

Counting Contours on Trees

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Abstract

We calculate the exact number of contours of size n containing a fixed vertex in d -ary trees and provide sharp estimates for this number for more general trees. We also obtain a characterization of the locally finite trees with infinitely many contours of the same size containing a fixed vertex.

Keywords: contours, trees, cut sets, Peierls, Catalan numbers

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Introduction

After the seminal paper of Rudolf Peierls [14], the standard technique to prove the existence of phase transitions in spin systems (Ising model type, for instance) goes by a *contour argument*. Roughly speaking, we need to define objects usually called *contours*, notions of size (length or surface) and interior for these objects. Furthermore, for a fixed vertex x of a graph G and, for each $n \in \mathbb{N}$, we need to estimate the number of contours of size n in G with x in their interiors.

A standard calculation in this approach is to control expressions as below:

$$\sum_{C \ni x} w(|C|) = \sum_{n=1}^{\infty} \sum_{\substack{C \ni x \\ |C|=n}} w(|C|) = \sum_{n=1}^{\infty} w(n) \sum_{\substack{C \ni x \\ |C|=n}} 1, \quad (1)$$

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where $|C|$ denotes the size of the contour C and $C \odot x$ denotes the fact that x belongs to the interior of C . Usually the function $w : \{\text{contours}\} \rightarrow \mathbb{R}^+$ depends only on the size of the contour and not on its position in the graph G . For the standard Ising model on \mathbb{Z}^2 , the function is given by $w(C) = w(|C|) = \exp(-2\beta|C|)$ where β is the inverse of the temperature. Then, to control (1) we need to estimate $\sum_{\substack{C \odot x \\ |C|=n}} 1$ and for this purpose

generating functions are very powerful tools. We can find similar expressions to (1) in almost all papers using the *Peierls argument*. The readers interested in the proof of the existence of phase transition using the Peierls contours can check standard books on the field [7, 11, 16, 22]. The original Peierls argument [14] was done for the Ising model on \mathbb{Z}^2 , but we can define contours for any \mathbb{Z}^d with $d \geq 3$ and the argument works as well. The estimates of the number of contours help us to give bounds for the critical temperature of the models, see [8, 13]. These facts show that the mathematical problem of counting contours on graphs has important consequences in statistical physics and naturally emerges.

Moreover, the problem of counting finite objects on graphs (subgraphs, paths with a fixed length, etc) is important for mathematicians and it is a classical problem in discrete mathematics. The history about the question of counting contours of the same size containing a fixed unit cube on \mathbb{Z}^d ($d \geq 2$) is the following: David Ruelle proved that there exist at most 3^n contours of size n containing a fixed unit cube; Lebowitz and Mazel [13] proved that there are between $(C_1 d)^{n/2d}$ and $(C_2 d)^{64n/d}$; and finally, differently from the previous approaches and using *generating functions*, Balister and Bollobás [8] improved these bounds showing that there are between $(C_3 d)^{n/d}$ and $(C_4 d)^{2n/d}$ contours of size n (C_1, C_2, C_3 and C_4 are constants).

In the last years, some attention was given to the Ising model on trees instead of \mathbb{Z}^d , and there is more than one definition of contour for trees and general graphs [2, 3, 13, 17, 18, 19, 20]. In our note we consider a definition proposed by Babson and Benjamini [3], where the contours are edge cuts which cut out exactly one finite component and they are minimal with this property. We will see that this definition on trees implies that the number of contours of size n coincides with the number of *external boundaries* with n vertices, a standard notion used by the combinatorics community. In the original paper they used the term *cut sets* as is usual for combinatorialists, the context was percolation theory, see also [4]. This definition was later considered in [2] in the study of bounds for the critical percolation probability p_c in general graphs.

