H-free subgraphs of dense graphs maximizing the number of cliques and their blow-ups

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Abstract

We consider the structure of H-free subgraphs of graphs with high minimal degree. We prove that for every $k > m$ there exists an $\epsilon := \epsilon(k, m) > 0$ so that the following holds. For every graph H with chromatic number k from which one can delete an edge and reduce the chromatic number, and for every graph G on $n > n_0(H)$ vertices in which all degrees are at least $(1-\epsilon)n$, any subgraph of G which is H-free and contains the maximum number of copies of the complete graph K_m is $(k-1)$-colorable.

We also consider several extensions for the case of a general forbidden graph H of a given chromatic number, and for subgraphs maximizing the number of copies of balanced blowups of complete graphs.

1 Introduction

The well known theorem of Turán ([19]) states that a K_k-free subgraph of the complete graph on n vertices with the maximum possible number of edges is $k-1$-chromatic. Erdős, Stone and Simonovits show in [14], [12] that for general H with $\chi(H) = k$ the maximum possible number of edges in an H-free graph on n vertices is at most $o(n^2)$ more than the number of edges in a $k-1$-chromatic graph on n vertices. In [4] it is shown that the same holds for H-free subgraphs of the complete graph that have the maximum possible number of copies of K_m for a fixed m such that $k > m \geq 2$.

Looking at subgraphs of general graphs G it is clear that a K_m-free subgraph of G with the maximum possible number of edges has at least as many edges as the largest $m-1$-partite subgraph. In [11] Erdős asked for which graphs there is an equality between the two. In [1] it is shown that this is the case for line graphs of bipartite graphs. In a different direction, in [8] it is proved that if a graph has a high enough minimum degree then any subgraph of it which is K_3-free and has the maximum possible number of edges is bipartite. In [7] a stronger bound is given on the minimum degree ensuring this. Before stating a generalization of these theorems we introduce some notation.

For a graph G, fixed graphs H and T and an integer k let $G_{part(k),T}$ be a k-partite subgraph of G with the maximum possible number of copies of T and let $\mathcal{G}_{ex}(T, H)$ be the family of subgraphs of G that are H-free and have the maximum possible number of copies of T. Let $\mathcal{N}(G, T)$ denote

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the number of copies of T in G. Call a graph H edge critical if there is an edge $\{u, v\} \in E(H)$ whose removal reduces the chromatic number of H.

In [3] the following theorem is proved, generalizing the results in [8] and [7]. Throughout the paper we denote by $\delta(G)$ the minimum degree in the graph G.

Theorem 1.1 ([3]). *Let H be a graph with $\chi(H) = k + 1$. Then there are positive constants $\gamma := \gamma(H)$ and $\mu := \mu(H)$ such that if G is a graph on $n > n_0(H)$ vertices with $\delta(G) > (1 - \mu)n$ then for every $G_{ex} \in \mathcal{G}_{ex}(K_2, H)$*

1. *If H is edge critical then $\mathcal{N}(G_{part(k),K_2}, K_2) = \mathcal{N}(G_{ex}, K_2)$*
2. *Otherwise, $\mathcal{N}(G_{part(k),K_2}, K_2) \leq \mathcal{N}(G_{ex}, K_2) \leq \mathcal{N}(G_{part(k),K_2}, K_2) + O(n^{2-\gamma})$.*

In the present short paper we prove two theorems for H-free subgraphs assuming H is edge critical. The first is for subgraphs maximizing the number of copies of K_m and the second for subgraphs maximizing the number of blow-ups of K_m. We also establish a proposition concerning graphs H that are not edge critical.

Theorem 1.2. *For every two integers $k > m$ and every edge critical graph H such that $\chi(H) = k$ there exist constants $\epsilon := \epsilon(k, m) > 0$ and $n_0 = n_0(H)$ such that the following holds. Let G be a graph on $n > n_0$ vertices with $\delta(G) \geq (1 - \epsilon)n$, then for every $G_{ex} \in \mathcal{G}_{ex}(K_m, H)$ the graph G_{ex} is $(k - 1)$-colorable.*

For integers m and t let $K_m(t)$ denote the t-blow-up of K_m, that is, the graph obtained by replacing each vertex of K_m by an independent set of size t and each edge by a complete bipartite graph between the corresponding independent sets.

Theorem 1.3. *For integers m and t and every edge critical H such that $\chi(H) = m + 1$ there exist constants $\epsilon := \epsilon(m, t)$ and $n_0 := n_0(H)$ such that the following holds. Let G be a graph on $n > n_0$ vertices with $\delta(G) > (1 - \epsilon)n$, then every $G_{ex} \in \mathcal{G}_{ex}(K_m(t), H)$ is m-colorable.*

Finally, for graphs H which are not edge critical we prove the following.

Proposition 1.4. *For every integers $m < k$ and t and graph H such that $\chi(H) = k$ there exists $\epsilon := \epsilon(m, t, k)$ and $n_0 := n_0(H)$ such that the following holds. Let G be a graph on $n > n_0$ vertices with $\delta(G) > (1 - \epsilon)n$ and assume that $t = 1$ or $k = m + 1$, then every $G_{ex} \in \mathcal{G}_{ex}(K_m(t), H)$ can be made $k - 1$ colorable by deleting $o(n^2)$ edges.*

Theorems 1.2 and 1.3 cannot be directly generalized to graphs H that are not edge critical as we can add to any $k - 1$-partite graph an edge without creating a copy of such H. On the other hand, we believe that the error term $o(n^2)$ in Proposition 1.4 can be improved to $O(n^{2-\delta})$ for some $\delta := \delta(H)$.

