Packing of partial designs

Noga Alon
Bellcore, Morristown, NJ 07960, USA
and Department of Mathematics
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel Aviv University, Tel Aviv, Israel

Abstract

We say that two hypergraphs H_1 and H_2 with v vertices each *can be packed* if there are edge disjoint hypergraphs H_1' and H_2' on the same set V of v vertices, where H_i' is isomorphic to H_i. It is shown that for every fixed integers k and t, where $t \leq k \leq 2t-2$ and for all sufficiently large v there are two (t,k,v) partial designs that cannot be packed. Moreover, there are two *isomorphic* partial (t,k,v)-designs that cannot be packed. It is also shown that for every fixed $k \geq 2t-1$ and for all sufficiently large v there is a (λ_1, t, k, v) partial design and a (λ_2, t, k, v) partial design that cannot be packed, where $\lambda_1 \lambda_2 \leq O(v^{k-2t+1} \log v)$. Both results are nearly optimal asymptotically and answer questions of Teirlinck. The proofs are probabilistic.

1 Introduction

Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two hypergraphs, where $|V_1| = |V_2| = v$. We say that H_1 and H_2 *can be packed* if there are edge disjoint hypergraphs H_1' and H_2' on the same set of v vertices, where H_i' is isomorphic to H_i for $i = 1, 2$.

For integers λ, t, k and v, where $v \geq k \geq t \geq 1$, a (λ, t, k, v) *partial design* is a k-uniform hypergraph H on v vertices so that no set of t vertices lies in more than λ edges of H. For brevity, a $(1,t,k,v)$ partial design is also called a (t,k,v) partial design. A (λ,t,k,v) partial design H is a *design* if every set of t vertices lies in precisely λ edges of H.

Various reasearchers studied the possibility of packing two given designs or partial designs as a function of their parameters. When $k = 1$ the problem is trivial. The case of $(\lambda, 1, 2, v)$ partial designs (which are just graphs on v vertices with maximum degree at most λ) is considered in various papers including [2], [4] and [8] (see also [3] and its references) and much is known here, although the main conjecture of [2] is still open. Much less is known for $k \geq 3$. Answering a problem of Doyen [5], Teirlinck showed in [10] that for $v \geq 7$ every two $(2, 3, v)$-designs (usually known as Steiner Triple Systems) can be packed. His proof can be easily modified to yield the same result for partial $(2, 3, v)$ designs as well. In [6] it is shown that if $k \geq 2t$ then every two (t, k, v) partial designs can be packed. The proof, by a simple probabilistic argument, proceeds as follows. If $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ are such designs, then the number of edges in each H_i is at most $\binom{v}{t}/\binom{k}{t}$. It is easy to check that the expected number of common edges of a random copy of H_1 and a random copy of H_2 in the complete hypergraph on v vertices is $|E_1| \cdot |E_2|/\binom{v}{k} < 1$ and hence there are two edge disjoint copies providing the required packing.

Our first result shows that the assumption that $k \geq 2t$ is almost tight.

Theorem 1.1 *For every fixed integers k and t where $t \leq k \leq 2t - 2$, there is a $v_0 = v_0(k, t)$ so that for all $v > v_0$ there are two (t, k, v)-partial designs that cannot be packed. Moreover, there is a single (t, k, v)-partial design so that two isomorphic copies of it cannot be packed.*

This answers a problem of Teirlinck [11]. Note that the case $k = 2t - 1$ for $t > 2$ remains open, and it seems plausible, in view of [10], that for this case a packing is always possible, provided v is sufficiently large.

For $v \geq k \geq t \geq 1$ let $c(t, k, v)$ denote the maximum integer c such that whenever $\lambda_1 \cdot \lambda_2 \leq c$ then every partial (λ_1, t, k, v) and (λ_2, t, k, v) designs can be packed. The probabilistic argument of [6] sketched above easily implies that for every fixed $k \geq 2t$ there is a constant $b = b(k, t) > 0$ so that

$$c(t, k, v) \geq bv^{k-2t}. \tag{1}$$

Teirlinck ([11]) observed that for every $k \geq t$ there is a $b' = b'(k, t) > 0$ so that

$$c(t, k, v) \leq b'v^{k-t}$$

and raised the problem of estimating the best possible power of v in the asymptotic behaviour of $c(t, k, v)$ for fixed k and t more accurately. Here we prove the following result that determines this power almost precisely.

