \frac{2}{\lambda+\sqrt{\lambda^2-4(d-1)}}$. Therefore $h(\lambda) \to \infty$ if $\lambda \to \infty$. Combining with the fact that $h(\lambda') = \mu < \mu_1$ and by the continuity of h, equation (10) that $h(\lambda) = \mu_1$ has a solution $\lambda'' \geq \lambda'$. Since $\lambda' > \max(2\sqrt{d-1}, \lambda_0)$, the existence of such a λ'' contradicts with the fact that λ_0 is the largest solution to (10).

We now prove the last statement. Let μ be any eigenvalue of G' which is less than $2\sqrt{d-1+\epsilon} - s/(d-1+\epsilon)$. Thus for any $\lambda \geq 2\sqrt{d-1+\epsilon}$ and any ℓ and $\epsilon \geq 0$,

$$h(\lambda) \geq \lambda - 2s/(\lambda + \sqrt{\lambda^2 - 4(d-1)}) \geq 2\sqrt{d-1+\epsilon} - s/(\sqrt{d-1+\epsilon} + \sqrt{\epsilon}) > \mu.$$

Thus (12) has no solution at least $2\sqrt{d-1+\epsilon}$, as desired. □

We are now ready to prove the main ingredient: Lemma 4.7.

Proof of Lemma 4.7. Let G'_0 be an N-lift of the complete graph K_{d+1} for sufficiently large N. For each vertex i in K_{d+1} where $1 \leq i \leq d+1$, let V_i be the set of vertices in the lift which are the pre-images of i through the covering map. Suppose G'_0 has girth $L \geq 100d/\epsilon$ and $\lambda_2(G'_0) \leq 2\sqrt{d-1} + \epsilon/2$. Such a graph exists by the work of Bordenave and Collins [8]. Since a random N-lift of K_t for $t \geq 4$ has the second largest eigenvalue at most $2\sqrt{t-2} + \epsilon/2$ with probability at least $1 - O(N^{-0.99})$ [8], we may further assume by a union bound that for each $1 \leq i \leq d-2$,

$$\lambda_2(G'_0[V_i \cup V_{i+1} \cup \cdots \cup V_{d+1}]) < 2\sqrt{d-i} + \epsilon/2. \tag{13}$$

By an abuse of notation, label the vertices in G'_0 as $1, 2, \ldots, |V(G'_0)| = (d+1)N$ such that vertices in V_i comes before vertices in V_{i+1} for each $1 \leq i \leq d$. For each $1 \leq i \leq |V(G'_0)|$, let G'_i be the graph obtained from G'_{i-1} by removing vertex i from $V(G'_0)$. Clearly for each $1 \leq t \leq d$, the graph G'_{tN} is $(d-t)$-regular since by removing vertices in $V_1 \cup \cdots \cup V_t$ from G'_0, the remaining graph is an N-lift of K_{d-t+1}.

Let $I = \lceil 10\sqrt{d-1}/\epsilon \rceil$. In a way similar to the previous argument, the interpolation procedure begins with the graph $G_0 = T^I(d)G'_0$. For each $0 \leq i \leq (d+1)N$, let $G_i = T^I(d)G'_i$. The procedure stops as soon as $i = dN$ or when the top eigenvalue of G_i is at most $2\sqrt{d-1} + \epsilon/2$.

Claim 4.11. *For each $0 \leq i \leq (d+1)N$, if $\lambda_1(G_i) > 2\sqrt{d-1} + \epsilon/10$, then $|\lambda_1(G_i) - \lambda_1(G_{i+1})| \leq 6d/L$.*

Proof. Note that by construction, G_{i+1} is obtained from G_i by removing vertex $i+1$, omitting several connected components each of which is a d-ary tree, and adding at most

d such components, joining their roots to the graph. Indeed by removing the vertex $i+1$ from G_i, there are several connected components each of which is a d-ary tree, connected in G_i by an edge from the root to the vertex $i+1 \in V(G_i)$. Let G_i'' be the induced subgraph of G_i obtained by removing these trees. Therefore G_i is the graph obtained from G_i'' and several d-ary trees so that for each such a tree, there is one edge between its root and the vertex $i+1$ in G_i''. Similarly, G_{i+1} can be considered as starting from the graph $G_i'' \setminus \{i+1\}$, and then for each vertex $u \in V(G_i'')$ which is a neighbor of vertex $i+1$ in G_i'', this vertex u is joined in G_{i+1} to the root of a new copy of a d-ary tree.

