Disjoint directed cycles

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Abstract

It is shown that there exists a positive ϵ so that for any integer k, every directed graph with minimum outdegree at least k contains at least ϵk vertex disjoint cycles. On the other hand, for every k there is a digraph with minimum outdegree k which does not contain two vertex or edge disjoint cycles of the same length.

1 Introduction

All graphs and digraphs considered here contain no parallel edges, unless otherwise specified, but may have loops. Throughout the paper, a cycle in a directed graph always means a *directed* cycle. For a positive integer k, let $f(k)$ denote the smallest integer so that every digraph of minimum outdegree at least $f(k)$ contains k vertex disjoint cycles. Bermond and Thomassen [6] conjectured that $f(k) = 2k - 1$ for all $k \geq 1$. Thomassen [9] showed that this is the case for $k \leq 2$, and proved that for every $k \geq 2$

$$f(k+1) \leq (k+1)(f(k)+k),$$

thus concluding that $f(k)$ is finite for every k and that $f(k) \leq (k+1)!$. Here we improve this estimate and show that $f(k)$ is bounded by a linear function of k.

Theorem 1.1 *There exists an absolute constant C so that $f(k) \leq Ck$ for all k. In particular, $C = 64$ will do.*

An easy corollary of this theorem is the following.

Corollary 1.2 *There exists a positive $\epsilon > 0$ so that for any r, every digraph with minimum outdegree at least r contains at least ϵr^2 edge disjoint cycles. In particular, $\epsilon = 1/128$ will do.*

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This strengthens a result in [4] where the above is proved for r-regular digraphs.

Thomassen [9] conjectured that for every integer $k \geq 1$ there is a (smallest) finite integer $g(k)$ such that any digraph with minimum outdegree at least $g(k)$ contains k pairwise disjoint cycles of the same length. Here we observe that this is false in the following strong sense.

Proposition 1.3 *For every integer r there exists a digraph with minimum outdegree r which contains no two edge disjoint cycles of the same length (and hence, of course, no two vertex disjoint cycles of the same length).*

For *undirected* graphs, it has been proved by Häggkvist [7] that for every integer $k \geq 1$ there is a (smallest) finite $h(k)$ such that every undirected graph with minimum degree at least $h(k)$ contains k vertex disjoint cycles of the same length. The results in [7] imply that $h(k) \leq 2^{O(k)}$. Thomassen [10] refined the arguments in [7] and his results imply that $h(k) \leq O(k^2)$. Here we show that in fact $h(k)$ is bounded by a linear function of k and conjecture that $h(k) = 3k - 1$.

The rest of this paper is organized as follows. In Section 2 we prove Theorem 1.1 by combining one of the ideas of Thomassen [9] with the probabilistic approach in [2], [4], together with some additional arguments. In Section 3 we consider the problem of finding disjoint cycles of the same length. The final section 4 contains some concluding remarks and open problems.

2 Disjoint cycles

In this section we prove Theorem 1.1. We make no attempt to optimize the constant C obtained in our proof, or optimize any of the other absolute constants that appear in the course of the proof. It is, in fact, not too difficult to improve the constant we get here considerably, but since it seems hopeless to apply our method to get the right value of C (conjectured to be 2 in [6]), we merely prove here that $f(k) \leq 64k$ for every k.

Note that this certainly holds for $k = 1$. Assuming it is not true for all k, let $k+1$ be the smallest integer violating the inequality. Then, clearly,

$$f(k+1) > f(k) + 64. \tag{1}$$

Put $r = f(k+1) - 1$ and let $D = (V, E)$ be a directed graph with minimum outdegree r which does not contain $k+1$ vertex disjoint cycles. Assume, further, that D has the minimum possible number of vertices among all digraphs as above, and subject to having these properties and this number of vertices assume it has the minimum possible number of edges. Clearly, in this case, every outdegree in D is precisely r. By the minimality of D, the indegree of each of its vertices is positive. Moreover, D contains no loops, since otherwise the digraph obtained from D by deleting a vertex incident with a loop cannot have k disjoint cycles, showing that $f(k+1) - 2 = r - 1 \leq f(k) - 1$, contradicting (1).

For a vertex v of D, let $N^-(v) = \{u, u \in V : uv \in E\}$ denote the set of all in-neighbors of v in D. The following claim is proved in [9]. For completeness we repeat its simple proof.

