Distinct Directions and Distinct Distances in $\mathbb{R}^d$

Noga Alon* Rom Pinchasi†

Abstract

We show that there exists an absolute positive constant $b(\geqslant \frac{1}{48})$ so that any set of n points in $\mathbb{R}^d$ that is d-dimensional determines at least bdn lines with pairwise distinct directions. As a consequence we prove that there are d-dimensional real norms $\|\cdot\|$ so that every set of $n > n_0(d)$ points that is d-dimensional determines at least $(bd - o(1))n$ distinct distances with respect to $\|\cdot\|$.

1 Introduction

The celebrated Gallai-Sylvester theorem asserts that n points in the plane that are not collinear must determine an *ordinary line*, that is a line passing through precisely two points of the set. Erdős noticed the following simple consequence. Any set of n points in the plane that are not collinear must determine at least n distinct lines, and equality happens only when $n-1$ of the points are collinear.

In the same spirit, Scott asked two similar questions in 1970:

1. Is it true that the minimum number of distinct directions of lines determined by n noncollinear points in $\mathbb{R}^2$ is $2\lfloor \frac{n}{2} \rfloor$?

2. What is the minimum number of distinct directions of lines determined by a 3-dimensional set of n points in $\mathbb{R}^3$?

In 1982 Scott's first question was answered in the affirmative by the celebrated theorem of Ungar [13] using the technique of allowable sequences invented by Goodman and Pollack.

Scott's second question was answered only much later in [11], where it is shown that a 3-dimensional set of n points determines at least $2n-5$ lines with distinct directions. This bound is sharp when n is odd.

It is natural now to wonder what happens in higher dimensions. The complexity of the solution of Scott's problem in $\mathbb{R}^3$, compared with the solution in $\mathbb{R}^2$, suggests that

*Department of Mathematics, Princeton University, Princeton, NJ 08544, USA. Research supported in part by NSF grant DMS-2154082. nalon@math.princeton.edu

†Mathematics Department, Technion, Haifa 32000, Israel. Supported in part by ISF grant No. 1091/21. room@technion.ac.il.

the question is not likely to be easier in higher dimensions. Similarly, the actual bound in three dimensions and the sharp examples suggest that one cannot expect a very simple expression as a tight answer for the same problem in d dimensions, nor a very simple construction achieving the minimum possible number of distinct directions.

It is however plausible to conjecture the following, which is suggested in [4], page 272, motivated by earlier related results of Jamison [6] and of Blokhuis and Seress [3].

Conjecture 1.1. *A d-dimensional set of n points determines at least $(d-1)n - O(d^2)$ lines with pairwise distinct directions.*

The example of $n-d+1$ collinear points plus additional $d-1$ points in general position not on this line shows that one cannot expect the answer to be more than

$$(d-1)(n-d+1) + \binom{d-1}{2} + 1 = (d-1)n - O(d^2)$$

even if we wish to bound from below the total number of lines determined by a d-dimensional set of n points, regardless of the directions of these lines.

In the present note we prove a modest result supporting Conjecture 1.1 and show that the number of distinct directions determined by a d-dimensional set of n points is at least linear in dn.

Theorem 1.2. *There exists an absolute constant $b > 0$ so that for every $d \geqslant 2$, any set of n points in $\mathbb{R}^d$ that is not contained in a hyperplane determines at least bdn line segments with pairwise distinct directions.*

Our proof shows that the result above holds with $b = \frac{1}{48}$. This estimate can be easily improved, but since our method does not suffice to get the best possible b (which should be close to 1 for large d, if Conjecture 1.1 holds), we make no attempt to optimize it.

Unlike in the two and three dimensional cases where the proofs of the sharp bounds are rather involved, our proof is short and is derived as a consequence of a deep result by Dvir, Saraf, and Wigderson ([7]), improving on an earlier similar result by Barak, Dvir, Wigserson, and Yehudayoff ([2]).

As suggested in the solution of Scott's problem in 3-dimensions ([11]), it is helpful to consider pairwise *non-convergent* segments, rather than pairwise non-parallel segments, as the former property, unlike the latter one, is much better preserved under projective transformations.

