A Note on Competitive Diffusion Through Social Networks

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Abstract

We introduce a game-theoretic model of diffusion of technologies, advertisements, or influence through a social network. The novelty in our model is that the players are interested parties outside the network. We study the relation between the diameter of the network and the existence of pure Nash equilibria in the game. In particular, we show that if the diameter is at most two then an equilibrium exists and can be found in polynomial time, whereas if the diameter is greater than two then an equilibrium is not guaranteed to exist.

1 Introduction

Social networks such as Facebook and Twitter are modern focal points of human interaction. The pursuit of insights into the nature of this interaction calls for a game-theoretic analysis. Indeed, a number of papers (see, e.g., [5]) investigate variations on the following setting. The social network is represented by an undirected graph, where the vertices are users and edges connect users who are in a social relationship. Suppose, for example, that there are several competing applications, e.g., voice over IP systems, that are not interoperable. The users play a *coordination game*, where if two neighbors adopt the same system they get some reward that is based on the inherent quality of the system. The goal is to study the diffusion of technologies through the social network. The point of view here is completely decentralized, and the players in the game are the users of the social network.

We propose a different, global point of view regarding the incentives that govern the diffusion process. Suppose we have several firms that would like to advertise competing products via "viral marketing". Each firm initially targets a small subset of users, in the hope that the rumor about its product would spread throughout the network. However, a user that adopts one product is reluctant to adopt another, hence the campaign of one firm negatively affects the success of another firm's campaign. To the best of our knowledge our model is the first game-theoretic model to deal with

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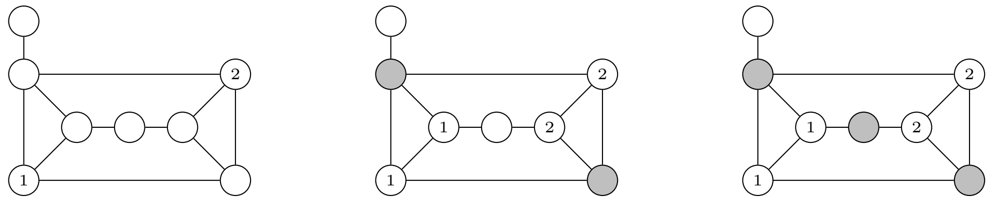

(a) Time 1. (b) Time 2. (c) Time 3, the process terminates.

Figure 1: An illustration of the diffusion process, with $N = \{1, 2\}$.

the incentives of interested parties outside the social network. Note that some previous papers did consider the problem of choosing an influential set of users as an optimization problem (see, e.g., [6]), but not in a competitive game-theoretic setting. Other papers, which deal with Voronoi games on graphs, provide a game-theoretic study of a facility location problem that does not involve a diffusion process, where rather each vertex is assigned to the closest agent and the utility of an agent is the number of vertices assigned to it (see, e.g., [3, 7]).

The model. Let $G = \langle V, E\rangle$ be an undirected graph. Furthermore, let $N = \{1, \ldots, n\}$ be the set of agents (the interested parties). The diffusion process unfolds as follows. There are $n + 2$ colors: a color for each agent $i \in N$, as well as two additional colors: white and gray. Initially, at time 1, some of the vertices are colored in the colors of N, while the others are white. At time $t + 1$ each white vertex that has neighbors colored in color i, but does not have neighbors colored in color j for any $j \in N \setminus \{i\}$, is colored in color i. A white vertex that has two neighbors colored by two distinct colors $i, j \in N$ is colored gray. In other words, we assume that if two agents compete for a user at the same time they "cancel out" and the user is removed from the game. The process continues until it reaches a fixed point, that is, all the remaining white vertices are unreachable due to gray vertices. See Figure 1 for an illustration of the diffusion process.

A game $\Gamma = \langle G, N\rangle$ is induced by a graph G, representing the underlying social network, and the set of agents N. The *strategy space* of each agent is the set of vertices V in the graph, that is, each agent i selects a single node that is colored in color i at time 1. Note that if two or more agents select the same vertex at time 1 then that vertex becomes gray. A *strategy profile* is a vector $\mathbf{x} = \langle x_1, \ldots, x_n\rangle \in V^n$, where $x_i \in V$ is the initial vertex selected by agent i. We also denote $\mathbf{x}_{-i} = \langle x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\rangle$.

