Feasible schedules for rotating transmissions

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Abstract

Motivated by a scheduling problem that arises in the study of optical networks we prove the following result, which is a variation of a conjecture of Haxell, Wilfong and Winkler.

Let k, n be two integers, let $w_{sj}, 1 \leq s \leq n, 1 \leq j \leq k$ be non-negative reals satisfying $\sum_{j=1}^{k} w_{sj} < 1/n$ for every $1 \leq s \leq n$ and let d_{sj} be arbitrary non-negative reals. Then there are real numbers $x_1, x_2, \ldots, x_n$ so that for every j, $1 \leq j \leq k$, the n cyclic closed intervals $I_s^{(j)} = [x_s + d_{sj}, x_s + d_{sj} + w_{sj}]$, $(1 \leq s \leq n)$, where the endpoints are reduced modulo 1, are pairwise disjoint on the unit circle.

The proof is based on some properties of multivariate polynomials and on the validity of the Dyson Conjecture.

1 Introduction

Motivated by the study of information transmission in optical networks, the authors of [3] considered several variants of the following problem. Given n transmitters $T_1, T_2 \ldots, T_n$ and k receivers $R_1, R_2, \ldots, R_k$, our objective is to design a rotating schedule that will enable the transmitters to transmit information to the receivers. We scale time so that the total length of the period in our periodic protocol is 1. We assume that each transmitter T_s has to transmit data that occupies time w_{sj} to receiver R_j, so that the total time it has to transmit to all receivers satisfies

$$w_s = \sum_{j=1}^{k} w_{sj} < w. \tag{1}$$

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The transmitter T_s starts to send all this information in time x_s in each period, and the time in which the information reaches receiver number j is governed by a delay d_{sj}. Therefore, the time interval in which R_j receives information from T_s is the interval $[x_s + d_{sj}, x_s + d_{sj} + w_{sj}]$. Since the communication is periodic, the endpoints of the intervals are computed modulo 1, and the intervals are considered to be cyclic ones. At any given point of time, each R_j can receive information from at most one transmitter. Therefore, for each fixed j, the n cyclic closed intervals $[x_s + d_{sj}, x_s + d_{sj} + w_{sj}]$, $(1 \leq s \leq n)$, are required to be pairwise disjoint. A *feasible schedule* is a choice of $x_1, x_2, \ldots, x_n$ so that all members in each of these k families of intervals are indeed pairwise disjoint.

The problem considered is how large can w be, so that for any choice of numbers w_{sj} that satisfy (1), and for any choice of delays d_{sj}, there is always a feasible schedule. Obviously w cannot exceed $1/n$, since it may be the case that all transmitters have to communicate all their data to the first receiver, which will thus have to be able to allocate to them pairwise disjoint intervals in its rotating schedule. Our main result in this note is that this is tight. If $w = 1/n$ then there is always a feasible schedule. This is stated in the following theorem.

Theorem 1.1 *Let k, n be two integers, let $w_{sj}, 1 \leq s \leq n, 1 \leq j \leq k$ be non-negative reals satisfying*

$$w_s = \sum_{j=1}^{k} w_{sj} < 1/n \quad \textit{for every} \quad 1 \leq s \leq n, \tag{2}$$

and let d_{sj} be arbitrary non-negative reals. Then there are real numbers $x_1, x_2, \ldots, x_n$ so that for every j, $1 \leq j \leq k$, the n cyclic closed intervals $[x_s + d_{sj}, x_s + d_{sj} + w_{sj}]$, $(1 \leq s \leq n)$, where the endpoints are reduced modulo 1, are pairwise disjoint on the unit circle.

The (short) proof, presented in the next section, is algebraic. It is based on some simple properties of multivariate polynomials, and on a result in enumerative combinatorics known as the Dyson Conjecture. Interestingly (and unfortunately) the proof is nonconstructive in the sense that it provides no efficient algorithmic way of finding a feasible schedule $x_1, \ldots, x_n$ for given sets of time durations w_{sj} and delays d_{sj}.