Two line segments in $\mathbb{R}^d$ are called *convergent* if they are two opposite edges of a convex 2-dimensional quadrilateral.

Theorem 1.3. *A set of n points in $\mathbb{R}^d$ $(d \geqslant 2)$ that is not contained in a hyperplane determines at least $\frac{1}{48}dn$ line segments no two of which are convergent or collinear.*

Notice that since two parallel line segments that are not collinear are convergent, Theorem 1.3 implies that a d-dimensional set of n points in $\mathbb{R}^d$ determines at least $\frac{1}{48}dn$ lines with pairwise distinct directions.

Using Theorem 1.2, we obtain a new result about the distinct distances problem for d-dimensional sets in typical d-norms. The distinct distances problem in the Euclidean plane is one of the best known classical open problems in Discrete Geometry, raised by Erdős in 1946 [8]. This is the problem of determining or estimating the minimum possible number of distinct distances determined by n points in the Euclidean plane. Although this problem has been settled by Guth and Katz up to a $\sqrt{\log n}$ multiplicative factor [9], the problem for d-dimensional Euclidean norms for $d \geqslant 3$ remains wide open. Indeed, for each $d \geqslant 3$ the conjecture is that the minimum possible number of distinct distances determined by n points in $\mathbb{R}^d$ with respect to the Euclidean norm is $\Theta(n^{2/d})$. The d-dimensional integer box with edges of length $n^{1/d}$ shows that this is an upper bound, but the best known lower bound does not provide the correct exponent of n, see [12]. The same problem for general norms has been considered as well, see [1] and the references therein. Call a norm $\|\cdot\|$ on $\mathbb{R}^d$ a d-norm. A recent result proved in [1] asserts that there are d-norms in which any set of $n > n_0(d)$ points determines at least $(1-o(1))n$ distinct distances with respect to $\|\cdot\|$. In fact, this holds for all typical d-norms, where the notion of a typical norm is defined as follows.

Identify a norm with its unit ball. The Hausdorff distance between any two such unit balls A and B is the maximum of all distances between a point of A and the set B and between a point of B and the set A. This distance defines a metric, and hence a topology, on the space of all d-norms. A set S in this space is nowhere dense if every non-empty open set contains a nonempty open set which does not intersect S. A meagre set is a countable union of nowhere dense sets. A space is called a Baire space if the complement of each meagre set in it is dense. It is known that the space of all d-norms endowed with the Hausdorff metric as above is a Baire space. See e.g. [1] for additional relevant details and references. In this terminology it is proved in [1] that in all d-norms but a meagre set, any set of $n > n_0(d)$ points determines at least $(1-o(1))n$ distinct distances, where the $o(1)$-term tends to 0 as n tends to infinity. This is tight up to the $o(1)$-error term, as in every d-norm, any set of n points along an arithmetic progression on a line determines exactly $n-1$ distinct distances.

Note, however, that this extremal example that appears in any d-norm is one-dimensional. It seems natural to consider the distinct distances problem for configurations of n points in a d-norm that are d-dimensional, that is, do not all lie in an affine hyperplane. For this case, we suggest the following conjecture.

Conjecture 1.4. *For every fixed d the following holds for all d-norms $\|\cdot\|$ but a meagre set. For all $n > n_0(d)$, any set of n points in $\mathbb{R}^d$ that do not all lie in an affine hyperplane determine at least $(d-o(1))n$ distinct distances with respect to $\|\cdot\|$, where the $o(1)$-term* *tends to* 0 *as* n *tends to infinity.*

This is, of course, trivial for $d = 1$ but is open already for $d = 2$, as the result in [1] only ensures $(1 - o(1))n$ distinct distances for every fixed dimension d.

Here we observe that by combining the assertion of Theorem 1.2 with the arguments in [1] we get the following weaker version of the conjecture.