Theorem 1.2 *For every fixed $k \geq 2t-1$, there is a positive constant $b" = b"(k,t) > 0$ so that for all v*

$$c(t,k,v) \leq b" v^{k-2t+1} \log v.$$

Note that Theorem 1.1 implies that for $k \leq 2t-2$, $c(t,k,v) = 0$. It would be interesting to close the gap between the upper bound for $c(t,k,v)$ given in the last theorem, and the lower bound provided by (1). By the result in [8] (see also [4]) one can easily show that $c(1,2,v) = \Theta(v)$ but the cases of larger t and k seem much more difficult.

The proofs of the two theorems above are probabilistic. The proof of Theorem 1.1 is more subtle and is given in Section 2. The (easier) proof of Theorem 1.2 is presented in Section 3.

2 Partial designs

In this section we prove Theorem 1.1. For simplicity of presentation we first prove that there are two partial (not necessarily isomorphic) designs that cannot be packed and then briefly comment on the modifications required in the proof in order to establish the stronger assertion dealing with two copies of the same partial design. The proof is probabilistic and applies the following correlation inequality of Suen ([9]). Let B_i, $i \in I$ be a finite family of events in an arbitrary probability space. Let p_i denote the probability of B_i and define $M = \Pi_{i \in I}(1-p_i)$. Note that if the events B_i are independent then M is simply the probability that no B_i holds. The inequality of Suen shows that M supplies a reasonable estimate for this probability even when the events are "mostly independent". More precisely, let us call a graph G on the set of vertices I a *superdependency graph* for the events B_i if the following condition holds. For any two disjoint subsets J_1 and J_2 of I, so that there are no edges of G between J_1 and J_2,

$$Prob(\bigwedge_{i \in J_1 \cup J_2} B_i) = Prob(\bigwedge_{i \in J_1} B_i) \cdot Prob(\bigwedge_{j \in J_2} B_j).$$

For $i, j \in I$ let $N_G(i,j)$ denote the set of all vertices of G adjacent to either i or j (or both).

Theorem 2.1 (Suen ([9])) *If B_i, $i \in I$ are events in a probability space, $p_i = Prob(B_i)$, $M = \Pi_{i \in I}(1-p_i)$ and $G = (I, E(G))$ is a superdependency graph for these events then the probability* *that no event B_i holds deviates from M by at most*

$$M(exp\ (\sum_{ij\in E(G)} y(i,j)) - 1),$$

where

$$y(i,j) = 2(Prob(B_i \wedge B_j) + p_i p_j)\ \Pi_{l\in N_G(i,j)}(1-p_l)^{-1}.$$

Proof of Theorem 1.1 Let k and t be fixed positive integers, $t \le k \le 2t-2$. Throughout the proof we assume, whenever this is needed, that v is a sufficiently large integer (as a function of k and t). Let V_1 and V_2 be two sets of v vertices each and put

$$p = v^{-\frac{3k-2t-1}{4}}.$$

Let $H_1 = (V_1, E_1)$ be a random (t,k,v) partial design obtained as follows. For each k-subset K of V_1, randomly and independently, *mark* K with probability p. E_1 is the set of all marked subsets of V_1 such that there is no other marked subset that intersects K by at least t elements. This is clearly a (t,k,v) partial design. Let $H_2 = (V_2, E_2)$ be another random (t,k,v) partial design obtained by applying the above marking procedure to the k-subsets of V_2.

To complete the proof it suffices to show that with positive probability H_1 and H_2 cannot be packed. For simplicity, denote $V = V_1$. We must show that with positive probability, for every bijection $\pi : V_2 \mapsto V$, the two sets E_1 and $\pi(E_2)$ intersect, where here $\pi(E_2) = \{\pi(e) : e \in E_2\}$, and $\pi(e) = \{\pi(v); v \in e\}$.