The largest eigenvalue of $G_i'' \setminus \{i+1\}$ is at most $\lambda_1(G_i)$ by eigenvalue interlacing. The largest eigenvalue of each tree is at most $2\sqrt{d-1}$. Applying Lemma 3.5 d times, we have $\lambda_1(G_{i+1}) \le \lambda_1(G_i) + 4d/L$.

For the other direction, by applying Lemma 3.5 at most d times to $T \cup G_i''$ and G_i,

$$|\lambda_1(G_i) - \max(\lambda_1(T), \lambda_1(G_i''))| \le 4d/L. \tag{14}$$

If $\lambda_1(G_i'') \le 2\sqrt{d-1}$, then since the eigenvalues of T are also no more than $2\sqrt{d-1}$, (14) implies $\lambda_1(G_i) \le 2\sqrt{d-1} + 4d/L < 2\sqrt{d-1} + \epsilon/10$, a contradiction. Therefore we can assume $\lambda_1(G_i'') > 2\sqrt{d-1}$, and thus is larger than $\lambda_1(T)$. This fact together with (14) imply $\lambda_1(G_i'') \ge \lambda_1(G_i) - 4d/L$. Let A, A'' be the adjacency matrices of G_{i+1}, G_i'' respectively, and by adding zero entries to vertices in $V(G_i'') \setminus V(G_i+1)$ and $V(G_{i+1}) \setminus V(G_i'')$ respectively. Let $\mathbf{v}, \mathbf{v}''$ be the top normal eigenvectors of G_{i+1}, G_i'' respectively. Thus $\lambda_1(G_{i+1}) - \lambda_1(G_i'') \ge \mathbf{v}''^T(A - A'')\mathbf{v}''$, which effectively is a sum of at most d terms of the form $\pm\mathbf{v}''(u)\mathbf{v}''(i+1)$ for several different vertices u's. By Claim 2.2, each such a term has absolute value at most $2/(L+1)$. Thus $\lambda_1(G_{i+1}) - \lambda_1(G_i'') \ge -2d/(L+1)$. Combining with the lower bound on $\lambda_1(G_i'')$, it follows that $\lambda_1(G_{i+1}) - \lambda_1(G_i) \ge -6d/L$. □

As $\lambda_1(G_0) = d$ and G_{dN} is the disjoint union of trees which has top eigenvalue at most $2\sqrt{d-1}$, by Claim 4.11, there is an i^* such that $|\lambda_1(G_{i^*}) - z| < \epsilon/2$. Furthermore, by Claim 4.3, for any $\ell \ge \lceil 10\sqrt{d-1}/\epsilon \rceil$, $|\lambda_1(T^\ell(d)G'_{i^*}) - \lambda_1(T^I(d)G'_{i^*})| \le \epsilon/2$. Therefore for any $\ell \ge \lceil 10\sqrt{d-1}/\epsilon \rceil$, $|\lambda_1(T^\ell(d)G'_{i^*}) - z| \le \epsilon$, as desired.

It remains to prove Statement 2. Fix any ℓ. Suppose G'_{i^*} has maximum degree at most $t+1$ but has some vertex of degree t. Here $0 \le t \le d-1$. Thus G'_{i^*} is an induced subgraph of the $(t+1)$-regular graph $G'_0[V_{d-t} \cup \cdots \cup V_{d+1}]$ by the construction. Passing down to the connected component of G_{i^*} with the largest top eigenvalue, call its $\tilde{G}_{i^*}$. This connected component is an induced subgraph of some connected component of $T^\ell_{d-t}(d)G'_0[V_{d-t} \cup \cdots \cup V_{d+1}]$. If $t \ge 2$, then by (13), $\lambda_2(G'_0[V_{d-t} \cup \cdots \cup V_{d+1}]) \le 2\sqrt{t} + \epsilon/2$. When $t \le 1$, then each connected component of $G'_0[V_{d-t} \cup \cdots \cup V_{d+1}]$ is a cycle or an edge, thus has second largest eigenvalue smaller than $2\sqrt{t}$. If ε' is chosen such that $2\sqrt{d-1+\varepsilon'}-(d-t)/\sqrt{d-1+\varepsilon'} \geq 2\sqrt{t}+\epsilon/2$, then by Lemma 4.8, each connected component of $T^{\ell}_{d-t}(d)G'_0[V_{d-t}\cup\cdots\cup V_{d+1}]$ has second largest eigenvalue at most $2\sqrt{d-1+\varepsilon'}$. The inequality holds for all $t \leq d-1$ if ε' is chosen such that $2\sqrt{d-1+\varepsilon'} \geq 1/\sqrt{d-1}+2\sqrt{d-1}+\epsilon/2$. Thus by eigenvalue interlacing, $\lambda_2(T^{\ell}(d)\tilde{G}_{i^*}) \leq 2\sqrt{d-1}+\frac{1}{\sqrt{d-1}}+\epsilon/2$, as desired.

