A note on degenerate and spectrally degenerate graphs

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Abstract

A graph G is called spectrally d-degenerate if the largest eigenvalue of each subgraph of it with maximum degree D is at most $\sqrt{dD}$. We prove that for every constant M there is a graph with minimum degree M which is spectrally 50-degenerate. This settles a problem of Dvořák and Mohar.

1 Introduction

The spectral radius $\rho(G)$ of a (finite, simple) graph G is the largest eigenvalue of its adjacency matrix. A graph is d-degenerate if any subgraph of it contains a vertex of degree at most d. A result of Hayes [5] asserts that any d-degenerate graph with maximum degree at most D has spectral radius at most $2\sqrt{dD}$. In fact, the result is a bit stronger, as follows.

Proposition 1.1 ([5]) *Let G be a graph having an orientation in which every outdegree is at most d and every indegree is at most D. Then $\rho(G) \leq 2\sqrt{dD}$.*

For completeness we include a simple proof (which is somewhat shorter than the one given in [5]).

Proof: Denote the vertices of G by $\{1, 2, \ldots, n\}$, and let d_i^+ and d_i^- be the indegree and outdegree of vertex i in an orientation of G in which every outdegree is at most d and every indegree is at most D. The spectral radius of G is the maximum possible value of the sum $\sum_{ij \in E} 2x_i x_j$ where E denotes the set of oriented edges of G and the maximum is taken over all vectors $(x_1, x_2, \ldots, x_n)$ satisfying $\sum_i x_i^2 = 1$. For each oriented edge $ij \in E$,

$$\frac{2x_i x_j}{\sqrt{dD}} \leq \frac{2x_i x_j}{\sqrt{d_i^+ \cdot d_j^-}} \leq \frac{x_i^2}{d_i^+} + \frac{x_j^2}{d_j^-}.$$

The desired result now follows by summing over all oriented edges. □

Following Dvořák and Mohar [3], call a graph G spectrally d-degenerate, if for every subgraph H of G, $\rho(H) \leq \sqrt{dD(H)}$, where $D(H)$ is the maximum degree of H. Thus, by Proposition 1.1,

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every d-degenerate graph is spectrally $4d$-degenerate. The authors of [3] proved the following rough converse:

(*) Any spectrally d-degenerate graph with maximum degree at most $D \geq 2d$ contains a vertex of degree at most $4d \log_2(D/d)$.

They further showed that the dependence on D cannot be eliminated if the dependence on d is subexponential and asked whether there is a function f mapping positive integers to positive integers such that for every d, any spectrally d-degenerate graph contains a vertex of degree at most $f(d)$.

In this note we settle this problem and show that there is no such function by proving the following.

Theorem 1.2 *For every positive integer M there is a spectrally 50-degenerate graph G in which every degree is at least M.*

Our proof combines the approach of [3], which is based on a construction of [6], with some additional probabilistic arguments.

The constant 50 can be reduced, and we make no attempt to optimize it, or the other absolute constants that appear in the proof. To simplify the presentation, we omit all floor and ceiling signs whenever these are not crucial.

2 Spectrally degenerate graphs of high degree

2.1 A probabilistic construction

In this subsection we describe a probabilistic construction which is similar to the one given in [6].

Theorem 2.1 *For every positive integer M and all sufficiently large $n > n_0(M)$ there exists a bipartite graph G with vertex classes A and B, satisfying the following properties.*

(i) $|B| \leq |A| = n$.

(ii) Every vertex of A has degree M and every vertex of B has degree larger than $1000M$.

(iii) Every subgraph of G with average degree at least 10 contains a vertex of degree at least $1000M$.

Proof: Fix $\epsilon > 0$, $\epsilon < 4^{-2M}$. Let B be a disjoint union of M sets $B_1, \ldots, B_M$, where $|B_i| = n^{1-4^i\epsilon}$. The graph G is a random graph constructed as follows. For each i, $1 \leq i \leq M$, let G_i be a bipartite graph on the vertex classes A and B_i consisting of a random set of $|A|$ edges obtained by picking, for each $a \in A$, a uniform random $b \in B_i$, taking ab to be an edge of G_i. The graph G consist of all edges of all graphs G_i. Note that the degree of every vertex of G that lies in A is exactly M, and that if n is sufficiently large then the degree of every vertex of G that lies in B is, asymptotically almost surely (a.a.s., for short), at least $n^{4\epsilon}/2 > 1000M$, provided n is sufficiently large. Here we say that a property holds a.a.s. if the probability it holds tends to 1 as n tends to infinity, and the claim about the degrees follows easily from the known estimates for binomial distributions, as the degree of each vertex in B_i is a binomial random variable with parameters n and $1/|B_i|$ whose expectation is $n^{4^i\epsilon}$.

Therefore the graph G satisfies the properties (i) and (ii) a.a.s. It remains to show that it satisfies property (iii) as well a.a.s.

Let H be a subgraph of G with average degree at least 10, and let s (≥ 20) denote the number of its vertices. We consider three possible cases.