Claim 1: For every vertex v of D, the induced subgraph of D on $N^-(v)$ contains a cycle.
Proof: It suffices to show that the minimum indegree in this induced subgraph is at least 1. Assume it is 0, then there is a vertex u with $uv \in E$ so that no other vertex of D dominates both u and v. But in this case, the digraph D' obtained from D by deleting all edges emanating from u except the edge uv and by contracting the edge uv has minimum outdegree r. By the minimality of D it contains $k+1$ disjoint cycles which easily supply $k+1$ disjoint cycles in D as well, a contradiction. This proves the claim. $\square$
We next show that the number of vertices of D is not too large.
Claim 2: $|V| \leq k(r^2 - r + 1)$.
Proof: Put $n = |V|$ and let G be the undirected graph on the set of vertices V in which two distinct vertices u and w are adjacent iff there is a vertex of D that dominates both. Define $m = n\binom{r}{2}$ and note that the number of edges of G is at most m. Therefore, as is well known (see e.g., [5] p. 282) it contains an independent set of size at least $n^2/(2m+n)$. If this number is greater than k there are $k+1$ vertices $u_1, \ldots, u_{k+1} \in V$ which are independent in G, that is, the sets $N^-(u_i)$ are pairwise disjoint. However, each of these sets contains a cycle, by Claim 1, and thus there are $k+1$ disjoint cycles in D, contradicting the assumption. Therefore

$$\frac{n^2}{nr(r-1)+n} = \frac{n^2}{2m+n} \leq k,$$

showing that $n \leq k(r^2 - r + 1)$, as needed. $\square$
We can now prove a nearly linear upper bound for $f(k)$. Although this is not essential for the proof of the linear bound we include this proof in the next claim, as it enables us to obtain a better constant C in the proof of Theorem 1.1, and as it illustrates the basic probabilistic approach we apply later.
Claim 3:

$$k(r^2 - r + 2)(1 - \frac{1}{k+1})^r \geq 1.$$

Proof: Assume this is false and

$$k(r^2 - r + 2)(1 - \frac{1}{k+1})^r < 1.$$

Assign to each vertex $v \in V$ randomly and independently, a color i, $1 \leq i \leq k+1$, where each of the $k+1$ choices are equally likely. Let V_i be the set of all vertices colored i and let D_i be the induced subgraph of D on the vertex set V_i. For each vertex v, let A_v denote the event that no out-neighbor of v has the same color as v. Clearly $Prob[A_v] = (1 - \frac{1}{k+1})^r$. Let B_i denote the event that $V_i = \emptyset$. Then $Prob[B_i] = (1 - \frac{1}{k+1})^n \leq (1 - \frac{1}{k+1})^{r+1}$. Therefore, by Claim 2

$$\sum_{v \in V} Prob[A_v] + \sum_{i=1}^{k+1} Prob[B_i] \leq n(1 - \frac{1}{k+1})^r + (k+1)(1 - \frac{1}{k+1})^{r+1}$$

$$\leq k(r^2 - r + 1)(1 - \frac{1}{k+1})^r + k(1 - \frac{1}{k+1})^r = k(r^2 - r + 2)(1 - \frac{1}{k+1})^r < 1.$$

It follows that with positive probability each D_i is nonempty and has a positive minimum outdegree, and hence contains a cycle. Thus, there is such a choice for the graphs D_i, and this gives $k+1$ disjoint cycles in D, contradicting the assumption, and proving the claim. $\Box$

Note that the above claim already supplies a nearly linear upper bound for $f(k)$ as it implies that

$$k(r^2 - r + 2) \geq e^{r/(k+1)},$$

showing that for large k,

$$f(k+1) - 1 = r \leq (3 + o(1))k \log_e k,$$

(where the $o(1)$ term here tends to 0 as k tends to infinity.) It also implies the following.

Corollary 2.1 *For every positive integer $h \leq 2^9$, $f(h) \leq 32h$.*

Proof: Otherwise, there is some $h = k+1 \leq 2^9$ so that $r = f(k+1) - 1 \geq 32(k+1)$. Define $b = r/(k+1)$ (≥ 32). Then, by Claim 3,

$$b^2 2^{27} \geq b^2(k+1)^3 \geq kr^2 \geq k(r^2 - r + 2) \geq e^{r/(k+1)} = e^b.$$

Thus $2^{27} \geq e^b/b^2$ and this is trivially false for all $b \geq 32$. $\Box$

The final ingredient needed for the proof of Theorem 1.1 is the following.