The required divisibility of the number of vertices of $T^{\ell}(d)\tilde{G}$ could be shown through the same interpolation analysis again, by noticing that within every d steps, there must be one step where the number of vertices is divisible by d. $\square$

5 Near Ramanujan graphs with localized eigenvectors

The proof of Theorem 1.6 is similar to the one described in the previous section, together with an additional quantitative analysis that establishes the eigenvector localization. The gadget to be applied to the patching lemma (Lemma 4.4) is as follows.

Lemma 5.1. *Fix integers $n_0, d \geq 3$. Let $\epsilon_1, \epsilon_2 > 0$ be sufficiently small in terms of d. There is a graph $\tilde{G}$ on at least n_0 vertices with maximum degree at most d, girth at least $0.5\log_d|V(\tilde{G})|$, such that for any $\ell \geq \lceil 100\sqrt{d-1}/\min(\epsilon_1,\epsilon_2)\rceil$, the largest eigenvalue of $T^{\ell}(d)\tilde{G}$ is in the interval $(2\sqrt{d-1}+\epsilon_1, 2\sqrt{d-1}+\epsilon_1+\epsilon_2)$. We may also assume the number of vertices of degree one in $T^{\ell}(d)\tilde{G}$ is divisible by d.*

Proof of Lemma 5.1. Let G'_0 be a d-regular bipartite graph on $n \geq 2n_0$ vertices which has girth at least $0.5\log_d n$. Such graphs exist by [13] (and explicit constructions are given in [25] where one can omit some of the generators to get the required degree if it is not of the form given in [25]). Since G'_0 is bipartite, it contains an independent set U with $|U| \geq n/2$. Fix such a U. Label the vertices of G'_0 from 1 to n such that the vertices in U are ranked the last. Define a sequence of graphs (G'_i) where for each $1 \leq i \leq n$, G'_i is obtained from G'_{i-1} by removing vertex i. Let $\epsilon = \min(\epsilon_1,\epsilon_2)$ and $I = \lceil 100\sqrt{d-1}/\epsilon\rceil$. Define a new graph sequence (G_i) where for each $0 \leq i \leq n$, $G_i = T^I(d)G'_i$. By a similar analysis to the one in the proof of Claim 4.11, when n is sufficiently large, $|\lambda_1(G_i)-\lambda_1(G_{i+1})| \leq 0.01\epsilon$ as long as $\lambda_1(G_i) \geq 2\sqrt{d-1}+0.5\epsilon_1$. Note that $\lambda_1(G_0)=d$. On the other hand, $\lambda_1(G_{n-|U|}) < 2\sqrt{d-1}$ since $G'_{n-|U|}$ is the independent set U and thus $G'_{n-|U|}$ is a disjoint union of d-ary trees. Thus there is an $1 \leq i \leq n-|U|$ such that $\lambda_1(G_i) \in (2\sqrt{d-1}+\epsilon_1+0.5\epsilon, 2\sqrt{d-1}+\epsilon_1+0.6\epsilon)$. Let $\tilde{G}$ be G'_i. As in Claim 4.3, for any $\ell \geq \lceil 100\sqrt{d-1}/\epsilon\rceil$, $|\lambda_1(T^{\ell}(d)G'_i)-\lambda_1(G_i)| \leq \epsilon/20$. Thus the desired bound on $\lambda_1(T^{\ell}(d)\tilde{G})$ holds. Note that by the construction, $\tilde{G}=G'_i$ has at least $|U| \geq n/2 \geq n_0$ vertices, and the girth of $\tilde{G}$ is at least $0.5\log_d|V(\tilde{G})|$. $\square$

Proof of Theorem 1.6. Assume $\beta > 0$ is sufficiently small in terms of d, ℓ is large in terms of d, β, and n_0 is large in terms of d, ℓ, β. By Lemma 5.1, there is a graph $\tilde{G}$ on $n \geq n_0$ vertices with maximum degree at most d, girth at least $0.5 \log_d |V(\tilde{G})|$ and $\lambda_1(T^\ell(d)\tilde{G}) = \mu_1 \in (2\sqrt{d-1} + 0.5\beta, 2\sqrt{d-1} + 0.8\beta)$.

