Disjoint Systems

Noga Alon *
and Benny Sudakov
Department of Mathematics
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel Aviv University, Tel Aviv, Israel

Abstract

A *disjoint system of type* $(\forall, \exists, k, n)$ is a collection $\mathcal{C} = \{\mathcal{A}_1, \ldots, \mathcal{A}_m\}$ of pairwise disjoint families of k-subsets of an n-element set satisfying the following condition. For every ordered pair $\mathcal{A}_i$ and $\mathcal{A}_j$ of distinct members of $\mathcal{C}$ and for every $A \in \mathcal{A}_i$ there exists a $B \in \mathcal{A}_j$ that does not intersect A. Let $D_n(\forall, \exists, k)$ denote the maximum possible cardinality of a disjoint system of type $(\forall, \exists, k, n)$. It is shown that for every fixed $k \geq 2$,

$$lim_{n \to \infty} D_n(\forall, \exists, k) \binom{n}{k}^{-1} = \frac{1}{2}.$$

This settles a problem of Ahlswede, Cai and Zhang. Several related problems are considered as well.

1 Introduction

In Extremal Finite Set Theory one is usually interested in determining or estimating the maximum or minimum possible cardinality of a family of subsets of an n element set that satisfies certain properties. See [5], [7] and [9] for a comprehensive study of problems of this type. In several recent papers (see [3], [1],[2]), Ahlswede, Cai and Zhang considered various extremal problems that study the maximum or minimum possible cardinality of a collection of families of subsets of an n-set, that satisfies certain properties. They observed that many of the classical extremal problems dealing with families of sets suggest numerous intriguing questions when one replaces the notion of a family of sets by the more complicated one of a collection of families of sets.

*Research supported in part by a United States Israel BSF Grant

In the present note we consider several problems of this type that deal with disjoint systems. Let $N = \{1, 2, \ldots, n\}$ be an n element set, and let $\mathcal{C} = \{\mathcal{A}_1, \ldots, \mathcal{A}_m\}$ be a collection of pairwise disjoint families of k-subsets of N. $\mathcal{C}$ is a *disjoint system of type* $(\exists, \forall, k, n)$ if for every ordered pair $\mathcal{A}_i$ and $\mathcal{A}_j$ of distinct members of $\mathcal{C}$ there exists an $A \in \mathcal{A}_i$ which does not intersect any member of $\mathcal{A}_j$. Similarly, $\mathcal{C}$ is a *disjoint system of type* $(\forall, \exists, k, n)$ if for every ordered pair $\mathcal{A}_i$ and $\mathcal{A}_j$ of distinct members of $\mathcal{C}$ and for every $A \in \mathcal{A}_i$ there exists a $B \in \mathcal{A}_j$ that does not intersect A. Finally, $\mathcal{C}$ is a *disjoint system of type* $(\exists, \exists, k, n)$ if for every ordered pair $\mathcal{A}_i$ and $\mathcal{A}_j$ of distinct members of $\mathcal{C}$ there exists an $A \in \mathcal{A}_i$ and a $B \in \mathcal{A}_j$ that does not intersect A.

Let $D_n(\exists, \forall, k)$ denote the maximum possible cardinality of a disjoint system of type $(\exists, \forall, k, n)$. Let $D_n(\forall, \exists, k)$ denote the maximum possible cardinality of a disjoint system of type $(\forall, \exists, k, n)$ and let $D_n(\exists, \exists, k)$ denote the maximum possible cardinality of a disjoint system of type $(\exists, \exists, k, n)$. Trivially, for every n,

$$D_n(\exists, \forall, 1) = D_n(\forall, \exists, 1) = D_n(\exists, \exists, 1) = n.$$

It is easy to see that every disjoint system of type $(\exists, \forall, k, n)$ is also a system of type $(\forall, \exists, k, n)$, and every system of type $(\forall, \exists, k, n)$ is also of type $(\exists, \exists, k, n)$. Therefore, for every $n \geq k$

$$D_n(\exists, \forall, k) \leq D_n(\forall, \exists, k) \leq D_n(\exists, \exists, k).$$

In this note we determine the asymptotic behaviour of these three functions for every fixed k, as n tends to infinity.