Given a strategy profile $\mathbf{x} \in V^n$, the utility of agent $i \in N$, denoted $U_i(\mathbf{x})$, is the number of nodes that are colored in color i when the diffusion process terminates. For instance, in the example given in Figure 1 the utility of each of the agents is two. A strategy profile $\mathbf{x}$ is a (pure strategy) *Nash equilibrium* of the game Γ if an agent cannot benefit from unilaterally deviating to a different strategy, i.e., for every $i \in N$ and $x'_i \in V$ it holds that $U_i(x'_i, \mathbf{x}_{-i}) \le U_i(\mathbf{x})$.

Our results. Given a graph G and $u, v \in V$, let $d(u, v)$ be the length of the shortest path between u and v (in terms of the number of edges). The *diameter* of the graph, denoted $D(G)$, is the maximum distance between a pair of vertices, that is, $D(G) = \max_{u,v\in V} d(u, v)$.

Our investigation focuses on the relation between the diameter of the graph and the existence of Nash equilibria in the induced diffusion game. Indeed, if we can find a Nash equilibrium then we can often predict the behavior of the agents and the outcome of this competitive diffusion process, or, alternatively, advise the agents how to play. Our first theorem is the following.

Theorem 2.1. Every game $\Gamma = \langle G, N \rangle$ where $D(G) \leq 2$ admits a Nash equilibrium. Furthermore, an equilibrium can be found in polynomial time.

Note that a random graph on n labeled vertices where each edge appears with probability p, usually denoted $G(n, p)$, has diameter at most two with high probability whenever $p \geq \sqrt{(c \ln n)/n}$ for $c > 2$ (see, e.g., [2] for more details about the diameter of random graphs). In particular (by taking $p = 1/2$) almost all graphs over n vertices have diameter at most two. Finally, social networks typically have a very small diameter. Therefore, it can be argued that assuming a diameter of two is not very restrictive.

It is now natural to ask whether the existence of Nash equilibria can also be guaranteed for diameters larger than two. It is not too difficult to construct a graph with diameter four that does not admit an equilibrium. Our second theorem gives a negative answer even with respect to diameter three.

Theorem 2.2. Let $N = \{1, 2\}$. There exists a graph G with $D(G) = 3$ such that the game $\Gamma = \langle G, N \rangle$ does not admit a Nash equilibrium.

The construction in the proof of Theorem 2.2 can easily be extended to a larger number of agents or to any (finite or infinite) diameter greater than three.

Discussion. In order to facilitate the game-theoretic analysis we consider a very simple model of diffusion. In particular, conflicts are deterministically resolved by introducing gray vertices, and each agent initially selects just one vertex. Richer (probabilistic) models of diffusion through a social network exist in the literature, e.g., [6, 4]. On the other hand, the assumption of discrete time steps is quite common.

Theorem 2.1 implies that with high probability a random graph (even a relatively sparse one) induces a game that admits a Nash equilibrium. However, social networks are normally not completely random, but rather often exhibit structure. Ideally one would be able to extend our result by showing that under a convincing random graph model of social networks (see, e.g., [1, 9]) the induced game admits a Nash equilibrium with high probability.

2 Proofs

We begin by proving Theorem 2.1; we subsequently discuss some implications of the proof.

Theorem 2.1. *Every game $\Gamma = \langle G, N \rangle$ where $D(G) \leq 2$ admits a Nash equilibrium. Furthermore, an equilibrium can be found in polynomial time.*

Proof. If $D(G) \leq 1$ then the graph is a clique and the theorem follows trivially. Therefore, we may assume that $D(G) = 2$.

Given a profile $\mathbf{x} \in V^N$, let $P(\mathbf{x}) = |\{\langle i, j \rangle : d(x_i, x_j) = 1\}|$, that is, the number of pairs with distance one from each other. Furthermore, denote the neighborhood of vertex $u \in V$ by $N_u = \{v : d(u, v) \leq 1\}$, and let $N(\mathbf{x}) = \bigcup_{i=1}^{n} N_{x_i}$. Consider the *potential function*

$$\Phi(\mathbf{x}) = |N(\mathbf{x})| \cdot n + P(\mathbf{x}) \quad .$$

It is sufficient to show that for every $\mathbf{x} \in V^n$, $i \in N$, and $x'_i \in V$,

$$U_i(x'_i, \mathbf{x}_{-i}) > U_i(\mathbf{x}) \Rightarrow \Phi(x'_i, \mathbf{x}_{-i}) > \Phi(\mathbf{x}) \quad . \tag{1}$$

