**High School Coalitions**

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# 1 The problem

Shay Moran showed me a few days ago a question posted by a woman named Ruthi Shaham in a Facebook Group focusing on Mathematics. She writes that her son has finished elementary school and is now moving to high school. When doing so, each child lists three friends, and the assignment of children into classes ensures that each child will have at least one of these three friends in his class. Ruthi further writes that her son heard from five of his schoolmates that they found that they can make their selections in a way that will ensure that all five will be scheduled to the same class. She tried to check with a paper and pencil and couldn't decide whether or not this is possible, but she suspects it is impossible. She is asking if this is indeed the case, and if so, whether a larger group of children can form such a coalition ensuring they will all necessarily be assigned to the same class.

In this brief write-up I will show that Ruthi is is indeed right, no coalition of five children can ensure they will share the same class. Moreover, no coalition of any size can ensure to share the same class. Amusingly, this is related to known problems and results in Combinatorics and Graph Theory, as are several variants of the problem mentioned below.

Here is a more formal formulation of the problem, with general parameters. Let $N = \{1, 2, \ldots, n\}$ be a finite set of size $n$, let $k$ and $r$ be integers, and suppose $n \geq k + 1$. For any collection of subsets $S_i$ of $N$, $(1 \leq i \leq n)$, with $i \notin S_i$, and $|S_i| = k$ for all $i$, let $P(S_1, S_2, \ldots, S_n)$ be a partition of $N$, so that :

$$\text{For any part } N_i \text{ of the partition and for any } j \in N, \text{ if } j \in N_i \text{ then } S_j \cap N_i \neq \emptyset. \qquad (1)$$

Here $N$ denotes the group of children, $S_i$ is the list of friends listed by child number $i$, and the partition of $N$ into parts $N_i$ is the partition of the set of children into classes. The function $P$ represents the way the children are partitioned into classes $N_i$ given their choices $S_i$, and the condition (1) is the one ensuring that each child will have at least one other child from his list in his class.

We say that a subset $R \subset N$ is a successful coalition, if there are choices $S_i, i \in R$ of sets $S_i$ satisfying $|S_i| = k$ and $i \notin S_i$ so that for any sets $S_j \subset N$ with $|S_j| = k$ for all $j \in N - R$, and for any function $P$ satisfying the conditions above, all elements of $R$ belong to the same part of the partition $f(S_1, S_2, \ldots, S_n)$. Note that by symmetry if a successful coalition of size $r$ is possible then any set of size $r$ can form such a coalition, and hence we may always assume that $R = \{1, 2, \ldots, r\}$.

The question of Ruthi is whether or not for $k = 3$ there can be a successful coalition $R$ of size $|R| = 5$.

**Theorem 1.1**

1. *For $k \leq 2$ and every integer $r > 1$, every set $R$ of size $r$ can form a successful coalition.*

2. *For any $k \geq 3$ and every $r > 1$ no set of size $r$ can form a successful coalition.*

## 2 Proofs

Before presenting the general proof, here is a short argument showing that for $k = 3$ no successful coalition of size 5 is possible. This proof is a simple application of the probabilistic method, which is a powerful method introduced by Paul Erdős some eighty years ago, and developed by him and by many other researchers since then. See, e.g., [3] for more about this method.

**Claim:** Suppose $n \geq 5$, $N = \{1, 2, \ldots, n\}$, $R = \{1, 2, \ldots, 5\}$, and let $S_1, \ldots, S_5$ be subsets of $N$, each of size 3, so that $i \notin S_i$ for all $1 \leq i \leq 5$. Then there are subsets $S_j \subset N$, for $5 \leq j \leq n$ and there is a partition $P(S_1, \ldots, S_n)$ of $N$ into two disjoint parts $N_1, N_2$ satisfying (1) such that $R$ intersects both $N_1$ and $N_2$.

**Proof:** Color the elements of $N$ randomly red and blue, where each $i \in N$ randomly and independently is red with probability $1/2$ and blue with probability $1/2$. The probability that all members of $R$ have the same color is $1/16$. For each fixed $i \leq 5$, the probability that the color of $i$ is different than that of all elements in $S_i$ is $1/8$. Therefore, with probability at least $1 - 1/16 - 5/8 > 0$ none of these events happens. Hence there is a coloring in which $R$ contains both red and blue elements, and every $i \in R$ has at least one member of $S_i$ with the same color as $i$. Fix such a coloring. Without loss of generality 1 is colored red and 2 is colored blue. Let $N_1$ be the set of all elements colored red and let $N_2$ be the set of all elements colored blue. For each $j \in N_1 - R$ let $S_j$ contain 1 and for each $j \in N_2 - R$ let $S_j$ contain 2. It is easy to see that the partition $N = N_1 \cup N_2$ satisfies (1) but $R$ intersects both $N_1$ and $N_2$, completing the proof. $\square$

Note that the above proof does not work for $r \geq 8$, thus the proof of Theorem 1.1 requires a different method, which we show next.