The rest of this short paper is organized as follows. In Section 2 we state several known results and prove some helpful lemmas. Section 3 contains the proof of of Theorem 1.2. Theorem 1.3 is proved in Section 4 and the proof of Proposition 1.4 appears in Section 5. The final Section 6 contains some concluding remarks and open problems.

2 Preliminary results

We start by stating several results about H-free graphs with high degrees and by deducing a corollary. Some of the theorems stated are simplified versions of the original results.

The first result about K_k-free graphs is by Andrásfai, Erdős and Sós.

Theorem 2.1 ([6])**.** *Let G be a graph on n vertice. If G is K_k-free and $\delta(G) \geq (1 - \frac{3}{3k-4})n$ then $\chi(G) \leq k-1$.*

A generalization of Theorem 2.1 proved in [13] is the following.

Theorem 2.2 ([13])**.** *Let H be a fixed edge critical graph which is not K_k and assume $\chi(H) = k$. If G is a graph on $n > n_0(H)$ vertices which is H-free and contains a copy of K_k then $\delta(G) \leq (1 - \frac{1}{k-3/2})n + O(1)$.*

This implies that if n is large enough, $\delta(G) \geq (1 - \frac{3}{3k-4})n \geq (1 - \frac{1}{k-3/2})n + O(1)$, and if G is H-free for some edge critical graph H with $\chi(H) = k$ then it must also be K_k-free. Together with Theorem 2.1 we get the following corollary:

Corollary 2.3. *Let H be a fixed edge critical graph such that $\chi(H) = k$. Let G be a graph on $n > n_0(H)$ vertices which is H-free and satisfies $\delta(G) \geq (1 - \frac{3}{3k-4})n$, then $\chi(G) \leq k-1$.*

We next state the graph removal lemma as it appears in [9] (see also [2], [18] and [16]) and prove a simple lemma using it. Throughout the paper we denote by $v(G)$ the number of vertices in the graph G.

Theorem 2.4 (The graph removal lemma)**.** *For any graph H with $v(H)$ vertices and any $\epsilon > 0$, there exists a $\delta > 0$ such that any graph on n vertices which contains at most $\delta n^{v(H)}$ copies of H can be made H-free by removing at most ϵn^2 edges.*

Throughout the paper, for fixed graphs T and H and an integer n we denote by $ex(n, T, H)$ the maximum possible number of copies of T in an H-free graph on n vertices.

Lemma 2.5. *Let H be a fixed graph such that $\chi(H) = k$ and let G be an H-free graph on n vertices, where $n > n_0(H)$. Then G can be made K_k-free by deleting $o(n^2)$ edges.*

Proof. Note, first, that the number of copies of K_k in G is $o(n^k)$. Indeed, in [4] it is shown that if a graph H is a subgraph of a blow-up of a graph T then $ex(n, T, H) = o(n^{v(T)})$.

Since $\chi(H) = k$, H is contained in a blow-up of K_k and hence $\mathcal{N}(G, K_k) \leq ex(n, K_k, H) \leq o(n^k)$. By the graph removal lemma G can be made K_k-free by removing $o(n^2)$ edges, as needed. □

We next prove two additional more technical lemmas.

Lemma 2.6. *Let G be a graph on n vertices satisfying $\delta(G) > (1 - \epsilon)n$ for some fixed $\epsilon > 0$, and let $m < k$ and t be integers.*

1. $\mathcal{N}(G_{part(k-1),K_m}, K_m) \geq (1 + o(1))(1 - \frac{m(m-1)}{2}\epsilon)\binom{k-1}{m}(\frac{n}{k-1})^m$
2. *Let $k = m + 1$ then* $\mathcal{N}(G_{part(m),K_m(t)}, K_m(t)) \geq (1 + o(1))(1 - c_2'\epsilon)n^{mt}\frac{1}{(t!m^t)^m}$

where $c_2' := c_2'(m,t)$.

Proof. To prove part 1 note that as $\delta(G) \geq (1-\epsilon)n$ the number of copies of K_m in G is at least

$$n \cdot (1-\epsilon)n \cdot (1-2\epsilon)n \dots (1-(m-1)\epsilon)n\frac{1}{m!} \geq (1+o(1))n^m(1-\frac{m(m-1)}{2}\epsilon)\frac{1}{m!}$$

Randomly partitioning the graph into $k-1$ sets yields a graph in which the expected number of copies of K_m is a least:

$$\begin{aligned}
&(1+o(1))n^m(1-\frac{m(m-1)}{2}\epsilon)\frac{1}{m!}\cdot\frac{k-2}{k-1}\cdot\frac{k-3}{k-1}\cdots\frac{k-m}{k-1}\\
=&(1+o(1))n^m(1-\frac{m(m-1)}{2}\epsilon)\frac{(k-2)!}{m!(k-1)^{m-1}(k-(m+1))!}\\
=&(1+o(1))(1-\frac{m(m-1)}{2}\epsilon)\binom{k-1}{m}(\frac{n}{k-1})^m.
\end{aligned}$$

Thus $G_{part(k-1),K_m}$ should have at least as many copies. This proves (1).