Theorem 1.5. *There exists an absolute positive constant* b *so that for any fixed* $d \geqslant 2$ *and all* d*-norms* $\|\cdot\|$ *but a meagre set the following holds. Any set of* $n > n_0(d)$ *points in* $\mathbb{R}^d$ *that do not all lie in an affine hyperplane determines at least* $(bd - o(1))n$ *distinct distances with respect to* $\|\cdot\|$*, where the* $o(1)$*-term tends to* 0 *as* n *tends to infinity.*

The rest of this note is organized as follows. In Section 2 we describe the proof of Theorem 1.3. Section 3 contains a sketch of the proof of Theorem 1.5. The final section 4 contains some concluding remarks and open problems.

2 Distinct directions in $\mathbb{R}^d$

In this section we prove Theorem 1.3. Denote by $f(d, n)$ the minimum number of distinct directions determined by a set of n points that has affine dimension d. We need to show that $f(d, n) \geqslant \frac{1}{48}nd$. The proof proceeds by induction on d. The case $d = 2$ follows from Ungar's theorem by which $f(2, n) \geqslant n - 1$. The case $d = 3$ was resolved in [11]. In fact, because $f(d, n) \geqslant f(2, n) \geqslant n - 1$, the assertion of Theorem 1.3 holds for every $d < 48$. Assuming the statement is true for $d - 1$, we prove it for d.

We use a result in [7], improving on a previous bound in [2]. We say that a line is special with respect to a set of points P if it contains at least 3 points of P. We will use Theorem 1.9 in [7]:

Theorem 2.1 (Theorem 1.9 in [7])**.** *Let* P *be a set of* n *points with the property that for every* $x \in P$ *at least* $\delta(n-1)$ *of the rest of the points lie on special lines through* x*. Then the affine dimension of the set* P *is at most* $\frac{c}{\delta}$*, where* $c = 12$*.*

An immediate consequence of Theorem 2.1 is the following.

Corollary 2.2. *Let* P *be a* d*-dimensional set of* n *points. Then there is a point* x *in* P *such that the number of lines connecting* x *to the other points in* P *is at least* $n(1 - \frac{c}{d})$*.*

Proof. Denote by M the maximum number such that there is a point in P with M distinct lines connecting it to the other point in P. If $M = n - 1$, there is nothing to prove. We assume therefore, that $M < n - 1$.

For every point $x \in P$ the number of points of $P \setminus \{x\}$ not lying on special lines through x is at most $M - 1$ (here we use the fact that $M < n - 1$). Consequently, the number of points of $P \setminus \{x\}$ that do lie on special lines through x is at least

$$n - 1 - (M - 1) = n - M = \frac{n - M}{n - 1}(n - 1).$$

Taking $\delta = \frac{n-M}{n-1}$ and applying Theorem 2.1, we get

$$d \leqslant c \frac{n - 1}{n - 1 - M}.$$

This implies

$$M \geqslant (1 - \frac{c}{d})(n - 1) + 1 \geqslant (1 - \frac{c}{d})n.$$

□

We remark that the exact bound in Corollary 2.2 is unknown and most likely difficult to achieve. See Section 4 for more details.

Recall that $c = 12$ in Theorem 2.1, however this value is not known to be tight and has been reported recently to be improved to $c = 4$. For this reason we incorporate the constant c in the rest of our calculations as a parameter rather than using its current best known published value $c = 12$.

We use Corollary 2.2 to find a point $x \in P$ that we can connect by at least $n(1 - \frac{c}{d})$ distinct lines to the other points in P. On each of these lines we take the longest segment possible delimited by two points of P. We thus get a set S_1 of at least $n(1 - \frac{c}{d})$ distinct segments determined by P, all containing the point x.

Next, we centrally project through x the set $P \setminus \{x\}$ onto a generic hyperplane H of dimension $d-1$. On H we get a set P' of at least $n(1 - \frac{c}{d})$ distinct points. Some of these points are multi-images of more than one point in P.