Indeed, given Equation (1) it clearly holds that any strategy profile $\mathbf{x} \in V^n$ that maximizes $\Phi(\mathbf{x})$ must be a Nash equilibrium. Moreover, in order to find one such profile we may start from some preference profile, and in each step attempt to find a profitable deviation for one of the agents. We terminate if there is no such deviation (which, by definition, means that we have found a Nash equilibrium). This algorithm terminates after a polynomial number of steps since $\Phi(\mathbf{x})$ is bounded from above by $n|V| + n^2$ for every $\mathbf{x}$, and by Equation (1) every profitable deviation by an agent increases the value of the potential function by at least one.

We turn to proving Equation (1). If the diameter of the graph is two then vertices can only be colored by an agent $i \in N$ at time 1 or 2. Specifically, the vertices colored by agent i are roughly the vertices in the neighborhood of x_i that are not neighbors of x_j for some $j \in N \setminus \{i\}$ (since these vertices are either gray or colored by j). Formally, define

$$A_i = \{\mathbf{x} : \exists j \in N \setminus \{i\} \ s.t. \ d(x_i, x_j) = 1\} \quad .$$

Assuming that $x_i \neq x_j$ for all $i \neq j$, the utility of agent i under the strategy profile $\mathbf{x} \in V^n$ is

$$U_i(\mathbf{x}) = |N_{x_i}| - |\bigcup_{j \neq i} (N_{x_i} \cap N_{x_j})| + \chi_{A_i}(\mathbf{x}) \quad ,$$

where χ_{A_i} is the indicator function that returns 1 if $\mathbf{x} \in A_i$ and 0 otherwise. The rightmost term is required since even if x_i is a neighbor of some x_j, it is still colored by agent i at time 1, but is nevertheless included in the middle term.

Now, suppose $U_i(x'_i, \mathbf{x}_{-i}) > U_i(\mathbf{x})$. It follows that

$$|N_{x'_i}| - |\bigcup_{j \neq i} (N_{x'_i} \cap N_{x_j})| + \chi_{A_i}(x'_i, \mathbf{x}_{-i}) > |N_{x_i}| - |\bigcup_{j \neq i} (N_{x_i} \cap N_{x_j})| + \chi_{A_i}(\mathbf{x}) \quad . \tag{2}$$

Since χ_{A_i} is a Boolean function, this implies that

$$|N_{x'_i}| - |\bigcup_{j \neq i} (N_{x'_i} \cap N_{x_j})| \geq |N_{x_i}| - |\bigcup_{j \neq i} (N_{x_i} \cap N_{x_j})| \quad . \tag{3}$$

We distinguish between two cases. If Equation (3) holds as a strict inequality then

$$\begin{aligned} |\bigcup_{j \neq i} N_{x_j} \cup N_{x'_i}| &= |\bigcup_{j \neq i} N_{x_j}| + |N_{x'_i}| - |\bigcup_{j \neq i} (N_{x'_i} \cap N_{x_j})| > |\bigcup_{j \neq i} N_{x_j}| + |N_{x_i}| - |\bigcup_{j \neq i} (N_{x_i} \cap N_{x_j})| \\ &= |\bigcup_{j \neq i} N_{x_j} \cup N_{x_i}| \quad , \end{aligned}$$

which implies that $|N(x'_i, \mathbf{x}_{-i})| \geq |N(\mathbf{x})| + 1$. In addition, a deviation of a single agent can decrease the number of adjacent pairs of agents by at most $n - 1$, i.e., $P(x'_i, \mathbf{x}_{-i}) > P(\mathbf{x}) - n$. We conclude that

$$\Phi(x'_i, \mathbf{x}_{-i}) = |N(x'_i, \mathbf{x}_{-i})|n + P(x'_i, \mathbf{x}_{-i}) \geq |N(\mathbf{x})|n + n + P(x'_i, \mathbf{x}_{-i}) > |N(\mathbf{x})|n + n + P(\mathbf{x}) - n = \Phi(\mathbf{x}) \quad .$$