Let F_1 be $T^\ell(d)\tilde{G}$. Note that F_1 has at most $d^{\ell+1}n$ vertices. Let F_0 be a d-regular Ramanujan graph on m vertices with girth at least $2\log m/3$ and $m = n^C$ where $C \geq 4$. We can assume $d^{\ell+1}n \leq n^{\sqrt{C}}$. The existence of such a graph and the fact that it can be constructed explicitly when $d = p+1$ where p is a prime is a result of Lubotzky, Phillips and Sarnak [25], and Margulis [28]. Applying the patching Lemma 4.4 to F_0 and F_1, we obtain a graph F where $\lambda_2(F) \in (2\sqrt{d-1} + 0.4\beta, 2\sqrt{d-1} + 0.9\beta)$.

Let $\mathbf{v}$ be the normal eigenvector of F corresponding to the second largest eigenvalue. We proceed to show that $\mathbf{v}$ is localized. Let X be the union of the leaves in F_1 and the set of vertices in $V(F) \setminus V(F_1)$ adjacent to these leaves.

Let A_C be the adjacency matrix for the induced subgraph of F on X, which is a disjoint union of stars, each having $d-1$ leaves. Let F_b be the big subgraph of F induced on $V(F) \setminus V(F_1)$ and let A_b be its adjacency matrix. Let A_s be the adjacency matrix of the relatively small graph F_1. Assume all those adjacency matrices are of dimension $|V(F)| \times |V(F)|$ by filling zeros in the additional columns and rows. Then $\mathbf{v}^t A_F \mathbf{v} = \mathbf{v}^t A_C \mathbf{v} + \mathbf{v}^t A_s \mathbf{v} + \mathbf{v}^t A_b \mathbf{v}$. We bound each of these three terms separately.

The first term satisfies $\mathbf{v}^t A_C \mathbf{v} \leq \sqrt{d-1} \sum_{u \in X} \mathbf{v}(u)^2 \leq 32(\sqrt{d-1})/\ell$, where the last inequality is by Lemma 3.4. The second term satisfies $\mathbf{v}^t A_s \mathbf{v} \leq \mu_1 \sum_{u \in V(F_1)} \mathbf{v}(u)^2$. It remains to bound the last term $\mathbf{v}^t A_b \mathbf{v}$. Let $\mathbf{v}'$ be a vector indexed by $V(F_0)$ which is equal to $\mathbf{v}$ on $V(F_b)$ and 0 elsewhere on $V(F_0)$. Write $\mathbf{v}' = c_1 1 + c_2 f$ where f is orthogonal to 1 and has norm one. Then $|c_1| = |\mathbf{v}'^t 1/|V(F_0)||$. Since $\mathbf{v}$ is orthogonal to 1, $|\mathbf{v}'^t 1| = |\sum_{u \in V(F_b)} \mathbf{v}(u)| = |\sum_{u \in V(F_1)} \mathbf{v}(u)|$, and thus $|c_1| \leq \sqrt{|V(F_1)|}/|V(F_0)|$ by Cauchy-Schwarz. Let A_0 be the adjacency matrix of F_0. Then $\mathbf{v}^t A_b \mathbf{v} = \mathbf{v}'^t A_0 \mathbf{v}' = c_1^2 1^t A_0 1 + c_2^2 f^t A_0 f$. Thus

$$\mathbf{v}^t A_b \mathbf{v} \leq (|V(F_1)|/|V(F_0)|^2) d|V(F_0)| + 2\sqrt{d-1}\|\mathbf{v}'\|_2^2 \leq \frac{d|V(F_1)|}{|V(F_0)|} + 2\sqrt{d-1} \sum_{u \in V(F_b)} \mathbf{v}(u)^2.$$

Here the inequalities are by the bounds on $|c_1|, |c_2|$ and the fact that $\lambda_2(F_0) = 2\sqrt{d-1}$.