**Proof of Theorem 1.1:** The case $k \leq 2$ is very simple. For $k = 1$ simply define $S_i = \{(i + 1) (\bmod\ r)\}$ to see that the coalition $R = \{1, 2, \ldots, r\}$ is successful. For $k = 2$ and $r = 2$, $S_1 = \{2, 3\}$, $S_2 = \{1, 3\}$ show that $\{1, 2\}$ is successful. For any larger $r$ add to the above $S_i = \{1, 2\}$ for all $3 \leq i \leq r$.

The more interesting part is the proof that for $k \geq 3$ no coalition of any size $r > 1$ can be successful. The case $r < k$ here is simple. One possible proof is to repeat the probabilistic argument described above for the case $k = 3, r = 5$. Since for $1 < r < k$, $k \geq 3$,

$$\frac{1}{2^{r-1}} + \frac{r}{2^k} \leq \frac{1}{2} + \frac{k-1}{2^k} \leq \frac{1}{2} + \frac{2}{8} < 1$$

the result follows as before. (It is also possible to give a direct simple proof for this case).

For $k \geq 3$, $r \geq k$ consider the digraph whose set of vertices is $N$, where for each vertex $i$ and each $j \in S_i$, $ij$ is a directed edge. Thus every outdegree in this digraph is exactly $k$. Given the sets $S_1, \ldots, S_r$ of outneighbors of the vertices in $R = \{1, 2, \ldots, r\}$ (representing the children attempting to form a successful coalition), define the sets $S_j$ for $j > r$ in such a way that the induced subgraph on $N - R$ is acyclic. (For example, we can define $S_j = \{1, 2, \ldots, k\}$ for each $j > r$, or $S_j = \{j-1, j-2, \ldots, j-k\}$ for each $j > r$. Note that here we used the fact that $r \geq k$).

The crucial result we use here is a theorem of Carsten Thomassen ([4], see also [1] for an extension). This Theorem asserts that any digraph with minimum outdegree at least 3 contains two vertex disjoint cycles. Let $A$ and $B$ be the sets of vertices of these two cycles. Note that both $A$ and $B$ must contain a vertex of $R$ (as $N - R$ contains no directed cycles). Let $A', B'$ be two sets of vertices satisfying $A \subset A'$, $B \subset B'$ with $|A'| + |B'|$ maximum subject to the constraint that every outdegree in $A'$ is at least 1 and every outdegree in $B'$ is at least 1. We claim that $A' \cup B'$ is the set $N$ of all vertices. Indeed, otherwise, every $v$ in $C = N - (A' \cup B')$ has no outneighbors in $A' \cup B'$ (otherwise we could have added it to either $A'$ or $B'$ contradicting maximality), so has at least $k \geq 3 > 1$ outneighbors in $C$ and then we can replace $A'$ by $A' \cup C$ contradicting maximality. This proves the claim. The assignment to two groups is now $N_1 = A'$ and $N_2 = B'$. Since both $A \subset A'$ and $B \subset B'$ contain elements of $R$, this shows that $R$ is not a successful coalition, completing the proof. □

# 3 Variants

1. What if every child is ensured to have at least two of his choices with him in his class ? In this case, even if $k$ is arbitrarily large (but $r$ is much larger) we do not know to prove that a coalition of $r$ cannot ensure they are all in the same group. This is identical to one of the open questions in [2]. On the other hand it is easy to see that this is impossible if $\frac{1}{2^{r-1}} + \frac{r(1+k)}{2^k} < 1$. Indeed, if so we can split the group of children randomly into two sets, red and blue. With positive probability the specific set of $r$ children trying to form a coalition is not monochromatic, and also for any child in the coalition there are at least two of his choices in his group. We can now fix the choices of all others outside the coalition to ensure they will also be happy with this partition. It follows that if in this version of the problem a successful coalition of size $r$ is possible, then $r$ has to be at least exponential in $k$.

2. Suppose we change the rules, and each child lists $k$ other children that he does *not* like, and wishes not to have many of them in his class. It can then be shown that for any $k$ there is an example of choices of the children in which each one lists $k$ others he prefers to avoid, so

that in any partition of the group of children into 2 classes, there will always be at least one poor child sharing the same class with all the $k$ he listed ! This is based on another result of Thomassen [5]: for every $k$ there is a digraph with minimum outdegree $k$ which contains no even directed cycle. If $D = (N, E)$ is such a digraph, and $N = V_1 \cup V_2$ is a partition of its vertex set into two disjoint parts, then, as observed in [2], there is a vertex in one of the classes having all its out-neighbors in the same class. Indeed, otherwise, starting at an arbitrary vertex $v_1$ we can define an infinite sequence $v_1, v_2, v_3, \ldots$, where each pair $(v_i, v_{i+1})$ is a directed edge with one end in $V_1$ and one in $V_2$. As the graph is finite, there is a smallest $j$ such that there is $i < j$ with $v_i = v_j$, and the cycle $v_i, v_{i+1}, \ldots, v_j = v_i$ is even, contradiction. On the other hand, by splitting the group of children into $s \geq 3$ disjoint groups, we can always ensure that each child will have in his own class at most $2k/s$ of the $k$ children he wants to avoid. This follows from a result of Keith Ball described in [2].

## References


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