Our contribution is to clarify the connection between contours on trees and natural objects in graph theory. Inspired by Balister and Bollobás [8], we show that in the case of regular trees (and d -ary trees) we can calculate the exact number of contours of size n containing a fixed vertex x . We also obtain a characterization for locally finite rooted trees with infinitely many contours of some fixed size n involving the root. In particular, we prove that we have infinitely many contours of the same size if and only if the tree has an *infinite independent path*. Nonamenable trees are the trees in which the length of the independent paths is uniformly bounded. In particular, trees which contain an infinite independent path are amenable trees. On the other hand, for nonamenable graphs with bounded degree, (in particular, d -ary trees) one possibility for the proof of the phase transition in Ising models and for the study of ground states is to count the number of connected components of a fixed size containing a vertex, instead of counting the number of contours, see [10, 12].

This note is organized as follows: in Section 1 we give some basic definitions of graph

theory, introduce the precise definition of a contour, and show the connection of these objects with external boundaries in graphs. In Section 2 we give explicit expressions for the number of contours of size n in regular and d -ary trees. In addition, we show that the binary trees are extremal objects with respect to the number of contours of a fixed size. More precisely, if we fix n , the number of contours of size n containing a fixed vertex is maximum in binary trees when we consider locally finite trees in which each vertex has at least two children. In Section 3 we give a geometric characterization of trees with infinitely many contours of the same size containing a fixed vertex. It turns out that this is equivalent to the existence of what's called an infinite independent path in the tree.

1 Definitions and Notations

The graphs $G = (V, E)$ considered are always simple, undirected, connected, with countably infinite number of vertices. All the graphs are locally finite, in other words, with finite degree for each vertex of V . The degree of a vertex x is the number of edges which are incident to x , denoted by $d(x)$. A *path* γ is an alternating sequence of vertices and edges $\gamma = (v_0, e_1, v_1, e_2, \dots, e_k, v_k)$ where $e_i = v_{i-1}v_i =: \{v_{i-1}, v_i\}$ and all vertices are distinct, with the possible exception of v_0, v_k . The vertices v_1, v_2, \dots, v_{k-1} are called *inner vertices* of γ . An *independent path* γ in a graph G is a path where all inner vertices of γ have degree two in G . When $v_0 = v_k$ we say that the path γ is a *cycle*. We say that a graph G is a *tree* if it is connected and has no cycles.

Given a vertex x and a subset of vertices $A \subset V$, let $d_G(x, A)$ denote the number $d_G(x, A) = \min\{|\gamma|; \gamma \text{ is a path in } G \text{ connecting } x \text{ to a vertex of } A\}$, where for each path γ in G , $|\gamma|$ denotes the number of edges of γ . Thus $d_G(x, A)$ is the usual distance in the graph G between x and A . The set $\partial_v^{ext} A = \{x \in V \setminus A : d_G(x, A) = 1\}$ is the *external boundary* of A .

Let $G = (V, E)$ be a graph, we say that a graph \tilde{G} is a *minor* of G , denoting by $\tilde{G} \preceq G$, when \tilde{G} is obtained from G after a sequence of the following operations: contracting some edges, deleting some edges and/or isolated vertices. We *contract* an edge $e = xy$ and obtain a graph that we denote by G/e when we delete the edge e from E , add to E the collection of edges $\{az; xz \in E \text{ or } yz \in E\}$ where a is a new vertex replacing the vertices x and y , and remove all resulting parallel edges. Thus $V(G/e) = V(G) \setminus (\{x, y\}) \cup \{a\}$. We *delete* an edge $e = xy$ when we remove the edge from the graph but keep the vertices on it, after the process we obtain a new graph $G \setminus e = (V, E \setminus \{e\})$, for a finite collection of edges C the procedure is the same, keeping the vertices and deleting the edges: $G \setminus C = (V, E \setminus C)$.

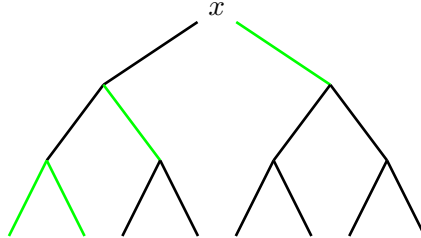
Definition 1. *Given a graph $G = (V, E)$, a finite set $C \subset E$ is called a contour if $G \setminus C$ has exactly one finite connected component, and it is minimal with respect to this property. That is, for all edges $e \in C$ the graph $(V, E \setminus (C \setminus e))$ does not have a finite connected component.*

If C is a contour in G then we denote by $G_C = (I_C, E_C)$ the unique finite connected component of $G \setminus C$.