To do so, we consider a fixed bijection π as above and estimate the probability that E_1 and $\pi(E_2)$ are disjoint. For each subset K of cardinality k of V, let B_K denote the event that K is in $E_1 \cap \pi(E_2)$. Note that this event occurs if and only if in the random marking process applied in the definition of H_1 and H_2, both $K \subset V$ and $\pi^{-1}(K) \subset V_2$ have been marked, and no other k-subset that intersects K or $\pi^{-1}(K)$ by at least t elements have been marked. By letting

$$s = \sum_{j=t}^{k-1} \binom{k}{j}\binom{v-k}{k-j} = O(v^{k-t})$$

denote the number of k-subsets of a v-element set that intersect a given k-subset by at least t elements we conclude that for each K

$$Prob(B_K) = p^2(1-p)^{2s}.$$

By the definition of p, and since $(3k-2t-1)/4 > k-t$ we conclude that $(1-p)^{2s} = 1+o(1)$. Therefore,

$$Prob(B_K) \sim p^2, \tag{2}$$

where here and from now on we write $f \sim g$ for two functions f and g of v if the ratio f/g tends to 1 as v tends to infinity.

Our objective is to apply Theorem 2.1 in order to estimate the probability that π provides a packing of H_1 and H_2, i.e., that none of the events B_K occurs. To this end, define a superdependency graph $G = (V(G), E(G))$ on the events B_K as follows. The set of vertices of G is the set $V(G) = \{K \subset V : |K| = k\}$ and two vertices K and K' are adjacent if and only if $|K \cap K'| \geq 2t-k$. (Note that this last inequality must hold if there is a k-subset that intersects both K and K' by at least t elements.)

We claim that G is a superdependency graph for the above events B_K. Indeed, if J_1 and J_2 are two disjoint subsets of $V(G)$ and there are no edges of G between J_1 and J_2 then there is no k-subset of V that intersects a member of J_1 and a member of J_2 by at least t elements each. Similarly, there is no k-subset of V_2 that intersects a member of $\pi^{-1}(J_1)$ and a member of $\pi^{-1}(J_2)$ by at least t elements each. However, this means that even if we know all the randomly chosen marks of the k-subsets that affect any of the events B_K, $K \in J_1$, this does not have any effect on the conditional probability of any Boolean function of the events $B_{K'}$,$K' \in J_2$, and hence G is indeed a superdependency graph, as claimed. Observe that the degree of every vertex of G is $O(v^{k-(2t-k)}) = O(v^{2k-2t})$.

By Theorem 2.1 we thus conclude that the probability that π provides a packing of H_1 and H_2 is at most

$$\Pi_{K \in V(G)}(1 - Prob(B_K)) \; exp(\sum_{KK' \in E(G)} y(K, K')),$$

where

$$y(K, K') = 2(Prob(B_K \wedge B_{K'}) + Prob(B_k)Prob(B_{K'}))\Pi_{K" \in N_G(K,K')}(1 - Prob(B_{K"}))^{-1}.$$

By (2), and since the maximum degree of G is $O(v^{2k-2t})$ we conclude that

$$y(K, K') \leq O(p^4),$$

where here we used the easy fact that for distinct K and K', $Prob(B_K \wedge B_{K'}) \sim p^4$. Therefore

$$Prob(\bigwedge_{K \in V(G)} \overline{B}_K) \leq exp(-(1+o(1))p^2 \cdot \binom{v}{k} + O(p^4)O(v^{3k-2t}))$$

$$\leq exp(-\Omega(v^{(2t+1-k)/2}) + O(v)) \leq exp(-\Omega(v^{3/2})).$$

We have thus proved that the probability that a fixed bijection π provides a packing of H_1 and H_2 is at most

$$exp(-\Omega(v^{3/2})).$$

Since the total number of bijections is only $v! = exp(O(v \log v))$, the probability that there exists a bijection that provides a packing is at most

$$v! \cdot exp(-\Omega(v^{3/2})) = o(1),$$

and hence almost surely, (i.e., with probability that tends to 1 as v tends to infinity), H_1 and H_2 cannot be packed. In particular, there exists a choice of partial (t, k, v)-designs H_1 and H_2 that cannot be packed, completing the proof of the theorem for the case of two (not necessarily isomorphic) partial designs.