We notice that the set P' is $(d-1)$-dimensional. This is because otherwise the set P cannot be d-dimensional. By the induction hypothesis, we find $f(d-1, n(1 - \frac{c}{d-1}))$ segments determined by points in P' no two of which are convergent. We replace, if necessary, each of these segments by the longest possible segment between two points of P' on the line containing it. Clearly, the resulting set S_2 of segments is still a collection of segments no two of which are convergent and no two are collinear.. Each segment $s \in S_2$ is a projection through x on H of at least one segment, that we choose arbitrarily if necessary and denote it by $\tilde{s}$, determined by two points in P. We set $\tilde{S}_2 = \{\tilde{s} \mid s \in S_2\}$.

Claim 2.3. *$S_1 \cup \tilde{S}_2$ is a collection of segments no two of which are convergent and no two are collinear.*

Proof. Clearly no two segments in S_1 are convergent because every two contain the point x. No two segments in S_1 are collinear as we choose only one segment for S_1 on every line through x.

We further notice that no two segments in $\tilde{S}_2$ are convergent because their central projections through x on H yield segments no two of which are convergent. Indeed, if

$\tilde{s}$ and $\tilde{s'}$ are convergent, then there is a point y that is the intersection of the two lines containing $\tilde{s}$ and $\tilde{s'}$. Notice that y must be different from x and the projection of y through x on H lies on the two lines containing s and s' but not on $s \cup s'$. This implies that s and s' are convergent contrary to the way they were chosen. We further notice that no two segments in $\tilde{S}_2$ are collinear because otherwise the central projections through x of two such segments would be collinear in S_2, which is impossible.

It is left to show that it is not possible that a segment $s \in S_1$ and a segment $\tilde{s'} \in \tilde{S}_2$ are convergent, as clearly no two such segments are collinear.

Assume to the contrary they are. The central projection through the point x on H takes s to a point in P' that lies on the line on H through s', but not on the segment s' itself. This is not possible because s' on H contains all the points in P' on the line through s', by our construction of S_2. $\square$

Based on Claim 2.3, we can now write $f(d,n) \geqslant n(1-\frac{c}{d}) + f(d-1, n(1-\frac{c}{d}))$. We will now prove by induction on d that $f(d,n) \geqslant \frac{dn}{4c}$. The constant 4 can be improved, and we make no attempt to optimize it here.

The inequality $f(d,n) \geqslant \frac{dn}{4c}$ is true for $d \leqslant 3c$ because of Ungar's theorem, that is $f(d,n) \geqslant f(2,n) \geqslant n-1$.

Assume therefore that $d > 3c$. We prove by induction on d that $f(d,n) \geqslant \frac{dn}{4c}$. For the induction step we need the following inequality to hold.

$$n(1-\frac{c}{d}) + \frac{(d-1)n(1-\frac{c}{d})}{4c} \geqslant \frac{dn}{4c}.$$

After simplification we get the following equivalent inequality to prove:

$$\frac{3}{4} + \frac{1}{4d} \geqslant \frac{1}{4c} + \frac{c}{d}$$

As we assume $d \geqslant 3c$, it is enough to show

$$\frac{5}{12} + \frac{1}{4d} \geqslant \frac{1}{4c}$$

One can easily check that this inequality holds for all $c \geqslant 1$. In our case $c = 12$ and we have therefore proved that $f(d,n) \geqslant \frac{dn}{48}$. This concludes the proof of Theorem 1.3, which implies Theorem 1.2.

3 Distinct distances for typical d-norms

In this section we sketch the proof of Theorem 1.5. The proof follows closely the one in [1], replacing Ungar's Theorem stated as Theorem 4.6 in [1] by our Theorem 1.2 here. Since the proof is very similar to the one in [1] we do not repeat here the identical parts, and merely describe the different points, referring to the arguments in [1] whenever needed. We start with the following modified version of Lemma 4.2 in [1].