This notion was originally defined by Babson and Benjamini in [3] where the authors used *minimal cut set* instead of *contour*. The definition was later used in [2] in the study of percolation problems on graphs.

Let \mathcal{F}_G be the set of all contours of G . We denote by \mathcal{F}_G^n the set of all contours of G of size n ; by $\mathcal{F}_G(x)$ ($\mathcal{F}_G^n(x)$) the set of all contours $C \in \mathcal{F}_G$ ($C \in \mathcal{F}_G^n$) such that $x \in I_C$.

Let T_d be a rooted tree such that all vertices have d children, i.e., the root has degree d and the other vertices have degree $d + 1$. The tree T_d is called *d-ary tree*. A 2-tree is called *binary tree*. When all the vertices of a tree have the same degree d we say that the tree is a *d-regular tree*.



Example of a contour of size four in a binary tree T_2

We conclude this section showing that in the case of trees there is a one-to-one correspondence between contours of size n in which the finite component contains the root and external boundaries of size n of sets containing the root. This proposition will allow us to conclude that for binary trees the number of contours of size n containing the root is the n -th Catalan number.

Proposition 1. *Let $T = (V, E)$ be a rooted, locally finite and infinite tree. Let x the root and suppose that T does not have leaves. Let*

$$\mathcal{B}_T^n(x) = \{B \subset V : B \text{ is finite, connected, } x \in B \text{ and } |\partial_v^{ext}(B)| = n\}$$

be the set of finite subtrees (induced by the vertices) of T containing x with external boundary of size n . Then there is a bijection between $\mathcal{B}_T^n(x)$ and $\mathcal{F}_T^n(x)$.

Proof. We will prove that for each $B \in \mathcal{B}_T^n(x)$ there exists a contour C such that $\partial_v^{ext}(B) = C$. We define the function $f : \mathcal{F}_T^n(x) \rightarrow \mathcal{B}_T^n(x)$ in the following way: let C be a contour in $\mathcal{F}_T^n(x)$. Remove all edges of C from the tree T . By definition of contour, we get a finite connected component B containing x . Define $f(C) = B$. To show that f is well defined, we shall prove that $|\partial_v^{ext}(B)| = n$. Actually, $B = I_C$.

By definition of contour, each edge in C has one endpoint in B and the other in $V \setminus B$. Let $V_e(C)$ be the set of endpoints in $C \cap (V \setminus B)$. As $|C| = n$ and the graph is a tree, we have $|V_e(C)| = n$. Clearly $V_e(C) \subseteq \partial_v^{ext}(B)$. If some element u of $\partial_v^{ext}(B)$ does not belong to $V_e(C)$, the edge connecting u with B does not belong to C , contradicting the fact that C is a contour. Thus f is well defined. It is not hard to see that f is a bijective function. \square

2 Contours on d-ary and regular trees

The main technique in this note is the use of generating functions in the investigation of counting problems on trees; this approach produces very clean proofs. Classical references to the technique are the two books of Richard P. Stanley [23, 24].

The well known *Catalan numbers* $C_{n-1} = \frac{1}{n} \binom{2n-2}{n-1}$ ($n \in \mathbb{N}$), have lots of interpretations in Combinatorics. In particular, see, e.g., [23, 24], these numbers count the number

of contours in binary trees by Proposition 1. In fact, let T_2 be the binary tree with root x . For all $n \geq 2$, we have $|\mathcal{F}_{T_2}^n(x)| = \frac{1}{n} \binom{2n-2}{n-1}$.