Theorem 1.1 *For every* $k \geq 2$

$$lim_{n \to \infty} D_n(\exists, \forall, k) \binom{n}{k}^{-1} = \frac{1}{k+1}.$$

Theorem 1.2 *For every* $k \geq 2$

$$lim_{n \to \infty} D_n(\forall, \exists, k) \binom{n}{k}^{-1} = \frac{1}{2}.$$

Corollary 1.3 *For every* $k \geq 2$

$$lim_{n \to \infty} D_n(\exists, \exists, k) \binom{n}{k}^{-1} = \frac{1}{2}.$$

Theorem 1.1 settles a conjecture of Ahlswede, Cai and Zhang [2], who proved it for $k = 2$ [1]. The main tool in its proof, presented in Section 2, is a result of Frankl and Füredi [8]. The proof of Theorem 1.2, which settles another question raised in [2] and proved for $k = 2$ in [1], is more complicated and combines combinatorial and probabilistic arguments. This proof and the simple derivation of Corollary 1.3 from its assertion are presented in Section 3.

2 Hypergraph decomposition and disjoint systems

In this section we prove Theorem 1.1. A *k-graph* is a hypergraph in which every edge contains precisely k vertices. We need the following result of Frankl and Füredi.

Lemma 2.1 ([8]) *Let $H = (U, \mathcal{F})$ be a fixed k-graph with $|\mathcal{F}| = f$ edges. Then one can place*

$$(1 - o(1))\binom{n}{k}/f$$

copies $H_1 = (U_1, \mathcal{F}_1), H_2 = (U_2, \mathcal{F}_2), \ldots$ of H into a complete k-graph on n vertices such that $|U_i \cap U_j| \leq k$ for all $i \neq j$, and if $|U_i \cap U_j| = k$ and $U_i \cap U_j = B$ then $B \notin \mathcal{F}_i$, $B \notin \mathcal{F}_j$. Here the $o(1)$ term tends to zero as n tends to infinity. $\Box$

Proof of Theorem 1.1 The lower bound for $D_n(\exists, \forall, k)$ is a direct corollary of Lemma 2.1. Let $(U, \mathcal{A})$ be the k-graph consisting of $k+1$ pairwise disjoint edges. By the lemma we can place $(1 - o(1))\binom{n}{k}/(k+1)$ edge disjoint copies of this graph into a complete k-graph on n vertices, so that any two copies will have at most k common vertices. Therefore if we take the edges of each copy as a family, we get $(1-o(1))\binom{n}{k}/(k+1)$ pairwise disjoint families which forms a disjoint system. Let $H_1 = (U_1, \mathcal{A}_1)$ and $H_2 = (U_2, \mathcal{A}_2)$ be two such families. Since $|U_1 \cap U_2| \leq k$ and the family H_1 consists of $k+1$ pairwise disjoint sets, there is a set $A \in \mathcal{A}_1$ which does not contain any point of $U_1 \cap U_2$. This A does not intersect any set of the family H_2. Therefore, our disjoint system is of type $(\exists, \forall, k, n)$.

We next establish an upper bound for $D_n(\exists, \forall, k)$. Let $\mathcal{C} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots\}$ be a disjoint system of type $(\exists, \forall\, k, n)$. We denote by $N_1, |N_1| = n_1$, the set of all families of $\mathcal{C}$ containing one element, by $N_2, |N_2| = n_2$, the set of those containing from two up to k elements, and by $N_3, |N_3| = n_3$, the set of those containing more than k elements. Since sets in any two one-element families are disjoint we have $n_1 \leq n/k$. Let $\mathcal{A}_i = \{A_1, \ldots, A_t\}$ be a family with $2 \leq t \leq k$ elements. By the definition of a system of type $(\exists, \forall, k, n)$ we conclude that any set B with the properties :

$$|B| = k \ ; \ \ B \subset \cup_{j=1}^{t} A_j \ ; \ \ B \cap A_j \neq \emptyset \ \ \forall j \qquad (1)$$

cannot be used as an element of any other family. We next bound the number of such sets B from below. Choose not necessarily distinct $a_j \in A_j$ for $2 \leq j \leq t$ such that $a_j \notin A_1$. Put $L = \{a_2, \ldots, a_t\}$, then $|L| \leq k-1$. Let $\mathcal{L}$ be the family of all sets of the form $L_r = L \cup Y_r$ where Y_r ranges over all $(k - |L|)$ element subsets of A_1. Clearly each such L_r satisfies the properties (1). Moreover $L_r \neq A_1$ and $|\mathcal{L}| \geq k$.

