Cleaning d-regular graphs with brushes

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Abstract

A model for *cleaning* a graph with brushes was recently introduced. We consider the minimum number of brushes needed to clean d-regular graphs in this model, focusing on the asymptotic number for random d-regular graphs. We use a degree-greedy algorithm to clean a random d-regular graph on n vertices (with dn even) and analyze it using the differential equations method to find the (asymptotic) number of brushes needed to clean a random d-regular graph using this algorithm (for fixed d). We further show that for any d-regular graph on n vertices at most $n(d+1)/4$ brushes suffice, and prove that for fixed large d, the minimum number of brushes needed to clean a random d-regular graph on n vertices is asymptotically almost surely $\frac{n}{4}(d + o(d))$, thus solving a problem raised in [15].

1 Introduction

The cleaning model, introduced in [13, 14], is a combination of the chip-firing game and edge-searching on a simple finite graph. Initially, every edge and vertex of a graph is *dirty* and a fixed number of brushes start on a set of vertices. At each step, a vertex v and all its incident edges which are dirty may be *cleaned* if there are at least as many brushes on v as there are incident dirty edges. When a vertex is cleaned, every incident dirty edge is traversed (i.e. cleaned) by one and only one brush, and brushes cannot traverse a clean edge. See Figure 1 for an example of this cleaning process. The initial configuration has only 2 brushes, both at a. The solid edges are dirty and the dotted edges are clean. The circle indicates which vertex is cleaned next.

The assumption in [14], and taken here, is that *a graph is cleaned when every vertex has been cleaned.* If every vertex has been cleaned, it follows that every edge has been cleaned. It may be that a vertex v has no incident dirty edges at the time it is cleaned, in which case no brushes move from v. Although this viewpoint might seem unnatural, it simplified much of the analysis in [14].

In this paper, we are interested in the asymptotic number of brushes needed to clean d-regular, and mainly random d-regular (finite, simple) graphs. At one extreme, the graph could consist of disjoint copies of K_{d+1}. From [14], K_{d+1} requires essentially $d^2/4$ brushes so that the whole graph requires approximately $nd/4$. At the lower end, if d is even then a ring of bipartite

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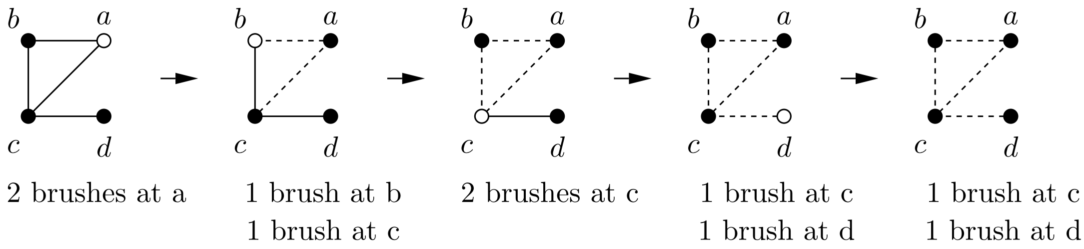


Figure 1: An example of the cleaning process for graph G.

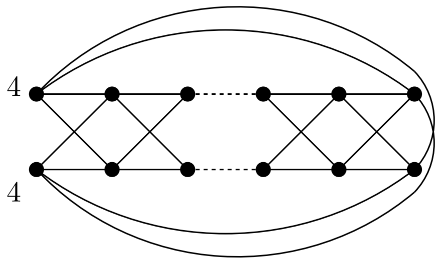


Figure 2: An example of the cleaning process for a 4-regular graph requiring 8 brushes.

graphs $K_{d/2,d/2}$ chained together (see Figure 2 for the case $d = 4$) requires only $d^2/4$ brushes regardless of the number of vertices (by working around the ring). If d is odd then every vertex has at least one brush in either the original or final configuration (see [14] for more details) so that a graph on n vertices requires at least $n/2$ brushes.

In Section 2 we introduce the formal definitions for the cleaning process and also include a description of the pairing model which is used in the results on random regular graphs, instead of working directly in the uniform probability space.

In Section 3 we describe some general upper and lower bounds for the minimum number of brushes needed to clean a graph, and show, in particular, that for any d-regular graph on n vertices, $n(d+1)/4$ brushes suffice if d is odd, and $\frac{n}{4}(d+1-\frac{1}{d+1})$ brushes suffice if d is even. These bounds are tight. We also show that for random d-regular graphs on n vertices, the minimum number of brushes needed is, asymptotically almost surely, at least $\frac{n}{4}(d - O(\sqrt{d}))$.