Here we present a proof where we calculate the exact number of contours in d -ary trees using generating functions, an alternative derivation can be found in [24]. Let $\mathbb{R}((z))$ be the ring of formal series defined by

$$\mathbb{R}((z)) = \left\{ \sum_{k \geq 0} a_k z^k : a_k \in \mathbb{R} \right\}.$$

We define the operator $[z^n]$ which extracts the coefficient of z^n in the series, that is, $[z^n](\sum_{k \geq 0} a_k z^k) = a_n$.

The Lagrange Inversion Theorem states that we can compute exactly the coefficients of a series under certain conditions. The reader can find a proof of this theorem in [9, 24].

Theorem (Lagrange Inversion Theorem, Lagrange – 1770). *Let $\phi \in \mathbb{R}((z))$ with $\phi(0) \neq 0$ and $f(z) \in z\mathbb{R}((z))$ defined by $f(z) = z\phi(f(z))$, then*

$$[z^n]f(z) = [z^{n-1}] \frac{1}{n} \phi(z)^n.$$

Proposition 2. *Let $d \geq 2$, $n \geq 1$, T_d be a d -ary tree with root x . Then $|\mathcal{F}_{T_d}^1(x)| = 0$ and*

$$|\mathcal{F}_{T_d}^n(x)| = \begin{cases} \frac{1}{n} \binom{\frac{d}{d-1}(n-1)}{\frac{1}{d-1}(n-1)}, & \text{if } n \equiv 1 \pmod{d-1}; \\ 0, & \text{otherwise,} \end{cases}$$

when $n \geq 2$.

Proof. For each edge with endvertex x , we can either include this edge in the contour or not. If we do not include it, we carry the root x to the other endvertex of this edge and apply again the same procedure. Consider $f(X) = \sum_{n \geq 1} a_n X^n$ the generating function in which the coefficients are $a_n = |\mathcal{F}_{T_d}^n(x)|$ for all $n \geq 1$. Then we have the following equation $f(X) = (X + f(X))^d$. Consider $h(X) = X + f(X)$, we have $h(X) = X + h(X)^d$, so $h(X) = X(1 - h(X)^{d-1})^{-1}$. Applying Lagrange's Theorem with $\phi(X) = (1 - X^{d-1})^{-1}$ we obtain $[X^n]h(X) = \frac{1}{n} [X^{n-1}]\phi(X)^n$. Now,

$$\phi(X)^n = (1 - X^{d-1})^{-n} = \sum_{k \geq 0} \binom{n+k-1}{k} X^{(d-1)k}.$$

Thus, if $n-1 = (d-1)k$ for some k , we have

$$[X^n]h(X) = \frac{1}{n} \binom{n+k-1}{k} = \frac{1}{n} \binom{\frac{d}{d-1}(n-1)}{\frac{1}{d-1}(n-1)}.$$

□

Remark: There is a geometric interpretation for the equation $h(X) = X + h(X)^d$. Let T_d be a d -ary tree with root x . Add an edge e for which x will be a leaf, and it will be an endpoint of e . Now we can either include the edge e in the contour or not. If we do not include it, we carry the root x to the other endvertex of this edge and apply the same procedure again.

Corollary 1. *Let $d \geq 2$, T_d be a d -ary tree with root x , and let $n \geq 1$ be such that $n \equiv 1 \pmod{d-1}$, and $k = (n-1)/(d-1)$. We have*

$$\frac{1}{n}d^k \leq |\mathcal{F}_{T_d}^n(x)| \leq \frac{1}{n}(ed)^k.$$

Proof. This is consequence of the following inequality. For all integers $0 \leq k \leq n$,

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

□

Proposition 3. *Let $d \geq 2$, $n \geq 1$, T be a $(d+1)$ -regular tree, and x be a vertex of T . Then*

$$|\mathcal{F}_T^n(x)| = a_{n-1} + \sum_{k=1}^{n-1} a_k a_{n-k},$$

where $a_n = |\mathcal{F}_{T_d}^n(x)|$.