We claim that no k-set can satisfy the properties (1) for two or more families from $\mathcal{C}$. Indeed assume this is false. Let B be a set which satisfies the properties (1) for two families $\mathcal{A} = \{A_1, \ldots, A_l\}$

and $\mathcal{F} = \{F_1, \ldots, F_m\}$. By the definition of a disjoint system of type $(\exists, \forall, k, n)$ there exists a set $A_i \in \mathcal{A}, 1 \leq i \leq l$ such that $A_i \cap F_j = \emptyset$ for all j. Since by (1) $B \subseteq \cup_{j=1}^m F_j$ we conclude that $B \cap A_i = \emptyset$, contradicting (1) and proving our claim.

Therefore with each family $\mathcal{A}_i$ in N_2 we can associate $k+1$ sets (k-sets of the form L_r as above together with the set A_1) which cannot be associated with any other family and are not members of any other family. In addition each family in N_3 contains at least $k+1$ k-sets. This implies that

$$(k+1)n_2 + (k+1)n_3 \leq \binom{n}{k}.$$

Therefore $(n_2 + n_3) \leq \binom{n}{k}/(k+1)$. Together with the fact that $n_1 \leq n/k$ we conclude that $D_n(\exists, \forall, k, n) \leq \frac{n}{k} + \binom{n}{k}/(k+1)$ completing the proof. $\square$

3 Random graphs and disjoint systems

In this section we prove Theorem 1.2. We need the following two probabilistic lemmas.

Lemma 3.1 (Chernoff, see e.g. [4], Appendix A) *Let X be a random variable with the binomial distribution $B(n, p)$. Then for every $a > 0$ we have*

$$Pr(|x - np| > a) < 2e^{-2a^2/n}. \ \square$$

Let L be a graph-theoretic function. L satisfies *the Lipschitz condition* if for any two graphs H, H' on the same set of vertices that differ only in one edge we have $|L(H) - L(H')| \leq 1$. Let $G(n, p)$ denote, as usual, the random graph on n labeled vertices in which every pair, randomly and independently, is chosen to be an edge with probability p. (See, e.g., [6].)

Lemma 3.2 ([4], Chapter 7) *Let L be a graph-theoretic function satisfying the Lipschitz condition and let $\mu = E[L(G)]$ be the expectation of $L(G)$, where $G = G(n, p)$. Then for any $\lambda > 0$*

$$Pr(|L(G) - \mu| > \lambda\sqrt{m}] < 2e^{-\lambda^2/2}$$

where $m = \binom{n}{2}$. $\square$

Proof of Theorem 1.2 Let n_1 be the number of families containing only one element. The same argument as in the proof of Theorem 1.1 shows that $n_1 \leq n/k$. This settles the required upper bound for $D_n(\forall, \exists, k)$, since all other families contain at least 2 sets.

We prove the lower bound using probabilistic arguments. We show that for any $\varepsilon > 0$ there are at least $\frac{1}{2}(1-\varepsilon)\binom{n}{k}$ families which form a disjoint system of type $(\forall, \exists, k, n)$, provided n is sufficiently large (as a function of ε and k). We first outline the main idea in the (probabilistic) construction and then describe the details. Let $G = G(n,p)$ be a random graph, where p is a constant, to be specified later, which is very close to 1. We use this graph to build another random graph G_1, whose vertices are all k-cliques in G. Two vertices of G_1 are adjacent if and only if the induced subgraph on the corresponding k-cliques in G is the union of two vertex disjoint k-cliques with no edges between them. Following the standard terminology in the study of random graphs we say that an event holds *almost surely* if the probability it holds tends to 1 as n tends to infinity. We will prove that almost surely the following two events happen. First, the number of vertices in G_1 is greater than $(1-\varepsilon/2)\binom{n}{k}$. Second, G_1 is almost regular, i.e., for every (small) $\delta > 0$ there exists a (large) number d such that the degree $d(x)$ of any vertex x of G_1 satisfies $(1-\delta)d < d(x) < (1+\delta)d$, provided n is sufficiently large.