Claim 4: If $r \geq 2^{12}$, then one can split the vertex set V of D into two nonempty disjoint subsets V_1 and V_2 so that the minimum outdegrees of the induced subgraphs of D on V_1 and on V_2 are both at least $\frac{r}{2} - r^{2/3}$.

Proof: Color each vertex of D either red or blue, choosing each color independently and uniformly at random. For each vertex $v \in V$ let A_v denote the event that v has less than $r/2 - r^{2/3}$ outneighbors with its own color. By the standard estimate of Chernoff for the distribution of binomial random variables (c.f., e.g., [1], Appendix A), for every v

$$Prob[A_v] \leq e^{-2r^{1/3}}.$$

Since there are $|V|$ vertices in D, where $r < |V| < r^3$, and since $2 \cdot 2^{-|V|} + r^3 e^{-2r^{1/3}} < 1$ for $r \geq 2^{12}$, we conclude that with positive probability both colors red and blue appear and none of the events A_v occurs. Given such a coloring, let V_1 be the set of vertices colored red and let V_2 be the set of vertices colored blue, completing the proof. $\Box$

Corollary 2.2 *If h is even and $f(h) > 2^{12}$ then*

$$f(h) - 2(f(h))^{2/3} \leq 2f(h/2).$$

Proof: Let $h = k+1$, put $r = f(k+1) - 1$ and let $D = (V, E)$ be as above. By the last claim D contains two vertex disjoint subgraphs D_1 and D_2, where the minimum outdegree of each D_i is at least $\lceil \frac{r}{2} - r^{2/3} \rceil$. If this number is at least $f((k+1)/2)$ then each D_i contains $(k+1)/2$ disjoint cycles giving a total of $k+1$ disjoint cycles in D, contradiction. Thus

$$\frac{f(h)-1}{2} - (f(h))^{2/3} \leq \frac{f(h)-1}{2} - (f(h)-1)^{2/3} = \frac{r}{2} - r^{2/3} \leq \lceil \frac{r}{2} - r^{2/3} \rceil \leq f(h/2) - 1,$$

implying the desired result. □

Corollary 2.3 *For every $h \geq 2^6$, $f(h) \leq 64h - 128h^{2/3}$.*

Proof: We apply induction on h. For h satisfying $2^6 \leq h \leq 2^9$ the desired result holds, since by Corollary 2.1

$$f(h) \leq 32h \leq 64h - 128h^{2/3}.$$

For $h > 2^9$, if $f(h) \leq 2^{12}$ then $f(h) \leq 8h < 64h - 128h^{2/3}$. Thus, we may and will assume that $f(h) > 2^{12}$. Hence, if h is even then, by Corollary 2.2 and by the induction hypothesis applied to $h/2$ (for which we already know the result)

$$f(h) - 2(f(h))^{2/3} \leq 2f(h/2) \leq 2(64\frac{h}{2} - 128(\frac{h}{2})^{2/3}) = 64h - 2^{1/3}128h^{2/3}.$$

If the desired result fails for h and $f(h) > 64h - 128h^{2/3}$, then, as $z - 2z^{2/3}$ is an increasing function for $z \geq 2^{12}$, we conclude that

$$f(h) - 2(f(h))^{2/3} > 64h - 128h^{2/3} - 2(64h - 128h^{2/3})^{2/3} > 64h - 128h^{2/3} - 2(64h)^{2/3} = 64h - 160h^{2/3}.$$

It follows that

$$64h - 160h^{2/3} < 64h - 2^{1/3}128h^{2/3}$$

implying that $2^{1/3} < 160/128 = 5/4$, which is false. Therefore the assumption that $f(h) > 64h - 128h^{2/3}$ is incorrect, completing the proof of the induction step for even h. If h is odd then, by Corollary 2.2 and by the induction hypothesis applied to $(h+1)/2$,

$$f(h+1) - 2(f(h+1))^{2/3} \leq 2f((h+1)/2) \leq 64(h+1) - 2^{1/3}128(h+1)^{2/3} \leq 64h + 64 - 2^{1/3}128h^{2/3}.$$