Proof. Let $g(X) = \sum_{n \geq 1} b_n X^n$ be the generating function with coefficients $b_n = |\mathcal{F}_T^n(x)|$, and $f(X) = \sum_{n \geq 1} a_n X^n$ be the generating function with coefficients $a_n = |\mathcal{F}_{T_d}^n(x)|$. Note that $g(X) = (X + f(X))^{d+1} = Xf(X) + f(X)^2$. The proof is a direct consequence of the previous proposition. □

A natural question is to compare the number of contours between different infinite trees. We next show that the binary tree is extremal in the class of all locally finite trees in which every vertex has at least two children.

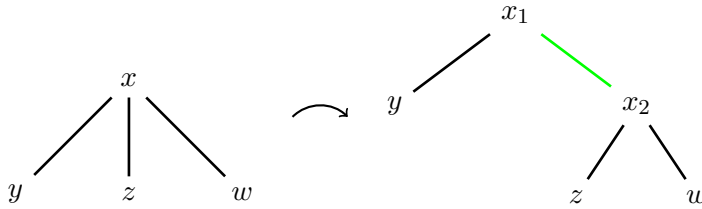
Theorem 1. *Let T be a locally finite and infinite rooted tree. Let x be the root of T and suppose that all vertices in T have at least two children. Then, for all $n \geq 1$, we have $|\mathcal{F}_T^n(x)| \leq |\mathcal{F}_{T_2}^n(x)|$.*

Proof. We will construct a binary labeled tree T' such that T is a minor of T' as follows. Starting from x we process the vertices of T according to a Breadth-First-Search order, that is, we start from the root x , then process its neighbors, followed by their neighbors and so on. When we process a vertex y of T that has $s > 2$ children, say z_1, z_2, \dots, z_s , we replace y by $s-1$ vertices y_1, y_2, \dots, y_{s-1} . For each i , the children of y_i are y_{i+1} and z_i , and the children of y_{s-1} are z_{s-1} and z_s . When a vertex y of T has 2 children, we keep the vertex y . Clearly T' is a binary tree. We call x' the root of T' . We will show that there exists an injective map f which takes each contour C in $\mathcal{F}_T^n(x)$ and produces a contour $f(C)$ in $\mathcal{F}_{T'}^n(x')$. In fact, for each edge of the form yz_i in C , we associate the edge $y_i z_i$ in T' (for yz_s take $y_{s-1} z_s$) and for y with $s = 2$ children we keep the edge yz_i . The collection of edges produced by this procedure is defined as $f(C)$. To simplify the argument let us call the new edges green edges, see the picture below.

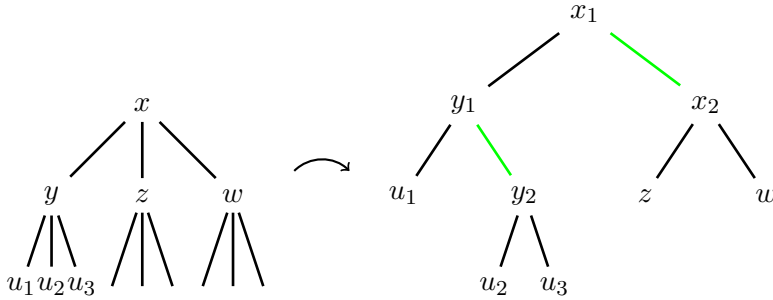
We should prove that $f : \mathcal{F}_T^n(x) \rightarrow \mathcal{F}_{T'}^n(x')$, in other words, that $f(C)$ belongs to $\mathcal{F}_{T'}^n(x')$. To see that $f(C)$ is a contour observe that, by construction, there are no green edges in $f(C)$. Suppose by contradiction that $T' \setminus f(C)$ has no finite connected component containing the root x' , then there exists an infinite path γ' in T' starting at the root x' of T' . When we contract all the green edges in T' , in particular in γ' , we obtain the original tree T and a path γ in T starting in the root x . Since there are no green edges in the path γ' , we have now an infinite path γ in T starting at the root