Adding the upped bounds on the three terms,

$$\mathbf{v}^t A_F \mathbf{v} \leq \frac{32}{\ell}\sqrt{d-1} + \frac{d|V(F_1)|}{|V(F_0)|} + 2\sqrt{d-1} \sum_{u \in V(F_b)} \mathbf{v}(u)^2 + \mathbf{v}^t A_s \mathbf{v}. \tag{15}$$

$$\leq \frac{32}{\ell}\sqrt{d-1} + \frac{d|V(F_1)|}{|V(F_0)|} + 2\sqrt{d-1} \sum_{u \in V(F_b)} \mathbf{v}(u)^2 + \mu_1 \sum_{u \in V(F_1)} \mathbf{v}(u)^2. \tag{16}$$

It is not difficult to see that by eigenvalue interlacing, $\mu_1 \leq \mathbf{v}^t A_F \mathbf{v}$. This together with (16) implies

$$\mu_1 \leq \frac{32}{\ell}\sqrt{d-1} + \frac{d|V(F_1)|}{|V(F_0)|} + \mu_1 \sum\nolimits_{u \in V(F_1)} \mathbf{v}(u)^2 + 2\sqrt{d-1} \sum\nolimits_{u \in V(F_b)} \mathbf{v}(u)^2$$

Subtracting from both sides $2\sqrt{d-1} = 2\sqrt{d-1}(\sum_{u \in V(F_1)} \mathbf{v}(u)^2 + \sum_{u \in V(F_b)} \mathbf{v}(u)^2)$, we get

$$\left(\mu_1 - 2\sqrt{d-1} - \frac{32}{\ell}\sqrt{d-1} - \frac{d|V(F_1)|}{|V(F_0)|}\right) \leq (\mu_1 - 2\sqrt{d-1}) \sum\nolimits_{u \in V(F_1)} \mathbf{v}(u)^2.$$

Since $\mu_1 \geq 2\sqrt{d-1} + 0.5\beta$ and $\frac{32}{\ell}\sqrt{d-1} + \frac{d|V(F_1)|}{|V(F_0)|} \leq \frac{32}{\ell}\sqrt{d-1} + \frac{d}{n} \leq 0.5\beta^2$ we have that

$$\sum\nolimits_{u \in V(F_1)} \mathbf{v}(u)^2 / \|\mathbf{v}\|_2^2 \geq 1 - \frac{0.5\beta^2}{0.5\beta} = 1 - \beta.$$

The desired result follows. $\square$

6 Remarks

- The quantitative estimates in Theorem 2.1 can be improved for values of n and d for which it is known that there are high girth Ramanujan graphs. In particular, by the constructions of Lubotzky, Phillips and Sarnak [25] and Margulis [28] for every $d = p + 1$ with p a prime congruent to 1 modulo 4, there are infinitely many values of n for which there are explicit d-regular Ramanujan graphs on n vertices with girth $\Omega(\log n / \log d)$. Plugging such a graph as G_0 in the proof we get the assertion of Theorem 2.1 in which $\log \log n$ is replaced by $\log n$. A similar remark applies to the proofs of Theorem 1.2 and Theorem 3.2.

- Conjecture 1.3 remains open. It is true, however, that if there is a d-regular graph H with top k eigenvalues $d = \mu_1 > \mu_2 \geq \ldots \geq \mu_k > 2\sqrt{d-1}$ then there are infinitely many connected d-regular graphs with the same sequence of k top eigenvalues. This follows from the result of Friedman and Kohler [16], see also [7], that all the new eigenvalues of random lifts of H are, with high probability, at most $2\sqrt{d-1} + o(1)$ where the $o(1)$-term tends to zero as the size of the lift tends to infinity.

- The problem of understanding the possible spectrum of finite d-regular graphs is challenging. Some aspects of this problem are considered here, other variants appear in [22], [34].

- The limit points of the spectral radii of sequences of (not necessarily regular) graphs have also been studied. In particulr, Shearer [33] proved that any real number greater than $\sqrt{2+\sqrt{5}}$ is such a limit, answering a question of Hoffman [19].

- Theorem 1.6 can be extended to yield near-Ramanujan regular graphs with multiple localized eigenvectors corresponding to eigenvalues strictly larger than $2\sqrt{d-1}$. This can be proved in a similar way, by patching multiple graphs. The detailed proof requires a more technical computation, and we thus decided not to include it here. The proof can be found on the second author's homepage [5].

Acknowledgment: We thank Peter Sarnak for suggesting the first problem considered here and also thank Vishal Gupta, Jiaoyang Huang, Michael Magee, Nike Sun, Michael Ren and two anonymous referees for helpful suggestions and comments.

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