Similarly to prove part 2 observe that the number of copies of $K_m(t)$ in G is at least

$$\frac{1}{m!(t!)^m}(n)_t((1-t\epsilon)n)_t\cdot\ldots\cdot((1-(m-1)t\epsilon)n)_t \geq (1+o(1))\frac{n^{mt}}{m!(t!)^m}(1-c_2'\epsilon).$$

Randomly partitioning G into m parts gives a graph in which the expected number of copies of $K_m(t)$ is at least

$$\begin{aligned}
&(1+o(1))(1-c_2'\epsilon)\frac{n^{mt}}{m!(t!)^m}(\frac{1}{m^{t-1}})(\frac{m-1}{m}\frac{1}{m^{t-1}})\dots(\frac{1}{m}\frac{1}{m^{t-1}})\\
=&(1+o(1))(1-c_2'\epsilon)n^{mt}\frac{1}{m!(t!)^m}\frac{m!}{m^{mt}}\\
=&(1+o(1))(1-c_2'\epsilon)n^{mt}\frac{1}{(t!m^t)^m}
\end{aligned}$$

and thus $G_{part(m),K_m}$ must have at least as many copies of $K_m(t)$. $\square$

Lemma 2.7. *Let G be a graph on n vertices with $\delta(G) > (1-\epsilon)n$ for some fixed $\epsilon > 0$ and let t and $m < k$ be integers. For a set $U \subseteq V(G)$ satisfying $|U| \geq \alpha n$ for some fixed $\alpha > 0$ let $f_{k,m,t}(U)$ be the maximum number of copies of $K_m(t)$ in a $k-1$-partite subgraph of $G[U]$. Then there exist constants $c_1 := c_1(k,m,\alpha)$ and $c_2 := c_2(m,t,\alpha)$ such that for every $v \in V(G)\setminus U$*

1. $f_{m,k,1}(U\cup\{v\}) \geq f_{m,k,1}(U) + (1+o(1))|U|^{m-1}\binom{k-1}{m}\frac{m}{(k-1)^m}(1-c_1\epsilon)$
2. $f_{m,m+1,t}(U\cup\{v\}) \geq f_{m,m+1,t}(U) + (1+o(1))|U|^{mt-1}\frac{mt}{(t!m^t)^m}(1-c_2\epsilon)$

Proof. Let $|U| = q \geq \alpha n$. We first prove part 1. Fix a partition of $G[U]$ into $k-1$ parts with $f_{m,k,1}(U)$ copies of K_m. By Lemma 2.6, part 1 the number of copies of K_m is at least:

$$(1+o(1))q^m(1-\frac{m(m-1)}{2}\epsilon)\binom{k-1}{m}\frac{1}{(k-1)^m}.$$

Averaging we get that there is a vertex, say $w \in U$, so that the number of copies of K_m it takes part in is at least

$$(1+o(1))\frac{m}{q}q^m(1-\frac{m(m-1)}{2}\epsilon)\binom{k-1}{m}\frac{1}{(k-1)^m} = (1+o(1))q^{m-1}(1-\frac{m(m-1)}{2}\epsilon)\binom{k-1}{m}\frac{m}{(k-1)^m}$$

Let $U_1, \ldots, U_{k-1}$ be the above fixed partition of U which has $f_{m,k,1}(U)$ copies of K_m and assume, without loss of generality, that $w \in U_{k-1}$. We add v to U_{k-1} and bound from below the number of copies of K_m we add by doing this. Let $b_i = |U_i|$ and let d_i be the number of neighbors w has in U_i which are not neighbors of v. Note that $\sum_{i\in[k-2]} d_i \leq \epsilon n$ and $\sum_{i\in[k-2]} b_i = q$.

For each U_i we estimate the number of copies of K_m in which w takes part that use vertices from U_i that are not neighbors of v. There are d_i of those, and in the worst case each such vertex is connected to all of the sets U_j for $j \neq i, k-1$. Thus the number of copies of K_m that w takes part in and v does not is at most

$$\begin{aligned}
&\sum_{i=1}^{k-2} d_i(\sum_{\{j_1,\ldots,j_{m-2}\}\subseteq[k-2]\setminus i} b_{j_1}\ldots b_{j_{m-2}})\\
\leq&(d_1+\cdots+d_{k-2})(b_1+\cdots+b_{k-2})^{m-2}\frac{1}{(m-2)!}\\
\leq&\epsilon n\cdot q^{m-2} \leq \frac{1}{\alpha}\epsilon q^{m-1}
\end{aligned}$$

And so by adding v the number of copies of K_m added is at least

$$\begin{aligned}
&(1+o(1))q^{m-1}[(1-\frac{m(m-1)}{2}\epsilon)\binom{k-1}{m}\frac{1}{(k-1)^m}-\frac{\epsilon}{\alpha}]\\
=&(1+o(1))q^{m-1}\binom{k-1}{m}\frac{1}{(k-1)^m}(1-c_1\epsilon)
\end{aligned}$$

The proof of (2) is similar. By Lemma 2.6 part 2, in any partition of $G[U]$ into m parts in which the number of copies of $K_m(t)$ is $f_{m,m+1,t}(U)$, this number is at least

$$(1+o(1))\frac{1}{(t!m^t)^m}(1-c_2'\epsilon)q^{mt}.$$

Let $U = U_1 \cup \ldots \cup U_m$ be such a partition. By averaging there must be a vertex, say $w \in U$, such that the number of copies of $K_m(t)$ it takes part in is at least:

$$(1+o(1))\frac{mt}{q}q^{mt}\frac{1}{(t!m^t)^m}(1-c_2'\epsilon) = (1+o(1))q^{mt-1}\frac{mt}{(t!m^t)^m}(1-c_2'\epsilon)$$

Assume, without loss of generality, that $w \in U_m$, and let us add v to U_m. Let $b_i = |U_i|$ and let d_i be the vertices in U_i that are neighbors of w and not of v. Then the number of copies of $K_m(t)$ in this partition that w takes part in and v does not is at most

$$\begin{aligned}
&\binom{b_m}{t-1}\sum_{i=1}^{m-1} d_i\binom{b_i}{t-1}\prod_{j\in[m-1]\setminus i}\binom{b_j}{t}\\
<&(\sum_{i=1}^{m-1} d_i)\binom{q}{mt-2} < c_2''\epsilon q^{mt-1}
\end{aligned}$$

where the last inequality is true for some $c_2'' := c_2''(m,t,\alpha)$. Thus when adding v to U_m the number of copies of $K_m(t)$ added is at least

$$q^{mt-1}[\frac{mt}{(t!m^t)^m}(1-c_2'\epsilon)-c_2''\epsilon] = q^{mt-1}\frac{mt}{(t!m^t)^m}(1-c_2\epsilon)$$

as needed. $\square$

3 Maximizing the number of cliques

In the proof of Theorem 1.2 we use the following result from [4].