Case 1: $s \leq |B_M| = n^{1-4^M\epsilon}$.
We show that in this case, a.a.s., the graph G contains no subgraph on s vertices with average degree at least 10. Indeed, such a subgraph can contain at most s edges incident with vertices of B_M (as each A-vertex has only one neighbor in B_M) and hence there are at least $4s$ edges of H incident with vertices in $\cup_{i<M} B_i$. But the probability that there is such a collection of $4s$ edges is at most

$$\sum_{s \leq n^{1-4^M\epsilon}} \binom{2n}{s} \binom{\binom{s}{2}}{4s} (\frac{1}{|B_{M-1}|})^{4s}$$

$$\leq \sum_{s \leq n^{1-4^M\epsilon}} (\frac{2en}{s})^s (\frac{es}{8})^{4s} (\frac{1}{|B_{M-1}|})^{4s}$$

$$\leq \sum_{s \leq n^{1-4^M\epsilon}} c^s n^s s^{3s} (\frac{1}{|B_{M-1}|})^{4s},$$

where $c = (2e)(e/8)^4$ is an absolute constant. The last quantity is at most

$$\sum_{s \leq n^{1-4^M\epsilon}} c^s n^s (n^{1-4^M\epsilon})^{3s} (n^{-(1-4^{M-1}\epsilon)})^{4s} = \sum_{s \leq n^{1-4^M\epsilon}} c^s n^{(-3 \cdot 4^M + 4^M)\epsilon s} < n^{-\epsilon}.$$

Therefore, a.a.s., there is no such H.

Case 2: $s \geq |B_1| = n^{1-4\epsilon}$.
In this case the graph has at least $4s$ edges incident with vertices in $\cup_{i \geq 2} B_i$. As the total size of these sets is smaller than $\frac{2s}{n^{12\epsilon}}$ the graph must have degrees at least $2n^{12\epsilon} > 1000M$, as needed.

Case 3: There is an i, $1 \leq i < M$, so that

$$n^{1-4^{i+1}\epsilon} = |B_{i+1}| \leq s < |B_i| = n^{1-4^i\epsilon}.$$

In this case the graph H has at most $2s$ edges incident with vertices in $B_i \cup B_{i+1}$. If its maximum degree does not exceed $1000M$, it has at most $1000M |\cup_{j \geq i+2} B_j| = o(s)$ edges incident with vertices in $\cup_{j \geq i+2} B_j$. It thus has at least $(3 - o(1))s$ edges incident with vertices in $\cup_{j<i} B_j$. The probability P that there is such a collection of edges can be bounded as in the first case. Indeed, P is at most the following, where in all sums over s, the parameter s ranges over all values between $|B_{i+1}|$ and $|B_i|$.

$$P \leq \sum_s \binom{2n}{s} \binom{\binom{s}{2}}{(3-o(1))s} (\frac{1}{|B_{i-1}|})^{3s-o(s)}$$

$$\leq \sum_s c^s n^s s^{(2-o(1))s} \left(\frac{1}{(n^{1-4^{i-1}\epsilon})^{(3-o(1))s}}\right),$$

for an appropriate absolute constant c. As $s \leq |B_i|$ in the above range, the last quantity is at most

$$\sum_s c^s (\; n^{1+(1-4^i\epsilon)(2-o(1))-(1-4^{i-1}\epsilon)(3-o(1))} \;)^s = \sum_s c^s n^{(o(1)-2\cdot 4^i\epsilon+3\cdot 4^{i-1}\epsilon)s} < n^{-\epsilon}.$$

This shows that the event in Case 3 also does not occur a.a.s., and completes the proof of the theorem. □

2.2 Bounding the spectral degeneracy

In order to prove Theorem 1.2 we need the following lemma, which is a bit stronger than a similar lemma proved in [3].

Lemma 2.2 *Suppose $M \geq M_0$, where M_0 is a large integer. Let H be a bipartite graph with vertex classes A and B in which every vertex of A has degree at least 10 and at most M, and every vertex of B has degree at least $D/100$ and at most D, where $D \geq 1000M$. Then there is a subset $A' \subset A$ so that the induced subgraph of H on $A' \cup B$ has minimum degree at least 10 and maximum degree smaller than $1000M$.*

Proof: It suffices to show that H contains an induced subgraph on $A' \cup B$, where $A' \subset A$, in which the degree of every vertex of A' is at least 10, and the degree of every vertex of B is at least $2M$ (> 10) and at most $800M$. We proceed to prove this statement using the Lovás Local Lemma proved in [4] (c.f. also, e.g., [2]). Our approach follows the one in [1]. Starting with $H_0 = H$, construct a sequence $H_i = (V_i, E_i)$ of induced subgraph of H, where for each $i \geq 1$, H_i is a random induced subgraph of H_{i-1} with vertex classes A_i and B, where A_i is obtained by picking each vertex in A_{i-1}, randomly and independently, to lie in A_i, with probability $1/2$.