x with $E(\gamma) \cap C = \emptyset$, a contradiction. To see that $f(C)$ has the minimality property suppose that there exists an edge $e' \in f(C)$ such that $E(T') \setminus (f(C) \setminus \{e'\})$ still has a finite component containing the root x' . When we contract all the green edges and add the corresponding edge $e \in C$ (the edge associated to e' by f), since C is a contour, there exists an infinite path α starting at the root x in T such that $e \in E(\alpha)$. We will construct, using the path α , an infinite path α' in T' starting at x' such that $e' \in E(\alpha')$, to get a contradiction. Indeed, consider the process to construct the tree T' in the vertices of α . Starting at the root x , for each edge $zy \in E(\alpha)$, where z is the father of y , after processing z there exists $1 \leq j \leq s - 1$ such that $z_j y$ is an edge of T' . Add $z_j y$ to $E(\alpha')$. If $j = 1$ we add the edge $z_1 y$ to α , if $j > 1$ we add the finite path starting in z_1 and ending in z_j , (which consists of green edges: $z_1 z_2, z_2 z_3, \dots, z_{j-1} z_j$) and the edge $z_j y$ to α' . Since the path α is infinite and $e \in E(\alpha)$ we construct an infinite path α' , starting in x' such that e' belongs to α' . This shows that $f(C)$ is indeed a contour.

It is also easy to check that f is injective and that $|C| = n$ implies $|f(C)| = n$.



Example first iteration



Part of the second iteration

□

By the theorem above we have an estimate for trees in which each vertex has at least r children, where $r \geq 2$. However, we have a better estimate for these trees. To prove this we use the inequality below, a classical theorem in extremal combinatorics proved in [6], see also [1] and its references for several variants and extensions.

Theorem (Bollobás, 1965). *Let $\{(A_i, B_i) : i \in I\}$ be a finite collection of pairs of finite sets such that $A_i \cap B_j = \emptyset$ if and only if $i = j$. Then*

$$\sum_{i \in I} \binom{|A_i| + |B_i|}{|A_i|}^{-1} \leq 1.$$

In particular, if for all $i \in I$ we have $|A_i| \leq a$ and $|B_i| \leq b$, then

$$|I| \leq \binom{a+b}{a}.$$

Theorem 2. *Let T be a locally finite infinite rooted tree with root x . Suppose that all vertices of T have at least r children, $r \geq 2$. Then, for all $n \geq 1$,*

$$|\mathcal{F}_T^n(x)| \leq \binom{n + \lfloor \frac{n-r}{r-1} \rfloor}{\lfloor \frac{n-r}{r-1} \rfloor}.$$

where $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$.

Proof. Let C be a contour of size n in T and let I_C be the finite connected component when we remove C from T . We will find an upper bound for the number of edges $|E(I_C)|$ in I_C . Let B be the induced subgraph of T on the union of I_C and C . Note that $B = (V, E)$ is a rooted finite subtree of T with n leaves, and each vertex of B that is not a leaf has at least r children. Let t be the number of vertices of B and consider the number $k = t - n$. Note that k is the number of vertices in I_C . Using the fact that all vertices of T have at least r children, we have

$$2(t-1) = \sum_{v \in V} d(v) \geq (k-1)(r+1) + r + n.$$

Thus, $k \leq (n-1)/(r-1)$.

Since I_C is a tree, the number of edges in I_C is $|E(I_C)| = k - 1 \leq (n-r)/(r-1)$.

To finish the proof we need the following:

Fact: If C_1 and C_2 are two distinct contours of a vertex x in T , each of size n , and if I_{C_1} is the finite connected component when we remove C_1 from T , then $E(I_{C_1}) \cap C_2 \neq \emptyset$.

Proof of fact: Suppose, by contradiction, that there exist two contours C_1 and C_2 as above in T , each of size n , such that $E(I_{C_1}) \cap C_2 = \emptyset$. Let I_{C_2} be the finite connected component when we remove C_2 from T . Then I_{C_1} is a subgraph of I_{C_2} and $I_{C_1} \neq I_{C_2}$. Since I_{C_1} and I_{C_2} are subtrees of T , and as all vertices in T have at least r children, we have $|\partial_e(I_{C_1})| < |\partial_e(I_{C_2})| = n$, a contradiction. This proves the fact.