Otherwise, Equation (3) holds as an equality, and hence $|N(x'_i, \mathbf{x}_{-i})| = |N(\mathbf{x})|$. It then follows from Equation (2) that $\chi_{A_i}(x'_i, \mathbf{x}_{-i}) > \chi_{A_i}(\mathbf{x})$. That is, agent i has no neighbors among $\mathbf{x}_{-i}$ under x_i but has at least one neighbor under x'_i. Thus the number of neighbors of agent i increases and the number of neighbors of agents $j \in N \setminus \{i\}$ does not decrease, i.e., $P(x'_i, \mathbf{x}_{-i}) > P(\mathbf{x})$. We conclude that

$$\Phi(x'_i, \mathbf{x}_{-i}) = |N(x'_i, \mathbf{x}_{-i})|n + P(x'_i, \mathbf{x}_{-i}) = |N(\mathbf{x})|n + P(x'_i, \mathbf{x}_{-i}) > |N(\mathbf{x})|n + P(\mathbf{x}) = \Phi(\mathbf{x}) \ .$$

This establishes Equation (1), and hence completes the proof of the theorem. □

What the proof of Theorem 2.1 essentially shows is that when the diameter of the graph is two the diffusion game is a *potential game* [8]; specifically, a function that satisfies (1) is known as a *generalized ordinal potential function.* Potential games have the property that *better response dynamics* converge to a Nash equilibrium; in other words, if at every stage the agents simply behave myopically, that is, some agent deviates to a more profitable strategy, then they will eventually reach an equilibrium.

We are now ready to prove our second theorem.

Theorem 2.2. *Let $N = \{1, 2\}$. There exists a graph G with $D(G) = 3$ such that the game $\Gamma = \langle G, N\rangle$ does not admit a Nash equilibrium.*

Proof. We first give our construction, then establish that it has diameter three and that it does not admit a Nash equilibrium.

The construction. Let $G = \langle V, E\rangle$ be defined as follows. The vertices of the graph are

$$V = \{v_1, \ldots, v_6\} \cup C_1 \cup C_2 \cup C_3 \ ,$$

where for $i = 1, 2, 3$, $C_i = C_{i1} \cup \cdots \cup C_{i5}$. Each C_{ij} contains ten vertices, that is, $|V| = 156$.

The edges of the graph are defined as follows. Each C_i, for $i = 1, 2, 3$, is a clique. There is an edge $\langle v_1, u\rangle$ for every $u \in C_{11} \cup C_{12} \cup C_{13} \cup C_{21} \cup C_{22} \cup C_{23}$; an edge $\langle v_2, u\rangle$ for every $u \in C_{11} \cup C_{14} \cup C_{15} \cup C_{21} \cup C_{24} \cup C_{25}$; an edge $\langle v_3, u\rangle$ for every $u \in C_{11} \cup C_{12} \cup C_{14} \cup C_{31} \cup C_{32} \cup C_{33}$; an edge $\langle v_4, u\rangle$ for every $u \in C_{11} \cup C_{13} \cup C_{15} \cup C_{31} \cup C_{34} \cup C_{35}$; an edge $\langle v_5, u\rangle$ for every $u \in C_{21} \cup C_{22} \cup C_{24} \cup C_{31} \cup C_{32} \cup C_{34}$; an edge $\langle v_6, u\rangle$ for every $u \in C_{21} \cup C_{23} \cup C_{25} \cup C_{31} \cup C_{33} \cup C_{35}$. An illustration of the graph G is given as Figure 2.

We refer to the vertices $v_1, \ldots, v_6$ as *hubs*; we say that v_1 and v_2 are *parallel* hubs, and so are v_3 and v_4, v_5 and v_6. If the hub v_i is connected by an edge to some of the vertices of clique C_j, we say that v_i is adjacent to C_j; for instance, v_1 and C_1 are adjacent, whereas v_1 and C_3 are not.

The construction possesses the following important properties:

1. Let v_i and v_j be two parallel hubs that are adjacent to a clique C_k. Then $(N_{v_i} \setminus N_{v_j}) \cap C_k$ contains exactly two of the sets C_{kl}, $l = 1, \ldots, 5$.

2. Let v_i and v_j be two nonparallel hubs that are adjacent to a clique C_k. Then $(N_{v_i} \setminus N_{v_j}) \cap C_k$ contains exactly one of the sets C_{kl}, $l = 1, \ldots, 5$.

Note that the construction is essentially symmetric with respect to the hubs.