Note that by construction, the degree of each vertex of A_i in H_i is exactly its degree in H, which, by assumption, is at least 10. For each vertex $v \in B$, let $d_i(v)$ denote the degree of v in H_i. We claim that with positive probability, as long as $D/2^i \geq 300M$, then $d_i(v)$ is close to $d_0(v)/2^i$. More precisely, let j be the maximum value of i so that $D/2^i \geq 300M$. Note that $D/2^j < 600M$. Define a sequence $a_0(v) = d_0(v)$ and $a_i(v) = \frac{a_{i-1}(v)}{2} + \frac{a_{i-1}(v)^{2/3}}{2}$ for all $i, 0 < i \leq j$, and similarly a sequence $b_0(v) = d_0(v)$ and $b_i(v) = \frac{b_{i-1}(v)}{2} - \frac{b_{i-1}(v)^{2/3}}{2}$. Then, with positive probability

$$\frac{D}{100 \cdot 2^i} 0.9 \leq \frac{b_0(v)}{2^i} \prod_{r \leq i-1} (1 - \frac{1}{b_r(v)^{1/3}}) = b_i(v) \leq d_i(v) \leq a_i(v) = \frac{a_0(v)}{2^i} \prod_{r \leq i-1} (1 + \frac{1}{a_r(v)^{1/3}}) \leq \frac{D}{2^i} 1.2$$

for all $i \leq j$ and all $v \in B$.

Indeed, the above statement is proved by induction on i. For $i = 0$ there is nothing to prove. Assuming the above holds for $i-1$, we establish the assertion for i using the Local Lemma. To do so consider, for each vertex $v \in B$, the event F_v that $d_i(v)$ fails to satisfy

$$\frac{d_{i-1}(v)}{2} - \frac{d_{i-1}(v)^{2/3}}{2} \leq d_i(v) \leq \frac{d_{i-1}(v)}{2} + \frac{d_{i-1}(v)^{2/3}}{2}.$$

Each event F_v is mutually independent of all other events F_u besides those for which u and v have a common neighbor in H_{i-1}. There are clearly at most $d_{i-1}(v)M < d_{i-1}(v)^2$ such vertices u, and hence the fact that $d_{i-1}(v) > M$, the known standard estimates of Binomial distributions (c.f., e.g., [2]) and the Local Lemma imply that with positive probability none of the events F_v holds. The graph H_j satisfies the conclusion of the lemma, completing its proof. $\Box$

We are now ready to prove the following result, which clearly implies Theorem 1.2.

Theorem 2.3 *Let G be a graph satisfying the assertion of Theorem 2.1, where $M > M_0$ and M_0 is as in Lemma 2.2. Then G is spectrally 50-degenerate and has minimum degree M.*

Proof: Assume this is false and let H be a subgraph of G with maximum degree D and spectral radius $\rho(H) > \sqrt{50D}$, where D is as small as possible. Clearly H cannot be 10-degenerate, since otherwise $\rho(H) < 2\sqrt{10D} < \sqrt{50D}$ and thus it contains a subgraph with minimum degree exceeding 10, implying, by Theorem 2.1, part (iii), that its maximum degree, and hence also D, are at least $1000M$.

We claim that there is a coloring of the edges of H by 2 colors, red and green, so that the following holds:

(a) In the graph H_r consisting of all red edges, the degree of every vertex of A is at most 10.

(b) In the graph H_g consisting of all green edges, the degree of every vertex of B is at most $D/100$.

To prove the claim, consider the following process of coloring vertices and edges of H by red and green. Starting with no colored vertex or edge, repeat the following two steps as long as it is possible to apply any of them:

(1) If there is a yet uncolored vertex v of B incident with at most $D/100$ uncolored edges, color it green, and color all uncolored edges incident with it green.

(2) If there is a yet uncolored vertex u of A incident with at most 10 uncolored edges, color it red, and color all uncolored edges incident with it red.

Note that only vertices of B can be green, and only vertices of A can be red. Moreover, once a vertex is colored, all edges incident with it get colors as well. In addition, a vertex of B has no green edges incident with it before it is colored, and a vertex of A has no red edges incident with it before it is colored. The construction thus implies that the degree of every vertex of B in the green graph is at most $D/100$, whereas the degree of every vertex of A in the red graph is at most 10.

The process terminates when either all edges are colored, or there are yet uncolored edges (and hence also vertices) and we cannot apply the rules (1) and (2) anymore. But if this is the case then in the induced subgraph on the yet uncolored vertices every vertex of B has degree exceeding $D/100$ (and at most D) and every vertex of A has degree exceeding 10 (and at most M). By Lemma 2.2 this contradicts the fact that G satisfies Theorem 2.1. Thus we have colored all edges, as claimed.

By the minimality of D, the graph H_g whose maximum degree is at most $D/100$ is spectrally 50-degenerate, and by Proposition 1.1 the graph H_R, which is 10-degenerate, has maximum eigenvalue at most $2\sqrt{10D}$. It follows that the spectral radius of H is at most $\sqrt{50D/100} + 2\sqrt{10D} < \sqrt{50D}$, contradicting the choice of H. This completes the proof. $\square$

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