Since $f(h) \leq f(h+1)$ and $z - 2z^{2/3}$ is increasing for $z \geq 2^{12}$, we conclude that

$$f(h) - 2(f(h))^{2/3} \leq 64h + 64 - 2^{1/3}128h^{2/3}.$$

Thus, if we assume that $f(h) > 64h - 128h^{2/3}$ it follows, as before, that

$$64h - 160h^{2/3} < f(h) - 2(f(h))^{2/3} \leq 64h + 64 - 2^{1/3}128h^{2/3},$$

implying that

$$2^{13}(2^{1/3} - \frac{5}{4}) < 128(2^{1/3} - \frac{5}{4})h^{2/3} < 64,$$

i.e., that

$$2^{1/3} < \frac{5}{4} + \frac{1}{128} = \frac{161}{128},$$

which is false. Therefore, $f(h) \leq 64h - 128h^{2/3}$, completing the proof. $\square$

Proof of Theorem 1.1: By Corollaries 2.1 and 2.3, for every positive integer k, $f(k) \leq 64k$. $\square$

Proof of Corollary 1.2: By Theorem 1.1 and its proof above, every digraph $D = D_0$ with minimum outdegree at least r contains at least $r/64$ vertex disjoint cycles. Omit the edges of these cycles from D to get a digraph D_1 whose minimum outdegree is at least $r-1$. This digraph contains, by Theorem 1.1, at least $(r-1)/64$ vertex disjoint cycles, whose edges may be omitted, yielding a graph D_2, with minimum outdegree $r-2$ at least. Repeating in this manner we obtain a collection of at least

$$\frac{r}{64} + \frac{r-1}{64} + \frac{r-2}{64} + \ldots + \frac{1}{64} \geq \frac{r^2}{128}$$

edge disjoint cycles in D, completing the proof. $\square$

3 Equicardinal disjoint cycles

3.1 Directed graphs

Proof of Proposition 1.3: For a positive integer r, construct a digraph $D = (V, E)$ on a set of $2^{r^2}r$ vertices as follows. Let $V_1, \ldots, V_{2^{r^2}}$ be pairwise disjoint sets of vertices, each of cardinality r, and put $V = V_1 \cup \ldots \cup V_{2^{r^2}}$. For each i satisfying $1 < i \leq 2^{r^2}$ every vertex of V_i is joined by a directed edge to every vertex of V_{i-1}. The resulting digraph is acyclic, and D is obtained from it by adding to it r^2 edges, which we call *special edges*, as follows. Put $V_1 = \{u_1, u_2, \ldots, u_r\}$. For each j, $1 \leq j \leq r$, there are r special edges $e_{j,1}, e_{j,2}, \ldots, e_{j,r}$ emanating from u_j, where $e_{j,s}$ joins u_j to an arbitrarily chosen member of $V_{2^{(j-1)r+s}}$. This completes the description of D. Define the *length* of $e_{j,s}$ to be $2^{(j-1)r+s}$ and note that the lengths of the r^2 special edges are distinct powers of 2.

Every outdegree in D is precisely r. Moreover, every cycle in D must contain at least one special edge and its length is precisely the sum of lengths of all the special edges it contains. Since these lengths are distinct powers of 2 it follows that two cycles in D have the same length iff they share exactly the same special edges. Thus there are no two edge (or vertex) disjoint cycles of the same length in D, completing the proof of Proposition 1.3. $\square$

3.2 Undirected graphs

Recall that $h(k)$ is the smallest integer for which every *undirected* graph with minimum degree at least $h(k)$ contains k vertex-disjoint cycles of the same length. As shown by Häggkvist [7], $h(k)$ is finite for every integer k. Thomassen [10] refined the methods of Häggkvist and proved that there is some absolute constant $C > 0$ such that any undirected graph with at least Ck^2 vertices and minimum degree at least $3k+1$ contains k pairwise disjoint cycles of the same length. This clearly implies that $h(k) \le O(k^2)$. Here we show that in fact $h(k)$ is bounded by a linear function of k. As in the previous section, we make no attempt to optimize the constant in our estimate and only show that $h(k) \le 1024k$ for every k. The constant 1024 can easily be improved considerably, but our method does not suffice to determine the precise value of $h(k)$, which we conjecture to be $3k-1$.