Section 4 concerns random d-regular graphs. Most of the results in this section form an extended version of the conference paper [15]. We first observe that if $d = 2$, then the brush number of a random d-regular graph on n vertices is a.a.s. $(1+o(1))\log n$; for $d = 3$, the brush number is equal to $n/2 + 2$ a.a.s.; for $d = 4$, $(1 + o(1))n/3$ brushes are enough to clean a graph a.a.s.; and for $d = 5$, roughly $0.644n$. In order to get an asymptotically almost sure upper bound on the brush number we use a degree-greedy algorithm [19] to clean the graph and then use the differential equation method, studied in [22] to find the asymptotic number of brushes required. We also consider the case of large d, and show that the typical brush number in this case is roughly $nd/4$, thus solving a problem raised in [15].

We conclude with a few open problems.

2 Definitions

The following cleaning algorithm and terminology was recently introduced in [14].

Formally, at each step t, $\omega_t(v)$ denotes the number of brushes at vertex v ($\omega_t : V \to \mathbb{N} \cup \{0\}$) and D_t denotes the set of dirty vertices. An edge $uv \in E$ is dirty if and only if both u and v are dirty: $\{u, v\} \subseteq D_t$. Finally, let $D_t(v)$ denote the number of dirty edges incident to v at step t:

$$D_t(v) = \begin{cases} |N(v) \cap D_t| & \text{if } \ v \in D_t \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.1 *The* ***cleaning process*** $\mathfrak{P}(G, \omega_0) = \{(\omega_t, D_t)\}_{t=0}^T$ *of an undirected graph* $G = (V, E)$ *with an* ***initial configuration of brushes*** ω_0 *is as follows:*

(0) *Initially, all vertices are dirty:* $D_0 = V$*; set* $t := 0$

(1) *Let* α_{t+1} *be any vertex in* D_t *such that* $\omega_t(\alpha_{t+1}) \geq D_t(\alpha_{t+1})$*. If no such vertex exists, then stop the process, set* $T = t$*, return the* ***cleaning sequence*** $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_T)$*, the* ***final set of dirty vertices*** D_T*, and the* ***final configuration of brushes*** ω_T

(2) *Clean* α_{t+1} *and all dirty incident edges by moving a brush from* α_{t+1} *to each dirty neighbour. More precisely,* $D_{t+1} = D_t \setminus \{\alpha_{t+1}\}$*,* $\omega_{t+1}(\alpha_{t+1}) = \omega_t(\alpha_{t+1}) - D_t(\alpha_{t+1})$*, and for every* $v \in N(\alpha_{t+1}) \cap D_t$*,* $\omega_{t+1}(v) = \omega_t(v) + 1$ *(the other values of* ω_{t+1} *remain the same as in* ω_t*)*

(3) $t := t + 1$ *and go back to* **(1)**

Note that for a graph G and initial configuration ω_0, the cleaning process can return different cleaning sequences and final configurations of brushes; consider, for example, an isolated edge uv and $\omega_0(u) = \omega_0(v) = 1$. It has been shown (see Theorem 2.1 in [14]), however, that the final set of dirty vertices is determined by G and ω_0. Thus, the following definition is natural.

Definition 2.2 *A graph* $G = (V, E)$ ***can be cleaned*** *by the initial configuration of brushes* ω_0 *if the cleaning process* $\mathfrak{P}(G, \omega_0)$ *returns an empty final set of dirty vertices* ($D_T = \emptyset$)*.*

Let the brush number, $b(G)$*, be the minimum number of brushes needed to clean* G*, that is,*

$$b(G) = \min_{\omega_0 : V \to \mathbb{N} \cup \{0\}} \Big\{ \sum_{v \in V} \omega_0(v) : G \text{ can be cleaned by } \omega_0 \Big\}.$$

Similarly, $b_\alpha(G)$ *is defined as the minimum number of brushes needed to clean* G *using the cleaning sequence* α*.*

It is clear that for every cleaning sequence α, $b_\alpha(G) \geq b(G)$ and $b(G) = \min_\alpha b_\alpha(G)$. (The last relation can be used as an alternative definition of $b(G)$.) In general, it is difficult to find $b(G)$, but $b_\alpha(G)$ can be easily computed. For this, it seems better not to choose the function ω_0 in advance, but to run the cleaning process in the order α, and compute the initial number of brushes needed to clean a vertex. We can adjust ω_0 along the way

$$\omega_0(\alpha_{t+1}) = \max\{2D_t(\alpha_{t+1}) - \deg(\alpha_{t+1}), 0\}, \quad \text{for } t = 0, 1, \ldots, |V| - 1, \tag{1}$$

since that is the number of brushes we have to add over and above what we get for free.

Our main results refer to the probability space of random d-regular graphs with uniform probability distribution. This space is denoted $\mathcal{G}_{n,d}$, and asymptotics (such as "asymptotically almost surely", which we abbreviate to a.a.s.) are for $n \to \infty$ with $d \geq 2$ fixed, and n even if d is odd.