Proposition 3.1 ([4])**.** *Let H be a graph such that $\chi(H) = k > m$ then*

$$ex(n, K_m, H) = (1+o(1))\binom{k-1}{m}(\frac{n}{k-1})^m$$

Proof of Theorem 1.2. Let G and H be as in the theorem and let $G_{ex} \in \mathcal{G}_{ex}(K_m, H)$. If $\delta(G_{ex}) \geq (1-\frac{3}{3k-4})n$, as H is edge-critical, by Corollary 2.3 $\chi(G_{ex}) \leq k-1$ and we are done. Thus assume towards contradiction that $\delta(G_{ex}) < (1-\frac{3}{3k-4})n$.

As any partition of G into $k-1$ parts is H-free, by Lemma 2.6, part 1, the number of copies of K_m in G_{ex} must be at least

$$(1+o(1))(1-\frac{m(m-1)}{2}\epsilon)\binom{k-1}{m}(\frac{n}{k-1})^m. \tag{1}$$

Consider the following iterative process of removing vertices from G_{ex}. Put $G_0 = G_{ex}$ and $n_0 = n$. Let $v_0 \in V(G)$ be an arbitrarily chosen vertex of G_0 satisfying $d(v_0) < (1-\frac{3}{3k-4})n_0$. Define $G_1 = G - v_0$ and $n_1 = n_0 - 1$. For $j \geq 1$ if the minimum degree in G_j satisfies $\delta(G_j) \geq (1-\frac{3}{3k-4})n_j$ then stop the process, otherwise take a vertex $v_j \in V(G_j)$ of degree $d_{G_j}(v_j) < (1-\frac{3}{3k-4})n_j$ and define $G_{j+1} = G_j - v_j$ and $n_{j+1} = n_j - 1$.

We first show that this process must stop after at most $n/2$ steps. To see this note that the number of copies of K_m removed with each deleted vertex is exactly the number of copies of K_{m-1} in its neighborhood. By Proposition 3.1 for any $(k-1)$-chromatic graph H', $ex(n, K_{m-1}, H') = (1+o(1))\binom{k-2}{m-1}(\frac{n}{k-2})^{m-1}$. As G_{ex} is H-free, the neighborhood of any vertex should be $(H-v)$-free, where $v \in V(H)$ is such that $\chi(H-v) = k-1$.

Thus at step j (starting to count from $j=0$), at most $(1+o(1))\binom{k-2}{m-1}(\frac{n_j(1-\frac{3}{3k-4})}{k-2})^{m-1}$ copies of K_m have been removed.

As the following equality holds

$$\frac{1}{k-1} - (1-\frac{3}{3k-4})\frac{1}{k-2} = \frac{1}{(3k-4)(k-2)(k-1)}$$

one can choose $\delta = \delta(k,m) > 0$ so that $((1-\frac{3}{3k-4})\frac{1}{k-2})^{m-1} = \frac{1}{(k-1)^{m-1}}(1-\delta)$. Thus the number of copies of K_m removed at step j is no more than:

$$(1+o(1))n_j^{m-1}\binom{k-2}{m-1}\frac{1}{(k-1)^{m-1}}(1-\delta) = (1+o(1))n_j^{m-1}\binom{k-1}{m}\frac{m}{(k-1)^m}(1-\delta) \tag{2}$$

Together with the fact that

$$\sum_{r=0}^{n/2-1}(n-r)^{m-1} \leq (1+o(1))(\frac{1}{m}-\frac{1}{m}\frac{1}{2^m})n^m$$

we conclude that the number of copies of K_m removed during the first $\frac{n}{2}$ steps is at most

$$\begin{aligned}&(1+o(1))\sum_{j=0}^{n/2-1} n_j^{m-1}\binom{k-1}{m}\frac{m}{(k-1)^m}(1-\delta)\\ \leq&(1+o(1))(1-\delta)\binom{k-1}{m}\frac{1}{(k-1)^m}(1-\frac{1}{2^m})n^m\end{aligned}$$

The graph $G_{n/2}$ has at most $ex(\frac{1}{2}n, K_m, H) = (1+o(1))\binom{k-1}{m}(\frac{1}{2}\frac{n}{(k-1)})^m$ copies of K_m, and hence the total number of copies of K_m in G_{ex} is at most

$$\begin{aligned}&(1+o(1))n^m(1-\delta)\binom{k-1}{m}\frac{1}{(k-1)^m}(1-\frac{1}{2^m}) + (1+o(1))n^m\binom{k-1}{m}\frac{1}{(k-1)^m}\frac{1}{2^m}\\ =&(1+o(1))n^m(1-\delta(1-\frac{1}{2^m}))\binom{k-1}{m-1}\frac{1}{(k-1)^m}\end{aligned}$$

But if ϵ is small enough this contradics (1). Thus the process must stop after $r+1 \leq \frac{n}{2}$ steps.

As $\delta(G_r) \geq (1-\frac{3}{3k-4})n_r$ and H is edge critical, Corollary 2.3 implies that $\chi(G_r) \leq k-1$. Define $V(G_r) = V_r$.

