Degrees and choice numbers

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Abstract

The *choice number* $ch(G)$ of a graph $G = (V, E)$ is the minimum number k such that for every assignment of a list $S(v)$ of at least k colors to each vertex $v \in V$, there is a proper vertex coloring of G assigning to each vertex v a color from its list $S(v)$. We prove that if the minimum degree of G is d, then its choice number is at least $(\frac{1}{2} - o(1)) \log_2 d$, where the $o(1)$-term tends to zero as d tends to infinity. This is tight up to a constant factor of $2 + o(1)$, improves an estimate established in [1], and settles a problem raised in [2].

1 Introduction

An undirected, simple graph $G = (V, E)$ is k-choosable if for every assignment of a list $S(v)$ of at least k colors to each vertex $v \in V$, there is a proper vertex coloring of G assigning to each vertex v a color from its list $S(v)$. The *choice number* $ch(G)$ of G, (which is also called the *list chromatic number* of G) is the minimum number k such that G is k-choosable.

The concept of choosability, introduced by Vizing in 1976 [6] and independently by Erdős, Rubin and Taylor in 1979 [4], received a considerable amount of attention recently. Many of the recent results can be found in the survey papers [1], [5] and their many references. By definition, the choice number $ch(G)$ of any graph G is at least as large as its chromatic number $\chi(G)$, and it is well known that strict inequality can hold. In fact, it is shown in [4] that the choice number of the complete bipartite graph with d vertices in each color class satisfies

$$ch(K_{d,d}) = (1 + o(1)) \log_2 d. \tag{1}$$

The *coloring number* $col(G)$ of $G = (V, E)$ is the minimum number d such that every subgraph of G contains a vertex of degree smaller than d. Equivalently, it is the minimum d such that there is an acyclic orientation of G in which every outdegree is smaller than d, or the minimum d such that G is $(d-1)$-degenerate. As observed already in [4], for every graph G, $ch(G) \leq col(G)$. In [1] a certain converse is proved: there is an absolute constant $c > 0$ such that if $col(G) > d$ then $ch(G) \geq c \frac{\log d}{\log \log d}$. In [2] it is conjectured that the $\log \log d$ term can be omitted. This is the main result of the present note, stated in the following theorem (in which the constants can be slightly improved).

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Theorem 1 *Let G be a simple graph with minimum degree at least d. If s is an integer and*

$$d > \frac{4(s^2+1)^2}{(\log_2 e)^2} 2^{2s} \tag{2}$$

then $ch(G) > s$.

This implies that the choice number of any graph with coloring number that exceeds d is at least $(\frac{1}{2} - o(1)) \log_2 d$. By (1) this is tight up to a constant factor of $2 + o(1)$.

2 The proof

Note, first, that there is a very simple characterization, given in [4], of graphs with choice number at most 2. By this characterization, each such graph contains a vertex of degree at most 2, implying the assertion of the theorem for $s \leq 2$. We thus may and will assume that s is at least 3. The proof of the theorem is probabilistic. Let $G = (V, E)$ be a simple graph with minimum degree at least d, and suppose (2) holds. Put $|V| = n$ and let $S = \{1, 2, \ldots, s^2\}$ be a set of colors. Our objective is to show that there are subsets $S(v) \subset S$, where $|S(v)| = s$ for all $v \in V$, such that there is no proper coloring $c : V \mapsto S$ that assigns to every $v \in V$ a color $c(v) \in S(v)$.

Let B be a subset of V where each $v \in V$, randomly and independently, is chosen to be a member of B with probability $\frac{1}{\sqrt{d}}$. For each $b \in B$, let $S(b)$ be a random subset of cardinality s of S, chosen uniformly and independently among all the $\binom{s^2}{s}$ subsets of cardinality s of S. Call a vertex $v \in V$ *good* if $v \notin B$ and for every subset $T \subset S$ of cardinality $|T| = \lceil s^2/2 \rceil$, there is a neighbor b of v in G such that $b \in B$ and $S(b) \subset T$. Note that for each fixed vertex $v \in V$, the probability that v is not good does not exceed

$$\frac{1}{\sqrt{d}} + (1 - \frac{1}{\sqrt{d}}) \binom{s^2}{\lceil s^2/2 \rceil} \left(1 - \frac{1}{\sqrt{d}} \frac{\lceil s^2/2 \rceil (\lceil s^2/2 \rceil - 1) \ldots (\lceil s^2/2 \rceil - s + 1)}{s^2(s^2-1)\ldots(s^2-s+1)}\right)^d \tag{3}$$

This is because the probability that $v \in B$ is at most $\frac{1}{\sqrt{d}}$. If it is not in B, then for each fixed subset T of cardinality $\lceil s^2/2 \rceil$ of S, and for each neighbor u of v in G, the probability that $u \in B$ and that $S(u) \subset T$ is precisely

$$\frac{1}{\sqrt{d}} \frac{\lceil s^2/2 \rceil (\lceil s^2/2 \rceil - 1) \ldots (\lceil s^2/2 \rceil - s + 1)}{s^2(s^2-1)\ldots(s^2-s+1)}.$$

As the degree of v is at least d, it follows that the probability that there is no neighbor u of v as above is at most

$$\left(1 - \frac{1}{\sqrt{d}} \frac{\lceil s^2/2 \rceil (\lceil s^2/2 \rceil - 1) \ldots (\lceil s^2/2 \rceil - s + 1)}{s^2(s^2-1)\ldots(s^2-s+1)}\right)^d,$$

and the estimate in (3) follows since there are

$$\binom{s^2}{\lceil s^2/2 \rceil}$$

possible choices for the subset T.