Instead of working directly in the uniform probability space of random regular graphs on n vertices $\mathcal{G}_{n,d}$, we use the *pairing model* of random regular graphs, first introduced by Bollobás [6], which is described next. Suppose that dn is even, as in the case of random regular graphs, and consider dn points partitioned into n labeled buckets $v_1, v_2, \ldots, v_n$ of d points each. A *pairing* of these points is a perfect matching into $dn/2$ pairs. Given a pairing P, we may construct a multigraph $G(P)$, with loops allowed, as follows: the vertices are the buckets $v_1, v_2, \ldots, v_n$, and a pair $\{x, y\}$ in P corresponds to an edge $v_i v_j$ in $G(P)$ if x and y are contained in the buckets v_i and v_j, respectively. It is an easy fact that the probability of a random pairing corresponding to a given simple graph G is independent of the graph, hence the restriction of the probability space of random pairings to simple graphs is precisely $\mathcal{G}_{n,d}$. Moreover, it is well known that a random pairing generates a simple graph with probability asymptotic to $e^{(1-d^2)/4}$ depending on d, so that any event holding a.a.s. over the probability space of random pairings also holds a.a.s. over the corresponding space $\mathcal{G}_{n,d}$. For this reason, asymptotic results over random pairings suffice for our purposes. One of the advantages of using this model is that the pairs may be chosen sequentially so that the next pair is chosen uniformly at random over the remaining (unchosen) points. For more information on this model, see [20].

3 Bounds

3.1 Lower bounds

When a graph G is cleaned using the cleaning process described in Definition 2.1, each edge of G is traversed exactly once and by exactly one brush.

Note that no brush may return to a vertex it has already visited, motivating the following definition.

Definition 3.1 *The* ***brush path*** *of a brush b is the path formed by the set of edges cleaned by b.*

By definition, G can be decomposed into $b_\alpha(G)$ brush paths. (Since no brush can stay at its initial vertex in the minimal brush configuration, these paths each have at least one edge.) Thus, the minimum number of paths into which a graph G can be decomposed yields a lower bound for $b(G)$. This is only a lower bound because some path decompositions would not be valid in the cleaning process. For example, K_4 can be decomposed into two edge-disjoint paths, but $b(K_4) = 4$.

Following Definitions 2.1 and 3.1, every vertex of odd degree in a graph G will be the endpoint of (at least) one brush path. This leads to a natural lower bound for $b(G)$ since a graph with d_o odd vertices cannot be decomposed into less than $d_o/2$ paths (see [14] for more details).

Theorem 3.2 *Given initial configuration ω_0, suppose G can be cleaned yielding final configuration ω_T. Then for every vertex v in G with odd degree, either $\omega_0(v) > 0$ or $\omega_T(v) > 0$. In particular, $b(G) \geq d_o(G)/2$ where $d_o(G)$ denotes the number of vertices of odd degree.*

The result can be improved a little if there is a lower bound on the vertex degrees (see Section 4.3 for details).

Another general lower bound for random d-regular graphs can be obtained as follows. By [14, Theorem 3.2],

$$b(G) \geq \max_j \min_{S \subseteq V, |S|=j} \{jd - 2|E(G[S])|\} = \max_j \min_{S \subseteq V, |S|=j} |E(S, V \setminus S)|, \tag{2}$$

where $E(S, V \setminus S)$ is the set of all edges between S and its complement, and $E(G[S])$ is the set of all edges in the induced subgraph of G on S. The proof is a simple corollary of the fact that the minimum above is a lower bound on the number of edges going from the first j vertices cleaned to elsewhere in the graph. So, suppose that x and y are functions of n such that the expected number $S(x, y)$ of sets S of xn vertices in $G \in \mathcal{G}_{n,d}$ with yn edges to the complement $V(G) \setminus S$ is $o(1)$. Then this theorem, together with the first moment principle, gives that the brush number is a.a.s. at least yn.

In order to find optimal values of x and y we use the pairing model. (This is essentially the same argument used by Bollobás [8] to obtain a lower bound on the isoperimetric number of random regular graphs, but since it is slightly simpler for our purposes, and we obtain a slightly different conclusion, we include the argument.) It is clear that

$$S(x, y) = \binom{n}{xn}\binom{xdn}{yn} M(xdn - yn) \binom{(1-x)dn}{yn} (yn)! M((1-x)dn - yn)/M(dn)$$

where $M(i)$ is the number of perfect matchings on i vertices, that is,

$$M(i) = \frac{i!}{(i/2)! 2^{i/2}}.$$

After simplification we get

$$S(x, y) = \frac{n!(xdn)!((1-x)dn)!(dn/2)!2^{yn}}{(xn)!((1-x)n)!(yn)!((xd-y)n/2)!(((1-x)d-y)n/2)!(dn)!}.$$