The $k-1$ partite subgraph of $G[V_r]$ with the maximum possible number of copies of K_m has at least as many copies of K_m as G_r. By Lemma 2.7, part 1 we can now add the vertices removed during the steps of the process starting from $j = r-1$ until $j = 0$, keeping the resulting subgraph $(k-1)$-partite, where with each such vertex we add at least $(1+o(1))n_j^{m-1}\binom{k-1}{m}\frac{m}{(k-1)^m}(1-c_1\epsilon)$ copies of K_m. Assuming that ϵ is small enough to ensure, say, $c_1\epsilon < \delta/2$ it follows that in each such step the number of added copies of K_m exceeds the number of copies removed in the corresponding removal step.

When all the vertices are back we obtain a $k-1$ partite subgraph of G containing more copies of K_m than G_{ex}. This subgraph is H-free, contradicting the maximality of G_{ex}. Thus the inequality $\delta(G_{ex}) \geq (1-\frac{3}{3k-4})n$ must hold and the desired result follows. □

4 Maximizing the number of blow-ups of cliques

To prove Theorem 1.3 we first need a good estimate on $ex(n, K_m(t), H)$ for H satisfying $\chi(H) = m+1$.

Proposition 4.1. *For integers m and t and any fixed graph H such that $\chi(H) = m+1$,*

$$ex(n, K_m(t), H) = (1+o(1))\binom{n/m}{t}^m$$

Proof. To show that $ex(n, K_m(t), H) \geq (1 + o(1))\binom{n/m}{t}^m$ it is enough to take the m-sided Turán graph (i.e. the m-partite graph with sides of nearly equal size). As $\chi(H) = m + 1$ it is H-free and has $(1 + o(1))\binom{n/m}{t}^m$ copies of $K_m(t)$.

As for the upper bound, in [4] it is shown that the graph which is K_{m+1} free and has the maximum possible number of copies of $K_m(t)$ is a complete multipartite graph. It is not difficult to see that the Turán graph maximizes the number of copies of $K_m(t)$ among these. Thus

$$ex(n, K_m(t), K_{m+1}) = (1 + o(1))\binom{n/m}{t}^m$$

Let H be as in the proposition, and let G be an H-free graph on n vertices with the maximum number of copies of $K_m(t)$. By Lemma 2.5 G can be made K_{m+1}-free by deleting $o(n^2)$ edges, and with them at most $o(n^2)O(n^{mt-2}) = o(n^{mt})$ copies of $K_m(t)$. Let G' be the graph obtained by removing those $o(n^2)$ edges.

As G' is K_{m+1}-free we get

$$\begin{aligned}(1 + o(1))ex(n, K_m(t), H) &= (1 + o(1))\mathcal{N}(G, K_m(T)) = \\ &= \mathcal{N}(G', K_m(t)) \leq ex(n, K_m(t), K_{m+1}) = (1 + o(1))\binom{n/m}{t}^m\end{aligned}$$

and so $ex(n, K_m(t), H) \leq (1 + o(1))\binom{n/m}{t}^m$ as needed. □

The idea of the proof of Theorem 1.3 is similar to the one of Theorem 1.2 but some of the estimates are more involved.

Proof of Theorem 1.3. Let G be a graph with $\delta(G) > (1 - \epsilon)n$ and let $G_{ex} \in \mathcal{G}(K_m(t), H)$. If $\delta(G_{ex}) \geq (1 - \frac{3}{3m-1})n$ then by Corollary 2.3, $\chi(G_{ex}) \leq m$, as H is edge critical and we are done.

Assume towards contradiction that $\delta(G_{ex}) < (1 - \frac{3}{3m-1})n$. Consider the following iterative process, similar to the one in the proof of Theorem 1.2. Put $G_0 = G_{ex}$ and $n_0 = n$. At step $j > 0$ if G_j satisfies $\delta(G_j) \geq (1 - \frac{3}{3m-1})n_j$ then stop the process, otherwise take $v_j \in V(G_j)$ of degree $d_{G_j}(v_j) \leq (1 - \frac{3}{3m-1})n_j$ and define $G_{j+1} = G_j - v_j$ and $n_{j+1} = n_j - 1$. We show that the process must stop after at most $\frac{n}{2}$ steps.

To bound the number of $K_m(t)$ removed at each step we take care of two cases. If in a copy of $K_m(t)$ there is a vertex in the same color class of v_i that is a neighbor of v_i in G_i, call this copy *dense*. If all of the vertices in the color class of v_i are non-neighbors of it in G_i call the copy *sparse*.

First we estimate the number of dense copies. Let $K^+_{m-1}(t)$ be the graph obtained by taking $K_{m-1}(t)$ and adding to it a vertex that is connected to all of the other vertices. The number of dense copies of $K_m(t)$ containing v_i is at most $\mathcal{N}(G[N(v)], K^+_{m-1}(t))n_i^{t-2}$.

As H is edge critical there is a vertex $v \in V(H)$ such that $\chi(H - v) = m$, let $H' = H - v$. By a result in [4] if H is contained in a blow-up of T then $ex(n, T, H) = o(n^{v(T)})$. As the neighborhood of v_i must be H'-free and as H' is contained in a blow-up of $K^+_{m-1}(t)$, it follows that $\mathcal{N}(G[N(v)], K^+_{m-1}(t)) = o(|N(v)|^{t(m-1)+1})$. Thus the number of dense copies of $K_m(t)$ in G_{j+1} containing v_i is $o(n_i^{tm-1})$.