Suppose $G_1 = (V,E)$ satisfies these properties. By Vizing's Theorem [10], the chromatic index $\chi'(G_1)$ of G_1 satisfies $\chi'(G_1) \le (1+\delta)d + 1$. Since for any $x \in G_1$ we have $d(x) \ge (1-\delta)d$, the number of edges $|E|$ of G_1 is at least $\frac{(1-\delta)d|V|}{2}$. Hence there exists a matching in G_1 which contains at least $\frac{(1-\delta)d|V|}{2}/\chi'(G_1) \sim \frac{(1-\delta)|V|}{2(1+\delta)}$ edges. This matching covers almost all vertices of G_1, as δ is small, providing a system of pairs of k-sets covering almost all the $\binom{n}{k}$ k-sets. Taking each pair as a family we have a disjoint system of size at least $\frac{1}{2}(1-\varepsilon)\binom{n}{k}$ and ε can be made arbitrarily small for all n sufficiently large.

We next show that the resulting system is a disjoint system of type $(\forall, \exists, k, n)$. Assume this is false and let $\mathcal{A} = \{A_1, A_2\}$ and $\mathcal{B} = \{B_1, B_2\}$ be two pairs where $A_1 \cap B_i \neq \emptyset$ for $i = 1, 2$. Choose $x_1 \in A_1 \cap B_1$ and $x_2 \in A_1 \cap B_2$. Since x_1 and x_2 belong to A_1 they are adjacent in $G = G(n,p)$. However, $x_1 \in B_1$, $x_2 \in B_2$ and this contradicts the fact that the subgraph of G induced on $B_1 \cup B_2$ has no edges between B_1 and B_2. Thus the system is indeed of type $(\forall, \exists, k, n)$ and

$$D_n(\forall, \exists, k) > \frac{1}{2}(1-\varepsilon)\binom{n}{k}$$

for every $\varepsilon > 0$, provided $n > n_0(k, \varepsilon_1)$, as needed.

The proof that indeed G_1 has the required properties almost surely will be deduced from the following two statements.

Fact 1. $G = G(n,p)$ satisfies the following condition almost surely. For every set X of k vertices of G, the number of vertices which do not have any neighbor in X is

$$(1+o(1))(1-p)^k(n-k),$$

where here the $o(1)$ term tends to zero as n tends to infinity.

Fact 2. For any $c > 0$, if n is sufficently large, $G = G(n,p)$ satisfies the following condition almost surely. For every set Y of n_1 vertices of G, where $cn \le n_1 \le n$, the number of k-cliques of the induced subgraph of G on Y is close to its expectation, i.e., is

$$(1+o(1))\binom{n_1}{k}p^{\binom{k}{2}},$$

where the $o(1)$ term tends to zero as n tends to infinity.

The proof of Fact 1 is a standard application of Lemma 3.1 and is thus left to the reader.

Proof of Fact 2. Let $H(Y,p)$ denote the induced subgraph of $G = G(n,p)$ on a fixed set Y of vertices, where $|Y| = n_1$. Let L be the graph-theoretic function given by

$$L(H') = \frac{1}{\binom{n_1}{k-2}} N(H'),$$

where H' is a graph on Y and $N(H')$ denotes the number of k-cliques in H'.

The expected value of $L(H(Y,p))$ is easily seen to be $\mu(L) = \frac{1}{\binom{n_1}{k-2}}\binom{n_1}{k}p^{\binom{k}{2}}$, and the expected value of $N = N(H(Y,p))$ is $\mu(N) = \binom{n_1}{k}p^{\binom{k}{2}}$. By the definition of L, if H_1 and H_2 are two graphs on Y which differ only in one edge than $|L(H_1) - L(H_2)| \le 1$, since the number of k-cliques of $H(Y,p)$ containing an edge is at most $\binom{n_1}{k-2}$. Thus, by Lemma 3.2

$$Pr[\ |L(H(Y,p)) - \mu(L)| > n^{\frac{3}{4}}\sqrt{\binom{n_1}{2}}\ ] < 2e^{-\frac{n^{\frac{3}{2}}}{2}}.$$

Consequently,

$$Pr[\ |N(H(Y,p)) - \mu(N)| > n^{\frac{3}{4}}\binom{n_1}{k-2}\sqrt{\binom{n_1}{2}}\ ] < 2e^{-\frac{n^{\frac{3}{2}}}{2}}.$$

Since k and p are constants, and $n_1 \ge cn$ we conclude that

$$n^{\frac{3}{4}}\binom{n_1}{k-2}\sqrt{\binom{n_1}{2}} = \gamma\binom{n_1}{k}p^{\binom{k}{2}} = \gamma\mu(N),$$

where $\gamma = \gamma(n, n_1, k, p)$ tends to 0 as n tends to infinity.