Using Stirling's formula ($n! \sim \sqrt{2\pi n}(n/e)^n$) and taking the exponential part we obtain

$$\begin{aligned} S(x, y) &\leq e^{o(n)} \frac{x^{x(d-1)n}(1-x)^{(1-x)(d-1)n} d^{dn/2}}{y^{yn}(xd-y)^{(xd-y)n/2}((1-x)d-y)^{((1-x)d-y)n/2}} \\ &= e^{-f(x,y,d)n + o(n)}, \end{aligned} \tag{3}$$

where

$$\begin{aligned} f(x, y, d) &= x(d-1)\ln x + (1-x)(d-1)\ln(1-x) + 0.5d\ln d - y\ln y \\ &\quad -0.5(xd-y)\ln(xd-y) - 0.5((1-x)d-y)\ln((1-x)d-y). \end{aligned}$$

Thus, if $f(x, y, d) < 0$, then $S(x, y)$ is exponentially small (n large) and the brush number is at least yn. Not surprisingly, the strongest bound is obtained for $x = 1/2$, in which case $f(x, y, d)$ becomes

$$(d-1)\ln(1/2) + (d/2)\ln d - y\ln y - (d/2-y)\ln(d/2-y)$$
$$= -\frac{d}{4}((1+z)\ln(1+z) + (1-z)\ln(1-z)) - \ln 2$$

where $y = (d/4)(1-z)$.

It is straightforward to see that this function is decreasing in z for $z \geq 0$. Let l_d/n denote the value of y for which it first reaches 0. Using the full power of the Stirling's formula, it is also not difficult to see that we can replace $e^{o(n)}$ by $O(n^{-1})$ in (3). This gives us the following asymptotically almost sure lower bounds l_d for the brush number of random d-regular graph: $l_4 = 0.22n$, $l_5 = 0.36n$, and $l_6 = 0.52n$. (In this paper, whenever we quote numerical values for computed constants such as l_d/n, we use three decimal places rounded down for lower bounds and up for upper bounds.)

In Figure 5, the values of l_d/dn have been presented for all d-values up to 100; we have also listed the first 30 and a few more values for higher d in Table 1 (see Section 4.6).

To obtain a result useful for all d, it is straightforward to show (since the Taylor expansion of $(1+z)\ln(1+z) + (1-z)\ln(1-z)$ is $z^2 + z^4/6 + \ldots$) that $l_d/n > (d/4)(1 - 2\sqrt{\ln 2}/\sqrt{d})$. This result has the following implication.

Corollary 3.3 *For $G \in \mathcal{G}_{n,d}$, a.a.s.*

$$b(G) \geq \frac{dn}{4}\left(1 - \frac{2\sqrt{\ln 2}}{\sqrt{d}}\right).$$

Alternatively, one can use the expansion properties of random d-regular graphs that follow from their eigenvalues to get a similar lower bound.

The adjacency matrix $A = A(G)$ of a given a d-regular graph G with n vertices, is an $n \times n$ real and symmetric matrix. Thus, the matrix A has n real eigenvalues which we denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. It is known that certain properties of a d-regular graph are reflected in its spectrum but, since we focus on expansion properties, we are particularly interested in the following quantity: $\lambda = \lambda(G) = \max(|\lambda_2|, |\lambda_n|)$. In words, λ is the largest absolute value of an eigenvalue other than $\lambda_1 = d$ (for more details, see the general survey [10] about expanders, or [5], Chapter 9).

The value of λ for random d-regular graphs has been studied extensively. A major result due to Friedman [9] is the following:

Lemma 3.4 ([9]) *For every fixed $\varepsilon > 0$ and for $G \in \mathcal{G}_{n,d}$,*

$$\mathbb{P}(\lambda(G) \leq 2\sqrt{d-1} + \varepsilon) = 1 - o(1).$$

The number of edges $|E(S, T)|$ between sets S and T is expected to be close to the expected number of edges between S and T in a random graph of edge density d/n, namely $d|S||T|/n$. A small λ (or large spectral gap) implies that this deviation is small. The following useful bound is essentially proved in [2] (see also [5]):

Lemma 3.5 (Expander Mixing Lemma) *Let G be a d-regular graph with n vertices and set $\lambda = \lambda(G)$. Then for all $S, T \subseteq V$*

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \lambda \sqrt{|S||T|} \, .$$

(Note that $S \cap T$ does not have to be empty; in general, $|E(S,T)|$ is defined to be the number of edges between $S \setminus T$ to T plus twice the number of edges that contain only vertices of $S \cap T$.)

For our purpose here it is better to apply a slightly stronger lower estimate for $|E(S, V \setminus S)|$, namely,

$$|E(S, V \setminus S)| \geq \frac{(d-\lambda)|S||V \setminus S|}{n} \tag{4}$$

for all $S \subseteq V$. This is proved in [4], see also [5].

From (4) and Lemma 3.4 we get immediately the following corollary. (In order to get the second part, it is enough to use (2) with $j = \lfloor n/2 \rfloor$. The second part is only slightly weaker than Corollary 3.3.)