As for the sparse copies, let $A(v_i)$ be the number of sparse copies of $K_m(t)$ in G_i containing v_i. Let $N^c_{G_i}(v_i) = V(G_i) \setminus (N_{G_i}(v_i) \cup \{v_i\})$ and $d = d_{G_i}(v_i)$, and let $H' = H - v$ for $v \in V(H)$ such that

$\chi(H') = m$. Using Proposition 4.1 we obtain the following bound on the number of sparse copies of $K_m(t)$ containing v_i

$$
\begin{aligned}
A(v_i) \leq & \sum_{u_1,\ldots,u_{t-1} \subseteq N^c_{G_i}(v_i)} ex\big(|N(v_i) \cap N(u_1) \cap \cdots \cap N(u_{t-1})|, K_{m-1}(t), H'\big) \\
& \leq \binom{n_i - d - 1}{t-1} ex(d, K_{m-1}(t), H') \\
& = (1+o(1)) \binom{n_i - d}{t-1} \binom{d/(m-1)}{t}^{m-1} \\
& \leq (1+o(1)) \frac{(n_i-d)^{t-1}}{(t-1)!} \Big(\frac{d^t}{(m-1)^t t!}\Big)^{m-1}
\end{aligned}
$$

To bound this quantity consider the following function $f(d) = d^{t(m-1)}(n_i - d)^{t-1}$. Note that $f(d)$ is a polynomial in d, $f(d) > 0$ for $0 < d < n_i$ and $f(n_i) = f(0) = 0$. Furthermore

$$
\begin{aligned}
f'(d) =& (d^{t(m-1)}(n_i-d)^{t-1})' \\
=& d^{t(m-1)-1}(n_i-d)^{t-2}[t(m-1)(n_i-d) - (t-1)d]
\end{aligned}
$$

Thus $f'(d) = 0$ for $d = 0$, $d = n_i$, $d = (\frac{t(m-1)}{t(m-1)+(t-1)})n_i = (1 - \frac{t-1}{tm-1})n_i =: \beta$ and is positive in $[0, \beta]$. It follows that between 0 and n $f(d)$ obtains its global maximum at the single values $0 < \beta < n$ for which $f'(\beta) = 0$, and it is increasing in $[0, \beta]$.

In our case $d < (1 - \frac{3}{3m-1})n_i$ and as $m > 2$ it follows that $1 - \frac{t-1}{tm-1} > 1 - \frac{3}{3m-1}$. We conclude that $f(d) \leq f((1 - \frac{3}{3m-1})n_i)$. Plugging this value it follows that

$$
\begin{aligned}
A(v_i) \leq & (1+o(1)) \frac{(\frac{3}{3m-1}n_i)^{t-1}}{(t-1)!} \Big(\frac{((1-\frac{3}{3m-1})n_i)^t}{(m-1)^t t!}\Big)^{m-1} \\
=& (1+o(1)) n_i^{mt-1} \Big(\frac{3}{3m-1}\Big)^{t-1} \Big(1 - \frac{3}{3m-1}\Big)^{t(m-1)} \frac{1}{(t-1)!(m-1)^{t(m-1)}(t!)^{m-1}}
\end{aligned}
$$

Next we bound $(\frac{3}{3m-1})^{t-1}(1 - \frac{3}{3m-1})^{t(m-1)}$. As $\frac{3}{3m-1} = \frac{1}{(3m-1)m} + \frac{1}{m}$ the following holds:

$$
\begin{aligned}
\Big(\frac{3}{3m-1}\Big)^{t-1}\Big(1 - \frac{3}{3m-1}\Big)^{t(m-1)} =& \Big(\frac{1}{m} + \frac{1}{(3m-1)m}\Big)^{t-1}\Big(\frac{m-1}{m} - \frac{1}{(3m-1)m}\Big)^{t(m-1)} \\
=& \frac{1}{m^{t-1}}\Big(\frac{m-1}{m}\Big)^{t(m-1)}\Big(1 + \frac{1}{3m-1}\Big)^{t-1}\Big(1 - \frac{1}{(3m-1)(m-1)}\Big)^{t(m-1)} \\
\leq& \frac{1}{m^{t-1}}\Big(\frac{m-1}{m}\Big)^{t(m-1)}\Big(1 + \frac{1}{3m-1}\Big)^{t-1} e^{-t/(3m-1)} \\
=& \frac{1}{m^{t-1}}\Big(\frac{m-1}{m}\Big)^{t(m-1)}\Big[\Big(1 + \frac{1}{3m-1}\Big)e^{-1/(3m-1)}\Big]^{(t-1)} e^{-1/(3m-1)} \\
\leq& \frac{1}{m^{t-1}}\Big(\frac{m-1}{m}\Big)^{t(m-1)}(1 - \delta)
\end{aligned}
$$

For an appropriate $\delta := \delta(m, d) > 0$, indeed such a δ exists as $e^{-1/(3m-1)} < 1$ and $(1 + \frac{1}{3m-1})e^{-1/(3m-1)} < 1$ for $m > 2$.

Therefore, the number of copies of $K_m(t)$ (both dense and sparse) removed at step i is at most

$$\begin{aligned}(1+o(1))n_i^{mt-1}(1-\delta)\frac{1}{m^{t-1}}(\frac{m-1}{m})^{t(m-1)}\frac{1}{(t-1)!(m-1)^{t(m-1)}(t!)^{m-1}}+o(n_i^{mt-1})\\ =(1+o(1))n_i^{mt-1}(1-\delta)\frac{mt}{m^{tm}(t!)^m}.\end{aligned} \tag{3}$$

If the process continued for $\frac{n}{2}$ steps, as $\sum_{r=0}^{n/2}(n-r)^{mt-1} \leq (1+o(1))\frac{n^{mt}}{mt}(1-\frac{1}{2^{mt}})$ the total number of copies of $K_m(t)$ removed is at most

$$\begin{aligned}&\sum_{r=0}^{(n/2)-1}(1+o(1))(n-r)^{mt-1}(1-\delta)\frac{mt}{m^{tm}(t!)^m}\\ \leq&(1+o(1))n^{mt}(1-\delta)(1-\frac{1}{2^{mt}})\frac{1}{m^{tm}(t!)^m}.\end{aligned}$$

By proposition 4.1 in the graph $G_{n/2}$ the number of copies of $K_m(t)$ is at most

$$(1+o(1))\binom{n/(2m)}{t}^m \leq (1+o(1))(\frac{(n/2)^t}{m^t t!})^m = (1+o(1))n^{tm}\frac{1}{m^{tm}(t!)^m}\frac{1}{2^{mt}}.$$

Thus the number of copies of $K_m(t)$ in G_{ex} is at most

$$n^{tm}\big((1-\delta(1-\frac{1}{2^{mt}}))\frac{1}{m^{tm}(t!)^m}\big)$$

in contradiction to the maximality of G_{ex}. And so the process must stop after $n/2$ steps.