Lemma 3.1. *There exists an absolute positive constant b so that the following holds. Let $d \geqslant 1$ be an integer and let $0 < \mu < 1$. Suppose that n is sufficiently large with respect to d and μ. Let $F \subseteq \mathbb{R}$ be a subfield of $\mathbb{R}$, and let V be a vector space over $\mathbb{R}$. Let $\mathbf{u}_1, \dots, \mathbf{u}_k \in V$ be non-zero vectors in V, and let $\mathbf{p}_1, \dots, \mathbf{p}_n \in V$ be distinct vectors such that not all of $\mathbf{p}_1, \dots, \mathbf{p}_n$ lie in a common $(d-1)$-dimensional affine subspace of V (as a vector space over $\mathbb{R}$). Suppose that for all $x, y \in \{1, \dots, n\}$ we have $\mathbf{p}_x - \mathbf{p}_y \in span_F(\mathbf{u}_i)$ for some $i \in \{1, \dots, k\}$. Then there exists a subset $I \subseteq \{1, \dots, k\}$, such that we have $\mathbf{u}_\ell \in span_F(\mathbf{u}_i : i \in I)$ for at least $d \cdot |I| + (bd - \mu) \cdot n + 1$ indices $\ell \in \{1, \dots, k\}$.*

Proof (sketch). Setting $m = \lceil (bd - \mu) \cdot n \rceil \leqslant bdn$, where b is the constant from Theorem 1.2, we want to prove that there is a subset $I \subseteq \{1, \dots, k\}$ with $\mathbf{u}_\ell \in \operatorname{span}_F(\mathbf{u}_i : i \in I)$ for at least $d \cdot |I| + m + 1$ indices $\ell \in \{1, \dots, k\}$. Suppose towards a contradiction that the desired subset $I \subseteq \{1, \dots, k\}$ does not exist. Then for every subset $I \subseteq \{1, \dots, k\}$, we have $\mathbf{u}_\ell \in \operatorname{span}_F(\mathbf{u}_i : i \in I)$ for at most $d \cdot |I| + m$ indices $\ell \in \{1, \dots, k\}$.

We may assume that for every $i \in \{1, \dots, k\}$ there exist distinct $x, y \in \{1, \dots, n\}$ with $\mathbf{p}_x - \mathbf{p}_y \in \operatorname{span}_F(\mathbf{u}_i)$ since otherwise, we can just omit all indices i for which this is not the case, and relabel the remaining indices.

By Theorem 1.2, there are at least bdn different line directions in $\operatorname{span}_{\mathbb{R}}(\mathbf{p}_1, \dots, \mathbf{p}_n) \subseteq V$ appearing among the differences $\mathbf{p}_x - \mathbf{p}_y$ with $1 \leqslant x < y \leqslant n$. For each of these differences we have $\mathbf{p}_x - \mathbf{p}_y \in \operatorname{span}_F(\mathbf{u}_i)$ for some $i \in \{1, \dots, k\}$. Hence, there must be at least bdn different vectors $\mathbf{u}_i$, so $k \geqslant bdn$.

We now construct a sequence of distinct indices $j_1, \dots, j_r \in \{1, \dots, k\}$ recursively exactly as described in the proof of Lemma 4.2 in [1]. We also define the subsets $H_0, H_1, \dots, H_r$ as in this proof and observe that the following two claims hold just as in that proof:

Claim 3.2. *We have $r \leqslant \frac{\mu}{3d} \cdot n$.*

Claim 3.3. *We have $|H_r| > \frac{\mu}{3d} \cdot n$.*

The only required difference between the proof of Claim 3.3 here and that of Claim 2 in [1] is the replacement of the penultimate line of this proof which is

$$k \geqslant n - 1 \geqslant \frac{2\mu}{3} \cdot n + \lceil (1 - \mu) \cdot n \rceil + 1 = d \cdot \frac{2\mu}{3d} \cdot n + m + 1 \geqslant d \cdot |I| + m + 1,$$

by the line

$$k \geqslant bdn \geqslant \frac{2\mu}{3} \cdot n + \lceil (bd - \mu) \cdot n \rceil + 1 = d \cdot \frac{2\mu}{3d} \cdot n + m + 1 \geqslant d \cdot |I| + m + 1,$$

which holds if n is sufficiently large with respect to μ. This contradicts our assumption that such a set I does not exist and completes the proof of the claim.

The rest of the proof of the lemma is identical to that of Lemma 4.2 in [1] where the only differences are as follows.