Finally, let us prove the desired result. Let $(C, E(I_C))$ be a pair of a contour C of size n , where I_C is the finite connected component when we remove C from T . We have $|C| = n$ and $|E(I_C)| \leq \lfloor (n-r)/(r-1) \rfloor$. The set $\{(C, E(I_C)) : C \in \mathcal{F}_T^n(x)\}$ is finite, and $C_1 \cap E(I_{C_2}) = \emptyset$ if and only if $C_1 = C_2$. By the theorem above,

$$|\mathcal{F}_T^n(x)| \leq \binom{n + \lfloor \frac{n-r}{r-1} \rfloor}{\lfloor \frac{n-r}{r-1} \rfloor},$$

concluding the result. \square

It is sometimes desirable to consider contours whose edges cover a fixed vertex, see [18]. We obtain some bounds for this case as well.

Definition 2. *Let T be an infinite tree with root x . A rooted contour is a contour C such that there exists an edge $l \in C$ incident with the root x .*

We denote by $\mathcal{F}_{r,T}^n(x)$ the set of rooted contours C on T of size n . We can calculate exactly $|\mathcal{F}_{r,T}^n(x)|$ for d -ary trees and regular trees. Clearly $|\mathcal{F}_{r,T}^n(x)| \leq |\mathcal{F}_T^n(x)|$.

Proposition 4. *Let T_d be a d -ary tree with root x . Then, for all $n \geq d$:*

$$|\mathcal{F}_{r,T_d}^n(x)| = a_n - \sum_{m_1+\dots+m_d=n} a_{m_1} \dots a_{m_d};$$

where $a_n = |\mathcal{F}_{T_d}^n(x)|$. (Note that $|\mathcal{F}_{r,T_d}^n(x)| = 0$ if $n < d$).

Proof. Let $f_{T_d}(X) = \sum_{n \geq 1} a_n X^n$ and $f(X) = \sum_{n \geq 1} c_n X^n$ be the generating functions with coefficients $a_n = |\mathcal{F}_{T_d}^n(x)|$ and $c_n = |\mathcal{F}_{r,T_d}^n(x)|$ respectively. For each edge incident with x we can add it or not to the contour C . Repeating the same process as we did in Proposition 2, if we do not add an edge we carry the root to the other endpoint of this edge. By the same proposition we know $f_{T_d}(X) = (X + f_{T_d}(X))^d$. Thus

$$f(X) = (X + f_{T_d}(X))^d - (f_{T_d}(X))^d = f_{T_d}(X) - (f_{T_d}(X))^d.$$

□

Proposition 5. *Let T be a $(d+1)$ -regular tree with root x . Then, for $n \geq d+1$:*

$$|\mathcal{F}_{r,T}^n(x)| = b_n - \sum_{m_1+\dots+m_{d+1}=n} a_{m_1} \dots a_{m_{d+1}};$$

where $a_n = |\mathcal{F}_{T_d}^n(x)|$ and $b_n = |\mathcal{F}_T^n(x)|$.

Proof. Using a similar argument to the one used in the previous proof, let $f(X) = \sum_{n \geq 1} d_n X^n$ be the generating function with coefficients $d_n = |\mathcal{F}_{r,T}^n(x)|$ and let $g(X) = \sum_{n \geq 1} b_n X^n$ be the generating function from Proposition 3. Then $f(X) = g(X) - (f_{T_d}(X))^{d+1}$. □

3 Infinitely many contours of size n

A natural question is to study when we have infinitely many contours for some size n whose finite connected component contains a fixed vertex x . We can characterize the trees with this property.

Notation 1. *Let $G = (V, E)$ be a graph. For each finite independent path γ of G linking two vertices x and y , remove all the edges (and inner vertices) of γ and add the edge xy . Denote this new graph that is a minor of G , possibly with fewer edges, by \tilde{G} .*

In the next lemma and proposition the notation \tilde{G} is used for this special case of minor.