Assume that we have stopped at step $r < n/2$ and let $V_r = V(G_r)$. By Corollary 2.3 G_r must be m-partite, thus the m-partite subgraph of $G[V_r]$ with the maximum possible number of copies of $K_m(t)$ has at least as many copies of $K_m(t)$ as G_r.

By Lemma 2.7, part 2, we can return the vertices removed in the process in a reverse order (starting from v_{r-1} until v_0) keeping the graph m-partite and adding with each vertex v_j at least $(1+o(1))n_j^{mt-1}\frac{mt}{(t!m^t)^m}(1-c_2\epsilon)$ copies of $K_m(t)$. Assuming that ϵ is small enough to ensure, say, $c_2\epsilon < \delta/2$ it follows that with each vertex v_j we add more copies of $K_m(t)$ than were removed at the corresponding step.

Thus when all vertices are returned we obtain an m-partite graph with more copies of $K_m(t)$ than G_{ex}. As an m-partite graph is H-free this contradicts the definition of G_{ex}. Thus it must be that $\delta(G_{ex}) \geq (1-\frac{3}{3m-1})n$ and G_{ex} is m-partite. □

5 Forbidding graphs that are not edge critical

The proofs of Theorems 1.2 and 1.3 actually give a stronger result than stated, as follows

Lemma 5.1. *Let $m < k$ and t be integers, let G be a graph on n vertices such that $\delta(G) > (1-\epsilon)n$, where $\epsilon := \epsilon(k,m,t) > 0$ is sufficiently small, and let $G_{ex} \in \mathcal{G}_{ex}(K_m(t), K_k)$. Assume that $k = m+1$ or $t = 1$. Then for every K_k-free subgraph of G on the same set of vertices, say $G^1 \subseteq G$, at least one of the following holds:*

1. $\mathcal{N}(G^1, K_m(t)) \le \gamma \mathcal{N}(G_{ex}, K_m(t))$ *for some* $\gamma := \gamma(k, m, t) < 1$*.*

2. G^1 *can be made* $k-1$*-chromatic by deleting* $o(n^2)$ *edges.*

Proof. If $\delta(G^1) \ge (1 - \frac{3}{3k-4})v(G^1)$ then by Theorem 2.1 G^1 is $k-1$-chromatic and hence case (2) holds and we are done. If $\delta(G^1) < (1 - \frac{3}{3k-4})v(G^1)$ we consider a similar process to the one in the proofs of Theorem 1.2 and 1.3. For step $j = 0$ of the process define $G^1_0 = G^1$, for steps $j > 0$ let v_j be a vertex of minimum degree in G^1_{j-1} and define $G^1_j = G^1_{j-1} - v_j$. The process stops when either $\delta(G^1_j) \ge (1 - \frac{3}{3k-4})v(G^1_j)$ or when $v(G^1_j) = \alpha n$ for $\alpha := \alpha(\gamma)$ small enough.

The calculations in the proofs of Theorems 1.2 and 1.3 (see equation (2) and (3)) yield that when removing a vertex of degree less than $(1 - \frac{3}{3k-4})v(G^1_j)$ the number of copies of $K_m(t)$ removed with it is at most

$$(1 + o(1))(1 - \delta)v(G^1_j)^{mt-1}\binom{k-1}{m}\frac{mt}{(k-1)^m(t!m^{t-1})^m}.$$

Assume that the process stops at step r. If $r = o(n)$ then by Theorem 2.1 the graph G^1_r is $k-1$-chromatic. In these r steps $o(n)$ vertices were deleted and with them no more than $o(n^2)$ edges, thus case (2) holds.

If $r = cn$ for some $c \le 1 - \alpha$ define the graph G_r as follows. If $c = 1 - \alpha$ take G_r to be a $k-1$-chromatic subgraph of G^1 on the vertices of G^1_r with the maximum possible number of copies of $K_m(t)$. It must be that $\mathcal{N}(G_r, K_m(t)) - \mathcal{N}(G^1_r, K_m(t)) < \alpha' n^{mt}$ for an appropriate $\alpha' = \alpha'(\alpha)$ which tends to 0 as α tends to 0. If $c > \alpha$ take $G_r = G^1_r$, by Theorem 2.1 this graph is $k-1$-chromatic.

As G_r is $k-1$-chromatic in both cases we can apply Lemma 2.7 to it and add back the vertices removed in the process, starting from $j = r-1$ to $j = 1$, while keeping the graph $k-1$-chromatic. We get that the number of copies of $K_m(t)$ added with each vertex v_j is at least

$$(1 + o(1))(1 - c\epsilon)v(G^1_j)^{mt-1}\binom{k-1}{m}\frac{mt}{(k-1)^m(t!m^{t-1})^m}.$$

Let G^2 be the graph obtained after adding back all the vertices.