The total number of possible sets Y is clearly less than 2^n. Hence, the probability that for some Y, $N(H(Y,p))$ deviates by more than $\gamma\mu(N(H(Y,p))$ from its expectation is less than $2^{n+1}e^{-\frac{n^{\frac{3}{2}}}{2}}$, which tends to zero as n tends to infinity. This completes the proof of Fact 2.

Returning to the proof of the theorem consider a k-clique X in G. The degree d of X as a vertex of G_1 is the number of k-cliques in the induced subgraph of G on the set of all vertices which have no neighbors in X. By Facts 1 and 2 each such degree is almost surely

$$(1+o(1))\binom{n_1}{k}p^{\binom{k}{2}}$$

where $n_1 = (1+o(1))(1-p)^k(n-k)$. Therefore, almost surely G_1 is almost regular and the degrees of its vertices tend to infinity with n.

In a similar manner, Fact 2 applied to the set Y of all vertices of G implies that the number of k-cliques in $G = G(n,p)$ (which is the number of vertices of G_1) is almost surely

$$(1+o(1))\binom{n}{k}p^{\binom{k}{2}}.$$

Fix $p < 1$ so that $p^{\binom{k}{2}} > 1 - \frac{\varepsilon}{4}$ for the required ε. With this p, almost surely the number of vertices in G_1 is more than

$$(1-\varepsilon/2)\binom{n}{k},$$

as needed. Therefore, our procedure produces, with high probability, a disjoint system of the required type with at least $\frac{1}{2}(1-\varepsilon)\binom{n}{k}$ pairs, completing the proof. $\square$

Remark. By combining our method here with the technique of [8] we can prove the following extension of Lemma 2.1, which may be useful in further applications. Since the proof is similar to the last one, we omit the details.

Proposition 3.3 *Let $H = (U, \mathcal{F})$ be a fixed k-graph with $|\mathcal{F}| = f$ edges and let g denote the maximum cardinality of an intersection of two distinct edges of H. Then one can place*

$$(1-o(1))\binom{n}{k}/f$$

pairwise edge-disjoint copies $H_1 = (U_1, \mathcal{F}_1), H_2 = (U_2, \mathcal{F}_2), \ldots$ of H into a complete k-graph on n vertices such that $|U_i \cap U_j| \leq k$ for all $i \neq j$, and such that if for some $i \neq j$ there is an $F_j \in \mathcal{F}_j$ so that $|F_j \cap U_i| \geq g+2$ then there is an $F_i \in \mathcal{F}_i$ so that $F_j \cap U_i \subset F_i$. Here the $o(1)$ term tends to zero as n tends to infinity.

Proof of Corollary 1.3 Let n_1 be the number of one element families in a disjoint system of type $(\exists, \exists, k, n)$. The trivial argument used in the proofs of Theorems 1.1 and 1.2 shows that $n_1 \leq n/k$ and thus implies that

$$D_n(\exists, \exists, k) \leq \frac{n}{k} + \frac{1}{2}\binom{n}{k}.$$

As observed in Section 1, $D_n(\forall, \exists, k) \leq D_n(\exists, \exists, k)$ and hence, by Theorem 1.2, the desired result follows. $\square$

Acknowledgment We would like to thank the referees for several helpful comments and suggestions.

References

[1] R. Ahlswede, N. Cai and Z. Zhang, *A new direction in extremal theory for graphs*, to appear in JCISS.

[2] R. Ahlswede, N. Cai and Z. Zhang, *Higher level extremal problems*, to appear.

[3] R. Ahlswede and Z. Zhang, *On cloud-antichains and related configurations*, Discrete Math. 85 (1990), 225-245.

[4] N. Alon and J. H. Spencer, **The Probabilistic Method**, Wiley, 1991.

[5] I. Anderson, **Combinatorics of Finite Sets**, Clarendon Press, 1987.

[6] B. Bollobás, **Random Graphs**, Academic Press, 1985.

[7] B. Bollobás, **Combinatorics**, Cambridge University Press, 1986.

[8] P. Frankl and Z. Füredi, *Colored packing of sets in combinatorial desgin theory*, Annals of Discrete Math. 34 (1987), 165-178.

[9] Z. Füredi, *Matchings and covers in hypergraphs*, Graphs and Combinatorics 4 (1988), 115-206.

[10] V. G. Vizing, *Coloring the vertices of a graph in prescribed colors* (in Russian), Diskret. Analiz. No. 29, Metody Diskret. Anal. v. Teorii Kodov i Shem 101 (1976), 3-10.