Lemma 1. *Let T be a tree with root x without leaves. Suppose that each independent path of T has finite length. Then $|\mathcal{F}_T^n(x)| < +\infty$ if and only if $|\mathcal{F}_{\tilde{T}}^n(x)| < +\infty$.*

Proof. For each contour $C = \{e_1, \dots, e_n\}$ of \tilde{T} , the contour is associated to a (unique) family of independent paths $\{\gamma_1, \dots, \gamma_n\}$ of T . Then,

$$\sum_{C \in \mathcal{F}_{\tilde{T}}^n(x)} \prod_{i=1}^n |\gamma_i| = |\mathcal{F}_T^n(x)|.$$

As the sum and the product are finite, we obtain $|\mathcal{F}_T^n(x)| < +\infty$. The converse is analogous. □

Thus we obtain the following characterization:

Theorem 3. *Let T be a locally finite rooted tree with a root x and without leaves. Then there exists $n \geq 1$ such that $|\mathcal{F}_T^n(x)| = +\infty$ if and only if T has an infinite independent path.*

Proof. If we assume that $|\mathcal{F}_T^n(x)| = +\infty$, the above is a consequence of Lemma 1 combined with Theorem 1. For the converse, take an infinite independent path γ . Let C be a contour of T that contains an edge e of γ . For all edge e' of $\gamma \setminus \{e\}$, we have that $C' = (C \setminus \{e\}) \cup \{e'\}$ is a contour of T and $|C| = |C'|$. Therefore, taking $n = |C|$ we obtain $|\mathcal{F}_T^n(x)| = +\infty$. \square

Proposition 6. *Let T be an infinite, locally finite rooted tree with root x without leaves. Suppose that T has an infinite independent path. Then there exists a sequence $(n_i)_{i \geq 1}$ such that $|\mathcal{F}_T^{n_i}(x)| = +\infty$ if and only if there is an infinite number of vertices in T with degree at least three.*

Proof. Suppose that there exist only a finite number of vertices in T with degree at least three. Take \tilde{T} constructed as in Notation 1. If an independent path is infinite, we replace this independent path by a leaf. This new tree we denote by T' . Since T has an infinite independent path, T' has at least one leaf. Moreover, T' is a finite tree because T has only a finite number of vertices with degree at least three. Let B be a subtree of T' such that $x \in B$ and B does not contain any leaf. Let C be the set of external boundary edges of B . For each C constructed in this way we obtain a family of contours of T of the same size and any contour in T comes from some external boundary edges for some such B . As we have a finite number of subtrees of T' that do not contain leaves, there exists $n_0 \geq 1$ such that for all $n \geq n_0$ we have $|\mathcal{F}_T^n(x)| = 0$.

For the converse, suppose that there exists an infinite number of vertices in T with degree at least three. Let E_k be the set of edges whose distance from x is k . E_k is a contour. Since the number of vertices in T with degree at least three is infinite, the number of edges in each E_k is tending to infinity when we increase k . Let $(k_i)_i$ be an increasing sequence of natural numbers such that $n_i = |E_{k_i}|$ is also an increasing sequence. Let γ be an infinite independent path. Note that there exists i_0 such that E_{k_i} contains an edge e_i of the infinite independent path γ for all $i \geq i_0$. Then, since we can replace e_i by any other edge of γ and obtain a new contour of the same size n_i , we have infinitely many contours of size n_i . \square

Final Remark

The Peierls strategy to look for contours involving a vertex fails if w in (1) depends only on the size of the contours when we have infinitely many contours of the same size. However, in [21] Rozikov studied an example of an Ising model type on \mathbb{Z} where we have infinitely many contours of size 2 involving the vertex 0. He adapted the Peierls argument to prove the phase transition for the model. In this case $w(C)$ must depend on the position of the contour C in the graph, this is the usual situation when the hamiltonian of the model it is not translation invariant, see [5, 15].

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