Assume that ϵ is small enough to ensure that $\delta - c\epsilon > c' > 0$ for some $c' := c'(\gamma)$. Let $n_j = v(G^1_j) = n - j$, and note that $\sum_{j=0}^{r} n_j^{mt-1} = \sum_{j=0}^{cn}(n-j)^{mt-1} \ge (1+o(1))n^{mt}\frac{1}{mt}(1-(1-c)^{mt})$. Thus the difference in the number of copies of $K_m(t)$ in G^1 and G^2 is at least

$$\begin{aligned}
\mathcal{N}(G^2, K_m(t)) - \mathcal{N}(G^1, K_m(t)) \ge& (1 + o(1))\sum_{j=0}^{r} c' n_j^{mt-1}\binom{k-1}{m}\frac{mt}{(k-1)^m(t!m^{t-1})^m} - \alpha' n^{mt}\\
\ge& (1 + o(1))(c' - \alpha')(1 - (1-c)^{mt})n^{mt}\binom{k-1}{m}\frac{1}{(k-1)^m(t!m^{t-1})^m}\\
=& (1 + o(1))(1 - \gamma)\mathcal{N}(G_{ex}, K_m(t))
\end{aligned}$$

where c' and α' are chosen so that the last equality holds.

As G^2 is a K_k-free subgraph of G, $\mathcal{N}(G_{ex}, K_m(t)) \ge \mathcal{N}(G^2, K_m(t))$ and thus $\gamma\mathcal{N}(G_{ex}, K_m(t)) \ge \mathcal{N}(G^1, K_m(t))$ and case (1) holds, as needed.

□

The proof of Proposition 1.4 is now a simple corollary of the last lemma.

Proof of Proposition 1.4. Let G be a graph on n vertices with $\delta(G) > (1-\epsilon)n$ and let $G_{ex(H)} \in \mathcal{G}_{ex}(K_m(t), H)$. By Lemma 2.6 $\mathcal{N}(G_{ex(H)}, K_m(t)) = \Theta(n^{mt})$. By Lemma 2.5 there is a graph $G_1 \subseteq G_{ex(H)}$ which is K_k-free and $e(G_{ex(H)}) - e(G_1) = o(n^2)$, and thus

$$\mathcal{N}(G_{ex(H)}, K_m(t)) = (1+o(1))\mathcal{N}(G_1, K_m(t)).$$

Let $G_{ex(K_k)} \in \mathcal{G}(K_m(t), K_k)$. To apply Lemma 5.1 we show that

$$\mathcal{N}(G_1, K_m(t)) = (1+o(1))\mathcal{N}(G_{ex(K_k)}, K_m(t)). \tag{4}$$

By theorems 1.2 and 1.3 $G_{ex(K_k)}$ is $k-1$-chromatic, and so it is H-free. Together with the fact that G_1 is K_k-free, we get

$$\mathcal{N}(G_1, K_m(t)) \leq \mathcal{N}(G_{ex(K_k)}, K_m(t)) \leq \mathcal{N}(G_{ex(H)}, K_m(t)) = (1+o(1))\mathcal{N}(G_1, K_m(t))$$

implying (4).

Thus case (1) in Lemma 5.1 does not hold for G_1, and so case (2) must hold, i.e. G_1 can be made $k-1$-chromatic by deleting $o(n^2)$ edges. As we got G_1 from $G_{ex(H)}$ by deleting $o(n^2)$ edges we get the required result. □

6 Concluding remarks and open problems

- Corollary 2.3 and Theorems 1.2 and 1.3 cannot be directly generalized for graphs H which are not edge critical. In [5] the following is shown (a weaker version of this statement is proved in [3])

  Theorem 6.1 ([5]). *Let H be a fixed graph on h vertices such that $\chi(H) = k \geq 3$ and let G be an H-free graph on n vertices with $\delta(G) \geq (1 - \frac{3}{3k-4} + o(1))n$, where n is large enough. Then one can delete at most $O(n^{2-1/(4(k-1)^{2/3}h)})$ edges from G and make it $k-1$-colorable.*

  This suggests that a stronger version of Proposition 1.4, stating that any extremal graph G_{ex} as in the proposition can be made $k-1$-chromatic by deleting $O(n^{2-\mu(H)})$ edges for some $\mu(H) > 0$, is likely to be true.

- Theorem 1.3 is limited to the case where $\chi(H) = m+1$, and from this we also get the condition in Proposition 1.4. One of the problems in extending it to graphs H with higher chromatic number is that of finding an explicit tight bound on $ex(n, K_m(t), K_k)$ for $k > m+1$. In [4] it is shown that the extremal graph is $k-1$-partite. However, it is not difficult to check that for $k \geq m+2$ such that $m \nmid k-1$ and large values of t, the parts are not of equal sizes.

- Theorems in the same spirit as those proven here may hold for other pairs of graphs T and H. In [4] it is observed that if H is not contained in any blow-up of T then $ex(n, T, H) = \Theta(n^{v(T)})$. This of course does not mean that the extremal graph is a blow-up of T, but in cases it is a similar behavior to that in the results proven here might be expected.

  A notable example is the case $T = C_5$ and $H = K_3$. In [15] and independently [17] it is shown that when $5|n$ the extremal graph is the equal sided blow-up of C_5. It might be true that this behavior holds for subgraphs of graphs of high minimum degree and not only for subgraphs of K_n, that is, the extremal subgraphs in this case may be subgraphs of the equal sided blow-up of C_5.

- The problem of obtaining the best possible bounds for the minimum degree ensuring that the results stated in Theorems 1.2, 1.3 and Propositon 1.4 hold is also interesting, but appears to be difficult. Even the very special case of Theorem 1.2 with $H = K_3$ and $m = 2$, conjectured in [7] to be $3/4 + o(1)$, is open.

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