Corollary 3.6 *Let $G \in \mathcal{G}_{n,d}$. For every $\varepsilon > 0$, a.a.s. all $S \subseteq V(G)$ satisfy the following condition*

$$|E(S, V \setminus S)| \geq \frac{(d - 2\sqrt{d-1} - \varepsilon)|S||V \setminus S|}{n} .$$

In particular, a.a.s.

$$b(G) \geq \frac{dn}{4} \left(1 - \frac{2}{\sqrt{d}} \right) .$$

Remark: The minimum number of edges in a cut that splits the vertex set of a graph into two equal parts is called its bisection width. In the above arguments we have used it as a lower bound for the brush number of the graph. It is worth noting that the $\frac{2}{\sqrt{d}}$ error term in the lower bound for the bisection width of a d-regular graph on n vertices is tight, up to a constant factor. Indeed, it is shown in [1] that for $n \gg d$, the bisection width of any d-regular graph on n vertices is at most $\frac{nd}{4}(1 - \Omega(\frac{1}{\sqrt{d}}))$.

3.2 A general upper bound

The following result provides an upper bound for the brush number of a general graph.

Theorem 3.7

$$b(G) \leq \frac{|E|}{2} + \frac{|V|}{4} - \frac{1}{4} \sum_{v \in V(G), \deg(v) \text{ is even}} \frac{1}{\deg(v)+1}$$

for any graph $G = (V, E)$.

Proof: Let π be a random permutation of the vertices of G taken with uniform distribution. We clean G according to this permutation to get the value of $b_\pi(G)$ (note that $b_\pi(G)$ is a random variable now). For a vertex $v \in V$, it follows from (1) that we have to assign to v exactly $X(v) = \max\{0, 2N^+(v) - \deg(v)\}$ brushes in the initial configuration, where $N^+(v)$ is the number of neighbors of v that follow it in the permutation (that is, the number of dirty neighbours of v at the time when v is cleaned). The random variable $N^+(v)$ attains each of the values $0, 1, \ldots, \deg(v)$ with probability $1/(\deg(v)+1)$; Indeed, this follows from the fact that the random permutation π induces a uniform, random permutation on the set of $\deg(v)+1$ vertices consisting of v and its neighbors. Therefore the expected value of $X(v)$, for even $\deg(v)$, is

$$\frac{\deg(v)+(\deg(v)-2)+\cdots+2}{\deg(v)+1}=\frac{\deg(v)+1}{4}-\frac{1}{4(\deg(v)+1)},$$

and for odd $\deg(v)$ it is

$$\frac{\deg(v)+(\deg(v)-2)+\cdots+1}{\deg(v)+1}=\frac{\deg(v)+1}{4}.$$

Thus, by linearity of expectation,

$$\mathbb{E}b_\pi(G)=\mathbb{E}\left(\sum_{v\in V}X(v)\right)=\sum_{v\in V}\mathbb{E}X(v)=\frac{|E|}{2}+\frac{|V|}{4}-\frac{1}{4}\sum_{v\in V(G),\deg(v)\text{ is even}}\frac{1}{\deg(v)+1},$$

which means that there is a permutation π_0 such that $b(G)\le b_{\pi_0}(G)\le \mathbb{E}b_\pi(G)$ and the assertion holds. ■

Note that the bound is tight when G is a union of cliques. From this we get immediately the following corollary.

Corollary 3.8 *Let $G=(V,E)$ be a d-regular graph on n vertices. If d is even, then*

$$b(G)\le\frac{n}{4}\left(d+1-\frac{1}{d+1}\right),$$

and if d is odd, then

$$b(G)\le\frac{n}{4}(d+1).$$

Both bounds are tight for every n and d satisfying $(d+1)|n$, as shown by a disjoint union of complete graphs K_{d+1}.

4 Cleaning random d-regular graphs

The differential equations method (described in [22]) is used here to find an upper bound on the number of brushes needed to clean a graph using a degree-greedy algorithm. We consider $d=2$ first, then state some general results, and apply them to the special cases of $3\le d\le 5$ before discussing higher values of d.

4.1 2-regular graphs

Let $Y=Y_n$ be the total number of cycles in a random 2-regular graph on n vertices. Since exactly two brushes are needed to clean one cycle, we need $2Y_n$ brushes in order to clean a 2-regular graph.

We know that the random 2-regular graph is a.a.s. disconnected; by simple calculations one can show that the probability of having a Hamiltonian cycle is asymptotic to $\frac{1}{2}e^{3/4}\sqrt{\pi}n^{-1/2}$ (see, for example, [20]).

We also know that the total number of cycles Y_n is sharply concentrated near $(1/2)\log n$. It is not difficult to see this by generating the random graph sequentially using the pairing model. The probability of forming a cycle in step i is exactly $1/(2n-2i+1)$, so the expected number of cycles is $(1/2)\log n + O(1)$. The variance can be calculated in a similar way. So we get that a.a.s. the brush number for a random 2-regular graph is $(1+o(1))\log n$.