Proposition 3.1 *Every undirected graph with minimum degree $r > 1$ contains at least $r/1024$ vertex-disjoint cycles of the same length.*

Proof: Let $G = (V, E)$ be an undirected graph with minimum degree r. If $r \le 1024$ the required result is trivial, hence we assume $r > 1024$. If $G = G_0$ contains a cycle C_1 of length at most, say, 20, delete its vertices from G to get a graph G_1. Note that the minimum degree in G_1 is at least $r-20$. If G_1 contains a cycle C_2 of length at most 20, delete its vertices from G_1 to get a graph G_2 whose minimum degree is at least $r-40$. Proceeding in this manner, if the process continues at least $\frac{r-1}{40}$ steps then we have at least $\frac{r-1}{40\cdot 20} > \frac{r}{1024}$ cycles of the same length, as needed. Thus the process terminates before that, ending with a graph G' whose minimum degree exceeds $r - 20\frac{r-1}{40} = \frac{r+1}{2}$ and whose girth exceeds 20. As the minimum degree in G' is an integer, it is at least $\frac{r}{2}+1$. Thus, for any fixed vertex v of G', the number of vertices of distance precisely 10 from v is bigger than $(r/2)^{10}$. It follows that the number of vertices of G', which we denote by n, is at least $r^{10}/1024$.

Let ϕ denote the smallest number of vertices of G' whose deletion leaves a forest. Note that $\phi > 1$. A well known result of Erdös and Posa (c.f., e.g., [8], Problem 10.18) implies that G' (and hence G) contains at least $\frac{\phi}{4\log_2 \phi}$ vertex disjoint cycles. If, say, $\phi \ge n^{11/20}\log n$ this implies that G' contains at least $n^{11/20}/4$ disjoint cycles. Among these, at most $n^{11/20}/8$ may have length at least $8n^{9/20}$ (as the total length of all cycles is at most n), and thus there are at least $n^{11/20}/8$ cycles of length at most $8n^{9/20}$, implying that at least $n^{1/10}/64 \ge r/128 > r/1024$ of them have the same length.

We may thus assume that $\phi < n^{11/20}\log n$, implying (since $n \ge (r/2)^{10} \ge 2^{90}$) that $\phi < n^{2/3}$ (as $n^{11/20}\log n$ is (much) smaller than $n^{2/3}$ for $n \ge 2^{90}$.) Fix a set S of ϕ vertices of G', so that $G' - S$ is a forest F. The number of vertices of F is at least $n - n^{2/3}$, and less than $0.1n$ of them have more than 20 neighbors in F. Therefore, there is a set $\{u_1, u_2, \ldots, u_t\}$ of $t \ge 0.9n - n^{2/3} > 0.7(\phi)^{3/2}$ vertices of F, each of which has at most 20 neighbors in F and hence at least $\frac{r}{2}+1-20 > 2$ neighbors in S. Construct an auxiliary multigraph H on the set of vertices S as follows. For each u_i, $1 \le i \le t$, pick arbitrarily two neighbors of u_i in S and join them by an edge in H. Note that H may have parallel edges. H has ϕ vertices and $t \ge 0.7(\phi)^{3/2}$ edges. Therefore, by a standard result from extremal graph theory (see, e.g., [8], Problem 10.36), it has a cycle of length at most 4. But this provides a cycle of length at most 8 in G', contradicting the fact that the girth of G' exceeds 20. Therefore, ϕ cannot be smaller than $n^{11/20}\log_2 n$, completing the proof. $\Box$

4 Concluding remarks and open problems

1. A close look at our proof of Theorem 1.1 shows that it implies the following statement:

   For every $C > 0$ and $\epsilon > 0$ there is a finite $k_0 = k_0(C, \epsilon)$ such that if $f(k) \leq Ck$ for all $k \leq k_0$ then $f(k) \leq (1+\epsilon)Ck$ for all k.