It seems plausible that the theorem can be generalized to the case in which not all the quantities w_s are bounded by the same real:

Conjecture 1.2 *Let k, n be two integers, let $w_{sj}, 1 \leq s \leq n, 1 \leq j \leq k$ be non-negative reals and put $w_s = \sum_{j=1}^{k} w_{sj}$. Suppose that $\sum_s w_s < 1$, and let d_{sj} be arbitrary non-negative reals. Then there are real numbers $x_1, x_2, \ldots, x_n$ so that for every j, $1 \leq j \leq k$, the n cyclic closed intervals*

$[x_s + d_{sj}, x_s + d_{sj} + w_{sj}]$, $(1 \leq s \leq n)$, *where the endpoints are reduced modulo* 1*, are pairwise disjoint on the unit circle.*

The special case of this conjecture in which for every fixed s, the k quantities w_{sj} are equal, has been conjectured (in a slightly different language) by Haxell, Wilfong and Winkler [3]. The special case in which for every fixed j, the n quantities w_{sj} are equal, is a special case of Theorem 1.1. The very special case in which all nk quantities w_{sj} are equal (which follows, of course, from Theorem 1.1), can be proved in a simpler way as well, using a simple greedy approach.

2 The proof

The proof of Theorem 1.1 uses (a special case of) the following result proved in [1], where it is called *Combinatorial Nullstellensatz.*

Theorem 2.1 ([1]) *Let F be an arbitrary field, and let $P = P(y_1, \ldots, y_n)$ be a polynomial in $F[y_1, \ldots, y_n]$. Suppose the degree $deg(P)$ of P is $\sum_{i=1}^n t_i$, where each t_i is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^n y_i^{t_i}$ in P is nonzero. Then, if $S_1, \ldots, S_n$ are subsets of F with $|S_i| > t_i$, there are $z_1 \in S_1, z_2 \in S_2, \ldots, z_n \in S_n$ so that*

$$P(z_1, \ldots, z_n) \neq 0.$$

We also need the following result, known as the Dyson Conjecture, which has been proved in [2], [4])(see also [5] for a more combinatorial proof.)

Theorem 2.2 ([2], [4]) *The coefficient of the monomial $\prod_{s=1}^n y_s^{(n-1)c_s}$ in the polynomial*

$$\prod_{1 \leq r < s \leq n} (y_r - y_s)^{c_r + c_s}$$

is

$$(-1)^{c_2 + 2c_3 + \ldots + (n-1)c_n} \frac{(c_1 + c_2 + \ldots + c_n)!}{c_1! c_2! \ldots c_n!}.$$

Proof of Theorem 1.1: Given k, n and real numbers w_{sj}, d_{sj} satisfying (2), let $p \equiv 1 \ mod(n)$ be a large prime. For every $1 \leq s \leq n, 1 \leq j \leq k$, define

$$c'_{sj} = \lceil w_{sj} p \rceil + 2$$

and let b_{sj} be the nearest integer to $d_{sj} p$. By (2) and by Dirichlet's Theorem on primes in arithmetic progressions we can make sure the prime p is as large as needed to ensure that for every fixed s, $1 \le s \le n$,

$$\sum_{j=1}^{k} c'_{sj} \le (p-1)/n.$$

Let c_{sj} be positive integers satisfying $c_{sj} \ge c'_{sj}$ for all s, j and

$$\sum_{j=1}^{k} c_{sj} = (p-1)/n$$

for all s, $1 \le s \le n$.

We construct the feasible schedule in a discrete fashion, by splitting our circular time unit into p equal pieces, and by allocating appropriate intervals of consecutive pieces for each required transmision.

Consider the following polynomial in n variables $y_1, y_2, \ldots, y_n$ over the finite field Z_p:

$$P(y_1, y_2, \ldots, y_n) = \prod_{1 \le r < s \le n} \prod_{j=1}^{k} \prod_{\ell=-c_{rj}}^{c_{sj}-1} (y_r + b_{rj} - y_s - d_{sj} - \ell).$$

The degree of this polynomial is precisely

$$\sum_{1 \le r < s \le n} \sum_{j=1}^{k} (c_{rj} + c_{sj}) = \sum_{1 \le r < s \le n} \frac{2(p-1)}{n} = (n-1)(p-1).$$

The coefficient of the monomial

$$\prod y_i^{(p-1)(n-1)/n}$$

in this polynomial is precisely the coefficient of this monomial in the polynomial

$$\prod_{1 \le r < s \le n} \prod_{j=1}^{k} \prod_{\ell=-c_{rj}}^{c_{sj}-1} (y_r - y_s) = \prod_{1 \le r < s \le n} (y_r - y_s)^{2(p-1)/n}.$$

By Theorem 2.2 (with $c_s = (p-1)/n$ for all s) this coefficient is, up to a sign,

$$\frac{(p-1)!}{[(p-1)/n)!]^n}$$

which is not zero in Z_p.