4.2 d-regular graphs ($d \geq 3$) — the general setting

In this subsection, we assume $d \geq 3$ is fixed with dn even. In order to get an asymptotically almost sure upper bound on the brush number, we study an algorithm that cleans random vertices of minimum degree. This algorithm is called *degree-greedy* because the vertex being cleaned is chosen from those with the lowest degree.

We start with a random d-regular graph $G = (V, E)$ on n vertices. Initially, all vertices are dirty: $D_0 = V$. In every step t of the cleaning process, we clean a random vertex α_t, chosen uniformly at random from those vertices with the lowest degree in the induced subgraph $G[D_{t-1}]$, where $D_t = D_{t-1} \setminus \{\alpha_t\}$. In the first step, d brushes are needed to clean a random vertex α_1 (we say that this is "phase zero"). Note that this is a.a.s. the only vertex whose degree in D_t is d at the time of cleaning. Indeed, if α_t $(t \geq 2)$ has degree d in $G[D_{t-1}]$, then $G[D_{t-1}]$ consists of some connected components of G and thus G is disconnected. It was proven independently in [7, 21] that G is disconnected with probability $o(1)$ (this also holds when d is growing with n, as shown in [12]). The induced subgraph $G[D_1]$ now has d vertices of degree $d-1$ and $n-d-1$ vertices of degree d.

In the second step, $d-2$ brushes are needed to clean a random vertex α_2 of degree $d-1$. Typically, in the third step, a vertex of degree $d-1$ is cleaned and in each subsequent step, a vertex of degree $d-1$ is cleaned until some vertex of degree $d-2$ is produced in the subgraph induced by the set of dirty vertices. After cleaning the first vertex of degree $d-2$, we typically return to cleaning vertices of degree $d-1$, but after some more steps of this type we may clean another vertex of degree $d-2$. When vertices of degree $d-1$ become plentiful, vertices of lower degree are more commonly created and these hiccups occur more often. When vertices of degree $d-2$ take over the role of vertices of degree $d-1$, we say (informally!) that the first phase ends and we begin the second phase. In general, in the kth phase a mixture of vertices of degree $d-k$ and $d-k-1$ are cleaned.

It is usually difficult to study the behaviour of a greedy algorithm at the end of the process. Fortunately, in this case we need to study only the first $\lfloor (d-1)/2 \rfloor$ phases since the rest of the vertices are cleaned 'for free'. The details of the following differential equations method have been omitted, but can be found in [19].

For $0 \leq i \leq d$, let $Y_i = Y_i(t)$ denote the number of vertices of degree i in $G[D_t]$. (Note that $Y_0(t) = n - t - \sum_{i=1}^{d} Y_i(t)$ so $Y_0(t)$ does not need to be calculated, but it is useful in the discussion.) Let $S(t) = \sum_{l=1}^{d} lY_l(t)$ and for any statement A, let δ_A denote the Kronecker delta function

$$\delta_A = \begin{cases} 1 & \text{if } A \text{ is true,} \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to see that

$$\begin{aligned}
&\mathbb{E}(Y_i(t) - Y_i(t-1) \mid G[D_{t-1}] \ \wedge \ \deg_{G[D_{t-1}]}(\alpha_t) = r) \\
&\qquad = f_{i,r}((t-1)/n, Y_1(t-1)/n, Y_2(t-1)/n, \ldots, Y_d(t-1)/n) \\
&\qquad = -\delta_{i=r} - r\frac{iY_i(t-1)}{S(t-1)} + r\frac{(i+1)Y_{i+1}(t-1)}{S(t-1)}\delta_{i+1\le d} \qquad (5)
\end{aligned}$$

for $i, r \in [d]$ such that $Y_r(t) > 0$. Indeed, α_t has degree r, hence the term $-\delta_{i=r}$. When a pair of points in the pairing model is exposed, the probability that the other point is in a bucket of degree i (that is, the bucket contains i unchosen points) is asymptotic to $iY_i(t-1)/S(t-1)$. Thus $riY_i(t-1)/S(t-1)$ stands for the expected number of the r buckets found adjacent to α_t which have degree i. This contributes negatively to the expected change in Y_i, whilst buckets of degree $i+1$ which are reached contribute positively (of course, only if this type of vertices (buckets) exist in a graph; thus $\delta_{i+1\le d}$). This explains (5).

Suppose that at some step t of the phase k, cleaning a vertex of degree $d-k$ creates, in expectation, β_k vertices of degree $d-k-1$ and cleaning a vertex of degree $d-k-1$ decreases, in expectation, the number of vertices of degree $d-k-1$ by τ_k. After cleaning a vertex of degree $d-k$, we expect to then clean (on average) β_k/τ_k vertices of degree $d-k-1$. Thus, in phase k, the proportion of steps which clean vertices of degree $d-k$ is $1/(1+\beta_k/\tau_k) = \tau_k/(\beta_k+\tau_k)$. If τ_k falls below zero, vertices of degree $d-k-1$ begin to build up and do not decrease under repeated cleaning vertices of this type and we move to the next phase.