- In the inequality

$$\sum_{j\in J}\lambda_j \geqslant \frac{\lambda_1+\cdots+\lambda_k - m\cdot\frac{3d}{\mu}}{d} \geqslant \frac{\frac{\mu}{24d}\cdot|H_r|^2 - n\cdot\frac{3d}{\mu}}{d}$$

  the last n in the numerator in the right-hand-side has to be replaced by bdn giving

$$\sum_{j\in J}\lambda_j \geqslant \frac{\lambda_1+\cdots+\lambda_k - m\cdot\frac{3d}{\mu}}{d} \geqslant \frac{\frac{\mu}{24d}\cdot|H_r|^2 - bdn\cdot\frac{3d}{\mu}}{d}.$$

- In the next sentence n has to be replaced, again, by bdn as written in the following three lines:

  By Claim 3.3 we have $bdn\cdot\frac{3d}{\mu} \leqslant bd\left(\frac{3d}{\mu}\right)^2\cdot|H_r| \leqslant \frac{\mu}{48d}\cdot|H_r|^2$ if n (and therefore also $|H_r|\geqslant\frac{\mu}{3d}\cdot n$) is sufficiently large with respect to d and μ. So we can conclude that

$$\sum_{j\in J}\lambda_j \geqslant \frac{\frac{\mu}{24d}\cdot|H_r|^2 - bdn\cdot\frac{3d}{\mu}}{d} \geqslant \frac{\frac{\mu}{48d}\cdot|H_r|^2}{d} = \frac{\mu}{48d^2}\cdot|H_r|^2.$$

The end of the proof of the lemma is identical to the one in [1]. □

The proof of Theorem 1.5 is now essentially identical to the proof of Theorem 1.3 as described in Section 5 of [1], where the only differences are as follows.

- Each of the appearances of the quantity $(1-\mu)n$ should be replaced by $(bd-\mu))n$, and in particular the parameter m which is $\lceil(1-\mu)n\rceil$ in Section 5 of [1] has to be $\lceil(bd-\mu)n\rceil$ here.

- The paragraph considering the case that all the points $\mathbf{p_1},\ldots,\mathbf{p_n}$ are on a common line should be omitted here, as in our case, by assumption, the affine dimension of this set of points is $d\geqslant 2$.

This completes the sketch of the proof of Theorem 1.5. □

4 Concluding remarks and open problems

- The discussion in Section 3 shows that while Conjecture 1.1, if true, does not imply Conjecture 1.4 as stated here, it does imply that any d-dimensional set of $n>n_0(d)$ in a typical d-norm determines at least $(d-1-o(1))n$ distinct distances.

  The argument together with the main result of [11] also shows that any 3-dimensional set of $n>n_0$ points in a typical 3-norm determines at least $(2-o(1))n$ distinct distances.

- A slightly modified version of an old conjecture of Dirac [5] asserts that for any set of n points in the plane, not all on a single line, there is a point that lies in at least $n/2 - O(1)$ distinct lines determined by the set. This Conjecture is still open but several weaker versions have been established over the years. See [10] for the best known bound and the history of the problem. Corollary 2.2 deals with the higher dimensional version of the problem. Its quality depends on the best known estimate for the constant c. It is worth noting that up to the constant c the bound it provides is essentially best possible. Indeed, to see an example assume for simplicity that d is odd and n is divisible by $d+1$. Then consider $(d+1)/2$ skew lines that affinely span $\mathbb{R}^d$. On each of these lines take $2n/(d+1)$ points. It is easy to check that for every point of the resulting set of points, the number of lines connecting it to the other points of the set is $n - 2n/(d+1) + 1 = n(1 - \frac{2}{d+1}) + 1$.

- Theorem 1.5 together with the simple example of $n - d + 1$ points in an arithmetic progression on a line and $d - 1$ additional points in general position not on this line, show that for typical d-norms, the minimum possible number of distinct distances determined by a d-dimensional set of $n > n_0(d)$ points is $\Theta(nd)$. Note that this expression is an increasing function of d. This is in contrast to the behavior of this minimum in Euclidean spaces, where the known upper bound (which is conjectured to be tight) is $O(n^{2/d})$-a decreasing function of d.

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