By Theorem 2.1 (with $t_i = (p-1)(n-1)/n$ ($< p$) and $S_i = Z_p$ for all i), there are some $z_s \in Z_p$ such that

$$P(z_1, z_2, \ldots, z_n) \ne 0.$$

We can now define $x_s = z_s/p$ and observe that by the definition of the numbers c_{sj}, b_{sj} and the polynomial P, this is a feasible schedule. Indeed, for every $r < s$ and every j there is an $\epsilon \in [-1, 1]$ such that

$$x_r + d_{rj} - x_s - d_{sj} = \frac{1}{p}(z_r + b_{rj} - z_s - b_{sj} + \epsilon).$$

As P does not vanish in $(z_1, z_2, \ldots, z_n)$, the quantity $z_r + b_{rj} - z_s - b_{sj}$, reduced modulo p, does not lie in the open interval $(-c_{rj} - 1, c_{sj})$. It follows that the quantity $x_r + d_{rj} - x_s - d_{sj}$, reduced modulo 1, does not lie in the interval $(\frac{-c_{rj}}{p}, \frac{c_{sj}-1}{p})$, which contains the closed interval $[-w_{rj}, w_{sj}]$. Therefore, the intervals $[x_r + d_{rj}, x_r + d_{rj} + w_{rj}]$ and $[x_s + d_{sj}, x_s + d_{sj} + w_{sj}]$ are disjoint, as needed. This completes the proof. □

3 Remarks, extensions and the algorithmic aspects

As is usually the case with applications of the Combinatorial Nullstellensatz, the proof it provides is non-constructive, and supplies no efficient algorithm for finding a required feasible schedule for a given set of time durations w_{sj} and delays d_{sj}. The fact that here we use the relatively simple special case of the theorem in which all sets S_i are the whole field (namely, we simply use the fact that the polynomial is not identically zero), does not seem to help in finding a solution efficiently.

It is worth noting that if, for $n > 1$, we replace the assumption that for every fixed s, $\sum_j w_{sj} < \frac{1}{n}$ by the stronger assumption that for every fixed s, $\sum_j w_{sj} < \frac{1}{2n-2}$ then a trivial greedy algorithm will provide a feasible schedule, since we can simply determine the numbers x_s one by one. Indeed, if $s > 1$ and the values of x_r for all $r < s$ have already been determined, there is always room for x_s, as the measure of all forbidden values for it is at most $\sum_{r:r<s} \sum_{j=1}^{k} w_{rj} + (s-1) \sum_{j=1}^{k} w_{sj} < 2(s-1)\frac{1}{2n-2} \leq 1$. Similar reasoning applies to the more general Conjecture 1.2, whose statement becomes easy if we strengthen, for $n > 1$, the assumption $\sum_{s=1}^{n} w_s < 1$ to $\sum_{s=1}^{n} w_s < \frac{n}{2n-2}$. (Here the greedy solution is obtained by determining the numbers x_s one by one, according to a non-increasing order of the quantities w_s).

It is not difficult to extend the statement of Theorem 1.1 by using the full power of Theorem 2.1 to get the following result, which enables us to put some restrictions on the numbers x_s.

Theorem 3.1 *Let k, n be two integers, let $w_{sj}, 1 \leq s \leq n, 1 \leq j \leq k$ be non-negative reals and put $w_s = \sum_{j=1}^{k} w_{sj}$. Suppose that $\sum_s w_s < 1$. Let r_s be non-negative reals such that $(n-1)w_s + r_s < 1$ for each s, $1 \leq s \leq n$, and let d_{sj} be arbitrary non-negative reals. Then for any given measurable sets J_s in $[0, 1]$, where the measure of J_s is r_s, there are real numbers $x_1, x_2, \ldots, x_n \in [0, 1]$ so that*

$x_s \notin J_s$ *and so that for every* j, $1 \leq j \leq k$, *the* n *cyclic closed intervals* $[x_s + d_{sj}, x_s + d_{sj} + w_{sj}]$, $(1 \leq s \leq n)$, *where the endpoints are reduced modulo* 1, *are pairwise disjoint on the unit circle.*

The proof is essentially identical to that of Theorem 1.1, with the only change that here we apply Theorem 2.1 with each $S_i \subset Z_p$ defined using the set J_i. This result may be useful in certain online scenarios, where some transmissions have already been scheduled, and we wish to add additional ones without changing the existing schedule. Here, too, the proof is not algorithmic.

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