From (5) it follows that

$$\begin{aligned}
\beta_k &= \beta_k(x, y_1, y_2, \ldots, y_d) &= f_{d-k-1,d-k}(x, y_1, y_2, \ldots, y_d) &= f_{d-k-1,d-k}(x, \mathbf{y}), \\
\tau_k &= \tau_k(x, y_1, y_2, \ldots, y_d) &= -f_{d-k-1,d-k-1}(x, y_1, y_2, \ldots, y_d) &= -f_{d-k-1,d-k-1}(x, \mathbf{y}),
\end{aligned}$$

where $x = t/n$ and $y_i(x) = Y_i(t)/n$ for $i \in [d]$. This *suggests* (see [22] for more information on the differential equations method) the following system of differential equations

$$\frac{dy_i}{dx} = F(x, \mathbf{y}, i, k)$$

where

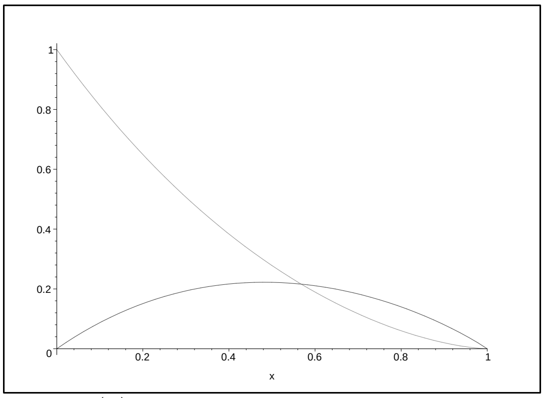
$$F(x, \mathbf{y}, i, k) = \begin{cases} \frac{\tau_k}{\beta_k+\tau_k} f_{i,d-k}(x, \mathbf{y}) + \frac{\beta_k}{\beta_k+\tau_k} f_{i,d-k-1}(x, \mathbf{y}) & \text{for } k \le d-2, \\ f_{i,1}(x, \mathbf{y}) & \text{for } k = d-1. \end{cases}$$

At this point we may formally define the interval $[x_{k-1}, x_k]$ to be phase k, where the termination point x_k is defined as the infimum of those $x > x_k$ for which at least one of the following holds: $\tau_k \le 0$ and $k < d-1$; $\tau_k + \beta_k = 0$ and $k < d-1$; $y_{d-k} \le 0$. Using final values $y_i(x_k)$ in phase k as initial values for phase $k+1$ we can repeat the argument inductively moving from phase to phase starting from phase 1 with obvious initial conditions $y_d(0) = 1$ and $y_i(0) = 0$ for $0 \le i \le d-1$.

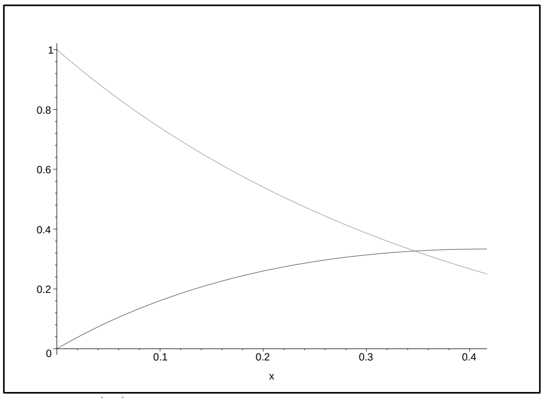
The general result [19, Theorem 1] studies a deprioritized version of degree-greedy algorithms, which means that the vertices are chosen to process in a slightly different way, not always the minimum degree, but usually a random mixture of two degrees. Once a vertex is chosen, it is treated the same as in the degree-greedy algorithm. The variables Y are defined in an analogous manner. The hypotheses of the theorem are mainly straightforward to verify but require several inequalities involving derivatives to hold at the termination of phases, for the full rigorous conclusion to be obtained. However, in practice, the equations are simply solved numerically in order to find the points x_k, since a fully rigorous bound is not obtained unless one obtains strict inequalities on the values of the solutions. The conclusion is that, for a certain algorithm using a deprioritized 'mixture' of the steps of the degree-greedy algorithm, with variables Y_i defined as above, we have that a.a.s.

$$Y_i(t) = ny_i(t/n) + o(n)$$

for $1 \le i \le d$ for phases $k = 1, 2, \ldots, m$, where m denotes the smallest k for which either $k = d - 1$, or any of the termination conditions for phase k hold at x_k apart from $x_k = \inf\{x > x_{k-1} : \tau_k \le 0\}$. We omit all details, pointing the reader to [19] and the general survey [22] about the differential equations method which is a main tool in proving [19, Theorem 1]. In addition, the theorem gives information on an auxiliary variable such as, of importance to our present application, the number of brushes used. Instead of quoting this precisely, we use it merely as justification for being able to use the above equations as if they applied to the greedy algorithm. (This is no doubt the case, but it is not actually proved in [19]. Instead, we know that they apply in the limit to a sequence of algorithms that use the steps of the degree-greedy algorithm.) The solutions to the relevant differential equations for $d = 3$ and 4 are shown in Figure 3.