G has diameter 3. Using Figure 2, it is easy to verify that G has diameter 3. For example, a path from v_1 to $u \in C_{32}$ is given by $v_1 \to w \to v_3 \to u$, where $w \in C_{11}$. A path from $u \in C_{13}$ to $w \in C_{24}$ is given by $u \to v_1 \to x \to w$, where $x \in C_{21}$.

G does not admit a Nash equilibrium. We consider strategy profiles $\langle x_1, x_2\rangle \in V^2$ for the two agents. The symmetries of our construction allow us to restrict our attention to six cases.

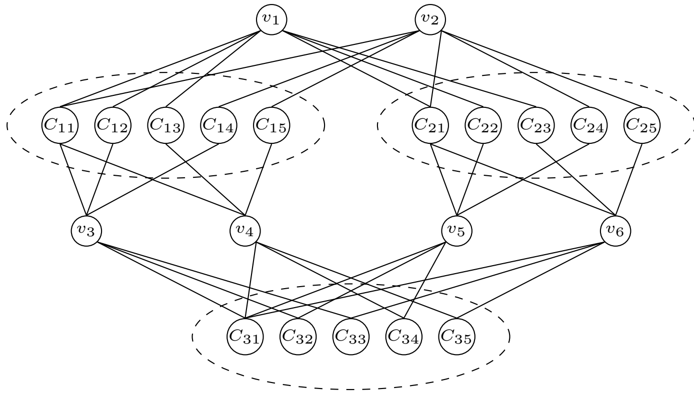

Figure 2: The construction of the proof of Theorem 2.2. The cliques C_1, C_2, C_3 are outlined by dashed ellipses, and the edges inside the cliques are not shown. An edge between v_i and C_{jk} implies that v_i is connected to all the vertices $u \in C_{jk}$.

Case 1: $x_1 = v_1, x_2 \in C_1$ (hub and adjacent clique). Agent 1 colors some of the vertices of C_2 and some hubs, that is, $U_1(x_1, x_2) < 60$. By deviating to $x_1' = v_5$, agent 1 colors $C_{21}, C_{22}, C_{24}, C_{31}, C_{32}, C_{34}$, i.e., $U_1(x_1', x_2) \geq 60$.

Case 2: $x_1 = v_1, x_2 \in C_3$ (hub and nonadjacent clique). Agent 2 colors the vertices of C_3 and some hubs, hence $U_2(x_1, x_2) < 60$. By deviating to $x_2' \in C_{11}$, agent 2 colors C_{14}, C_{15}, and C_3, thus $U_2(x_1, x_2') \geq 70$.

Case 3: $x_1 = v_1, x_2 = v_3$ (nonparallel hubs). Agent 1 colors C_{13}, C_2, and some hubs, therefore $U_1(x_1, x_2) < 70$. By deviating to $x_1' \in C_{11}$, agent 1 colors C_{13}, C_{15}, C_2, so $U_1(x_1', x_2) \geq 70$.

Case 4: $x_1 = v_1, x_2 = v_2$ (parallel hubs). Agent 1 colors $C_{12}, C_{13}, C_{22}, C_{23}$, and some hubs ($v_3, \ldots, v_6$ are gray and C_3 remains white), hence $U_1(x_1, x_2) < 50$. By deviating to $x_1' \in C_3$, agent 1 can guarantee a utility of at least 50 (since it colors C_3).

Case 5: $x_1 \in C_1$, $x_2 \in C_3$ (different cliques). If $x_1 \notin C_{11}, x_2 \notin C_{31}$, then agent 1 can benefit by deviating to $x_1' \in C_{11}$, since then it colors both v_1 and v_2 at time 2 (rather than just one of them), and colors twenty vertices of C_2 at time 3 (rather than ten). Hence we can assume without loss of generality that $x_1 \in C_{11}$. In that case, agent 2 colors C_3 and some hubs, therefore $U_2(x_1, x_2) < 60$. By deviating to $x_2' = v_5$, agent 2 colors at least $C_{21}, C_{22}, C_{24}, C_{31}, C_{32}, C_{34}$, hence $U_2(x_1, x_2') \geq 60$.

Case 6: $x_1 \in C_1$, $x_2 \in C_1$ (same clique). Since $C_1 \setminus \{x_1, x_2\}$ is gray, there are at most 108 vertices that are not gray, therefore it must hold that either $U_1(x_1, x_2) < 60$ or $U_2(x_1, x_2) < 60$. By deviating to v_5 an agent can guarantee a utility of at least 60. □

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