Clearly,

$$\frac{\lceil s^2/2\rceil(\lceil s^2/2\rceil-1)\ldots(\lceil s^2/2\rceil-s+1)}{s^2(s^2-1)\ldots(s^2-s+1)} \geq \frac{1}{2^s}\prod_{i=0}^{s-1}\frac{s^2-2i}{s^2-i}$$

$$=\frac{1}{2^s}\prod_{i=0}^{s-1}(1-\frac{i}{s^2-i}) \geq \frac{1}{2^s}\left(1-\frac{\sum_{i=0}^{s-1} i}{s^2-s}\right)=\frac{1}{2^{s+1}}.$$

Substituting in (3), and using the fact that for $s \geq 3$, $\binom{s^2}{\lceil s^2/2\rceil} \leq 2^{s^2}/4$, we conclude that the probability that v is not good does not exceed

$$\frac{1}{\sqrt{d}}+\frac{2^{s^2}}{4}(1-\frac{1}{\sqrt{d}2^{s+1}})^d \leq \frac{1}{\sqrt{d}}+\frac{2^{s^2}}{4}e^{-\frac{\sqrt{d}}{2^{s+1}}}<1/4,$$

where the last inequality follows from (2).

It follows that the expected number of vertices v which are not good is less than $n/4$ and hence, by Markov's inequality, the probability that there are at least $n/2$ good vertices exceeds $1/2$. As the expected size of B is $n/\sqrt{d}$, the probability that $|B| > 2n/\sqrt{d}$ is smaller than $1/2$. Therefore, with positive probability, $|B| \leq 2n/\sqrt{d}$ and there are at least $n/2$ good vertices.

Fix a choice of B and of $S(b), b \in B$ such that $|B| \leq 2n/\sqrt{d}$ and there is a set A of $g \geq n/2$ good vertices. For each $a \in A$ choose a set of colors $S(a) \subset S$, where each set $S(a)$ is chosen randomly independently and uniformly among all s-subsets of S. To complete the proof we show that with positive probability there is no proper coloring $c : V \mapsto S$ of G, assigning to each vertex $v \in A \cup B$ a color from its list $S(v)$.

There are at most $s^{|B|}$ possibilities for the restriction $c_{|B}$ of the coloring c to the vertices in B, satisfying $c(b) \in S(b)$ for each $b \in B$. Fix such a restriction, and let us estimate the probability that it can be extended to a proper coloring of the induced subgraph of G on $A \cup B$ assigning to each vertex a color from its list. The crucial observation is that as each $a \in A$ is good, the set T_a of all colors assigned by $c_{|B}$ to its neighbors in B is a set that intersects every subset of cardinality $\lceil s^2/2\rceil$ of S, and thus its cardinality is at least $\lfloor s^2/2\rfloor+1 \geq \lceil s^2/2\rceil$. If $S(a)$ is a subset of T_a, there is no proper color available for a in its list. Therefore, the probability that a can be colored is at most

$$1-\frac{\lceil s^2/2\rceil(\lceil s^2/2\rceil-1)\ldots(\lceil s^2/2\rceil-s+1)}{s^2(s^2-1)\ldots(s^2-s+1)} \leq 1-\frac{1}{2^{s+1}}.$$

The events corresponding to distinct good vertices a are mutually independent, by the independent choice of the sets $S(a)$. Therefore, the probability that a fixed partial coloring $c_{|B}$ can be extended to a proper one $c : A \cup B \mapsto S$ assigning to each vertex a color from its list is at most

$$(1-\frac{1}{2^{s+1}})^g \leq (1-\frac{1}{2^{s+1}})^{n/2} \leq e^{-n/2^{s+2}}.$$

Note that

$$s^{|B|}e^{-n/2^{s+2}} \leq e^{\frac{2n}{\sqrt{d}}\ln s - n/2^{s+2}},$$

which is less than 1, by (2) and the fact that $s \geq 3$.

Therefore, with positive probability there is no coloring of the desired type, implying that $ch(G) > s$ and completing the proof. $\square$

3 Concluding remarks

The choice of the total number of colors in the proof of Theorem 1 is motivated by the old results of Erdős [3] on uniform hypergraphs with chromatic number bigger than 2.

Theorem 1 and the discussion preceding it imply that the choice number of any graph G with coloring number $col(G) = d$ satisfies

$$(\frac{1}{2} - o(1)) \log_2 d \leq ch(G) \leq d.$$

As the coloring number of a given input graph can be easily determined in linear time, this provides an efficient approximation algorithm for finding an estimate of the choice number of a given graph. Although this is a very rough approximation, there is no known similar result for approximating the chromatic number of a given input graph.

In [2] it is shown that the choice number of a random bipartite graph with n vertices in each class in which each pair of vertices from distinct classes forms an edge, randomly and independently, with probability p, is almost surely (that is, with probability that tends to 1 as np tends to infinity) $(1 + o(1)) \log_2(np)$. Note that all degrees of such a graph are $(1 + o(1))np$, and hence these graphs also show that the estimate in Theorem 1 is tight, up to a multiplicative factor of $2 + o(1)$. It seems plausible that the choice number of any d-regular bipartite graph is $(1 + o(1)) \log_2 d$. This is related to a question mentioned in [2]. By the result here the choice number of each such graph is at least $(\frac{1}{2} - o(1)) \log_2 d$, and it is easy to show that it is at most $O(d/\log d)$.

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