The proof of the stronger assertion, that there is a single partial (t, k, v) design so that two copies of it cannot be packed is similar but requires a little more care. Here is a brief description. To simplify notation we omit all floor and ceiling signs when these are not crucial. Let p be the same probability as before and let $H_1 = (V, E_1)$ be the random partial design defined by the marking process described above. We claim that with high probability one cannot pack two copis of H_1. To prove this claim we first observe that the expected number of pairs of k-subsets of V that share at least t common elements and have both been marked is $p^2 O(v^{2k-t}) = o(pv^k)$ and hence almost surely there are only $o(pv^k)$ such pairs. On the other hand, by the standard estimates for Binomial distributions (cf., e.g., [1], Appendix A), for a fixed subset of $v/2$ vertices, the probability that the number of marked k-subsets of this subset is less than, say, half its expectation (i.e., less then $0.5p\binom{v/2}{k}$) is exponentially small in this expectation, i.e., it is much smaller than 2^{-v}. Thus almost surely in *every* subset of $v/2$ vertices there are at least $\Omega(pv^k)$ marked k-subsets, and since every pair among the $o(pv^k)$ ones mentioned above can cause the deletion of at most 2 marked k-subsets in the definition of H_1 we have proved the following:

Claim Almost surely every set of $v/2$ vertices of H_1 contains an edge.

We can thus assume that H_1 satisfies the property in the last claim. Consider, now, a fixed bijection $\pi : V \mapsto V$. If π has at least $v/2$ fixed points then since H_1 satisfies the above property π fixes at least one edge of H_1 (pointwise) and hence does not provide a packing of two copies of H_1. It remains to consider the bijections π that fix less than $v/2$ points. It is easy to see that for each such π there are two disjoint subsets U and W of V such that $|U| = |W| = v/6$ and π maps U onto W. Let H_U denote the induced subhypergraph of H_1 on U and let H_W be the induced subhypergraph on W. Observe that the k-subsets of W differ from these of U. Moreover, no k-subset of V intersects by at least t elements a k-subset of W and a k-subset of U. It follows that all the random choices in the marking process that generates H_U are independent of those in the process that generates H_W, and these are thus essentially two independent random partial $(t, k, v/6)$-designs. We can thus repeat the argument in the first part of the proof and conclude that for every fixed π, U and W as above, the probability that the restriction of π to a bijection from U to W packs H_U and H_W is at most $exp(-\Omega(v^{3/2}))$. As the total number of choices for π,U and W is less than $4^v v!$ we conclude that almost surely there is no such π. Hence there is a partial (t, k, v) design that cannot be packed with a copy of itself, completing the proof of the theorem. □

3 Designs with multiplicities

In this section we present the proof of Theorem 1.2. Let k and t be positive integers, $k \geq 2t - 1$. Observe that the case $k = t = 1$ is trivial as $c(1, 1, v) = 0$ for all v so suppose $k > 1$. As before assume, whenever this is needed, that v is sufficiently large. Throughout the proof we let $b_1, b_2, \ldots$ denote various positive constants that may depend on k and t but are independent of v.

Let V and V_2 be two sets of v vertices each and let $H_1 = (V, E_1)$ be an arbitrary partial (t, k, v)-design, where $|E_1| \geq b_1 v^t$. (The fact that such a partial design with

$$b_1 = (1 + o(1))\frac{1}{t!\binom{k}{t}}$$

exists is proved in [7]. However, if one does not need such a precise estimate of b_1 then this fact can be proved directly by a simple probabilistic argument which we omit).

Note that H_1 is a $(1, t, k, v)$ partial design. Define

$$p = b_2 \frac{\log v}{v^{t-1}},$$

where $b_2 = b_2(k,t)$ will be chosen later, and let $H_2 = (V_2, E_2)$ be a random k-uniform hypergraph obtained by choosing each k-subset K of V_2, randomly and independently with probability p, to be an edge of H_2. To complete the proof we establish the following two claims.

Claim 1 For sufficiently large b_2, almost surely no t-subset of V_2 is contained in more than $2p\binom{v-t}{k-t} \leq 2b_2 v^{k-2t+1} \log v$ edges of H_2. Thus H_2 is almost surely a (λ_2, t, k, v) partial design with $\lambda_2 = 2b_2 v^{k-2t+1} \log v$.