2. The conjecture of Bermond and Thomassen that $f(k+1) = 2k+1$ for every k remains open for all $k > 1$. Our results imply, however, that if the conjecture is false for some fixed value of k, then there is a counterexample with at most $4k^3 + 2k^2 + k$ vertices. Indeed, if there is a digraph with minimum outdegree $r = 2k+1$ and no $k+1$ disjoint cycles then, by taking a minimum example with these properties it follows from Claim 2 that its number of vertices is at most $k(r^2 - r + 1) = 4k^3 + 2k^2 + k$. This improves an estimate of Thomassen [9] who showed that if the conjecture is false for k then there is a counterexample with at most $(2k+2)^{k+3}$ vertices. Note that for the case $k = 1$ (two disjoint cycles), for which the conjecture has been proved in [9], our estimate above shows that it suffices to check it for digraphs with at most 7 vertices, and this is not a difficult task. The first open case is $k+1 = 3$, where it suffices to prove that any digraph with at most 42 vertices and minimum outdegree 5 contains three disjoint cycles. Even this relatively modest size seems to be too large to enable a brute force attack.

3. Our technique in Section 2 enables one to obtain reasonable upper bounds for $f(k)$ even for small values of k. We illustrate this fact by trying to estimate $f(4)$ (conjectured to be 7.) By Claim 3, $f(4) < r+1$ provided $3(r^2 - r + 2)(\frac{3}{4})^r < 1$. It is not difficult to check that this holds for $r = 27$ showing that $f(4) \leq 27$. This can be improved, however, using the known result of [9] that $f(2) = 3$, as follows. Suppose $f(4) > r$, and take, as in Section 2, a minimum digraph D with all outdegrees r and no 4 vertex disjoint cycles. By Claim 2 the number of vertices n of D is at most $3(r^2 - r + 1)$ (and at least $r+1$, say.) If one can partition the set of vertices of D into two disjoint parts so that the minimum outdegree of the induced subgraph on each part is at least 3, then it would follow that D contains 4 disjoint cycles, a contradiction. Therefore, by applying a random splitting as in Section 2, it follows that if

   $$2 \cdot 2^{-(r+1)} + 3(r^2 - r + 1)\frac{1 + r + \binom{r}{2}}{2^r} < 1,$$

   then $f(4) \leq r$. This inequality holds for $r = 17$ showing that in fact $f(4) \leq 17$.

4. In [4] it is conjectured that every r-regular directed graph contains at least $\binom{r+1}{2}$ edge-disjoint directed cycles (which, if true, is best possible), and it is proved that any such digraph contains at least $\frac{3}{2^{19}}r^2$ edge disjoint cycles. Corollary 1.2 and its proof here show that minimum outdegree at least r already ensures the existence of at least $r^2/128$ ($> \frac{3}{2^{19}}r^2$) edge disjoint cycles, and hence the regularity assumption is not needed to get a quadratic number of edge-disjoint

cycles. The regularity is crucial, however, if we want to place some conditions on the lengths of the cycles. By combining the techniques in [4] and in [3] it is not difficult to prove the following.

Proposition 4.1 *For any integer $s > 1$ there exists some $\epsilon(s) > 0$ so that any r-regular digraph contains at least $\lfloor \epsilon(s) r^2 \rfloor$ edge disjoint cycles, each of length divisible by s.*

A similar statement totally fails already for $s = 2$ for digraphs with large minimum outdegrees with no regularity assumption, as an example of Thomassen [11] shows that for every r there is a digraph with minimum outdegree r which does not contain even cycles at all.

5. It would be interesting to decide if for any two positive integers k and s there is a finite number $F = F(k, s)$ so that for every digraph $D = (V, E)$ with minimum outdegree F there is a partition of V into k pairwise disjoint subsets $V_1, \ldots, V_k$ such that the minimum outdegree of the induced subgraph of D on each V_i is at least s. It is not difficult to see that the finiteness of the function $f(k)$ considered here and in [9] implies that $F(k, 1)$ is finite for every k. We do not know if $F(2, 2)$ is finite. The corresponding function for the *undirected* case is finite for all admissible values of the parameters, as follows from observations by various researchers.

6. It would be interesting to determine the function $h(k)$ considered in Subsection 3.2 precisely. As shown by a complete graph on $3k - 1$ vertices, $h(k) \geq 3k - 1$ for every k, and we suspect that this is tight. The result of Thomassen [10] that any undirected graph with at least Ck^2 vertices and minimum degree at least $3k + 1$ contains k pairwise disjoint cycles of the same length may be useful in trying to prove this estimate.

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