Claim 2 For sufficiently large b_2, almost surely H_1 and H_2 cannot be packed.

Since H_1 is a $(1, t, k, v)$ partial design, these two claims clearly imply that $c(t, k, v) < 1 \cdot \lambda_2 = O(v^{k-2t+1} \log v)$ completing the proof of Theorem 1.2. It thus remains to prove the two claims.

Proof of Claim 1 Let T be a fixed t-subset of V_2. The number of k-subsets of H_2 that contain T is a Binomial random variable with parameters $\binom{v-t}{k-t}$ and p, and hence its expectation is $p\binom{v-t}{k-t}$. By the standard estimates for Binomial distributions (cf., e.g., [1], Appendix A), the probability that T is contained is more than $2p\binom{v-t}{k-t}$ edges of H_2 is at most

$$exp(-b_3 p \binom{v-t}{k-t}) \leq exp(-b_4 b_2 v^{k-2t+1} \log v).$$

Here b_4 is some constant dependning only on k and t, and hence, by choosing, e.g., $b_2 = b_2(k,t) \geq 2t/b_4$, and since $k - 2t + 1 \geq 0$, the last quantity is at most v^{-2t}.

Since there are $\binom{v}{t} \leq v^t$ subsets of cardianlity t of V_2, the probability that there is one contained in more than $2p\binom{v-t}{k-t}$ edges of H_2 is at most $v^t v^{-2t} = o(1)$, completing the proof of the claim. $\Box$

Proof of Claim 2 Let $\pi : V_2 \mapsto V$ be a fixed bijection between V_2 and V and let us estimate the probability that π supplies a packing of H_1 and H_2. This probability is precisely the probability that there is no edge K of H_1 so that $\pi^{-1}(K) \in E_2$. Since each k-subset of V_2 is an edge of H_2 with probability p and all these choices are independent, the above probabilty is precisely

$$(1-p)^{|E_1|} \leq exp(-p \cdot b_1 v^t) \leq exp(-b_1 b_2 v \log v).$$

If $b_2 = b_2(k,t) > 1/b_1$ the last quantity is smaller than v^{-v}. Since the number of possible bijections is $v!$, the probability that there exists a bijection which provides a packing of H_1 and H_2 is at most $v! \cdot v^{-v} = o(1)$, completing the proof of the claim, and implying the assertion of Theorem 1.2. $\Box$

Acknowledgement I would like to thank L. Teirlinck for helpful comments.

References

[1] N. Alon and J. H. Spencer, *The Probabilistic Method*, Wiley, 1991.

[2] B. Bollobás and S. E. Eldbridge, *Packing of graphs and applications to computational complexity*, J. Combinatorial Theory, Ser. B 25 (1978), 105-124.

[3] B. Bollobás, *Extremal Graph Theory*, Academic Press, London, 1978, Chapter 8.

[4] P. A. Catlin, *Subgraphs of graphs*, Discrete Math. 10 (1974), 225-233.

[5] J. Doyen, *Constructions of disjoint Steiner Triple Systems*, Proc. Amer. Math. Soc. 32 (1972), 409-416.

[6] P. Ganter, J. Pelikan and L. Teirlinck, *On small sprawling systems of equicardinal sets*, ARS Combinatoria 4 (1977), 133-142.

[7] V. Rödl, *On a packing and covering problem*, European J. Combinatorics 5 (1985), 69-78.

[8] N. Sauer and J. Spencer, *Edge disjoint placement of graphs*, J. Combinatorial Theory, Ser. B 25 (1978), 295-302.

[9] W. C. S. Suen, *A correlation inequality and a Poisson limit theorem for nonoverlapping balanced subgraphs of a random graph*, Random Structures and Algorithms 1 (1990), 231-242.

[10] L. Teirlinck, *On making two Steiner Triple Systems disjoint*, J. Combinatorial Theory, Ser. A 23 (1977), 349-350.

[11] L. Teirlinck, Talk in "Waterloo 92", an International Conference on Combinatorics and Optimization, Waterlo, Ontario, Canada, 1992.