(a) 3-regular graph, phase 1



(b) 4-regular graph, phase 1

Figure 3: Solution to the differential equations.

In the kth phase a mixture of vertices of degree $d - k$ and $d - k - 1$ are cleaned. Since $\max\{2l - d, 0\}$ brushes are needed to clean a vertex of degree l (see (1)), we need

$$u_d^k = (1 + o(1))n \left(\max\{d - 2k, 0\} \int_{x_{k-1}}^{x_k} \frac{\tau_k}{\tau_k + \beta_k} dx + \max\{d - 2k - 2, 0\} \int_{x_{k-1}}^{x_k} \frac{\beta_k}{\tau_k + \beta_k} dx \right)$$

brushes in phase k. Thus, the total number of brushes needed to clean a graph using the degree-greedy algorithm is a.a.s. equal to

$$u_d = \sum_{k=1}^{\lfloor (d-1)/2 \rfloor} u_d^k = (1 + o(1))n \Bigg(\sum_{k=1}^{\lfloor (d-1)/2 \rfloor} \left((d - 2k - 2)(x_k - x_{k-1}) + 2 \int_{x_{k-1}}^{x_k} \frac{\tau_k}{\tau_k + \beta_k} dx \right) + \delta_{d \text{ is odd}} \int_{x_{k-1}}^{x_k} \frac{\beta_k}{\tau_k + \beta_k} dx \Bigg).$$

4.3 3-regular graphs

Let $G = (V, E)$ be any 3-regular graph on n vertices. The first vertex cleaned must start three brush paths, the last one terminates three brush paths, and all other vertices must start or finish at least one brush path, so the number of brush paths is at least $n/2 + 2$.

The result mentioned above can be shown to result in an upper bound of $n/2 + o(n)$ for the brush number of a random 3-regular (i.e. cubic) graph. We do not provide details because of the following stronger result. It is known [18] that a random 3-regular graph a.a.s. has a Hamilton cycle. The edges not in a Hamilton cycle must form a perfect matching. Such a graph can be cleaned by starting with three brushes at one vertex, and moving along the Hamilton cycle with one brush, introducing one new brush for each edge of the perfect matching. Hence the brush number of a random 3-regular graph with n vertices is a.a.s. $n/2+2$. Note that this is also the brush number of any cubic Hamiltonian graph on n vertices.

4.4 4-regular graphs

For 4-regular graphs, we are interested in phase 1 only: we need two brushes to clean vertices of degree 3, but vertices of degree 2 are cleaned 'for free'. Note that $y_1(x) = y_2(x) = 0$. We have the following system of differential equations

$$\begin{aligned} \frac{dy_4}{dx} &= \frac{-6y_4(x)}{3y_3(x) + 2y_4(x)} \\ \frac{dy_3}{dx} &= \frac{-3y_3(x) + 4y_4(x)}{3y_3(x) + 2y_4(x)} \end{aligned}$$

with the initial conditions $y_4(0) = 1$ and $y_3(0) = 0$. The solution (see Figure 3 (b)) to these differential equations is

$$\begin{aligned} y_4(x) &= 5 - 4\sqrt{1+3x} + 3x \\ y_3(x) &= \frac{4(-3 + 3\sqrt{1+3x} - 5x + x\sqrt{1+3x})}{2 - \sqrt{1+3x}}, \end{aligned}$$

so $\beta_1 = -3 + 3\sqrt{1+3x}$ and $\tau_1 = 3 - 2\sqrt{1+3x}$. Thus phase 1 finishes at time $t_1 = 5n/12$ ($x_1 = 5/12$ is a root of the equation $\tau_1(x) = 0$) and the number of vertices of degree 3 cleaned during this phase is asymptotic to

$$n \int_0^{5/12} \frac{\tau_1}{\tau_1 + \beta_1} dx = n/6\,.$$

Since we need 2 brushes to clean one such vertex we get an asymptotically almost sure upper bound of $u_4 = (1 + o(1))n/3$.

On the other hand, it is true that a.a.s. a random 4-regular graph can be decomposed into two edge-disjoint Hamilton cycles [11], and hence four paths.

Note that the following two problems can be posed in general for any $d \geq 3$.

Open Problem 4.1 *Is it true that for the random case it is best to clean lowest degree vertices?*

In other words, if one is going to choose a random vertex of a given degree then one might as well choose a random vertex of minimum degree.

If Problem 4.1 is proven to be true, then the following problem should be considered. To get the brush number one might (in fact, probably should) choose non-random vertices during the cleaning process. But it might be true that a.a.s. one cannot save more than $o(n)$ brushes compared to the greedy algorithm under consideration.

Open Problem 4.2 *Is it true that a.a.s. the brush number for a random d-regular graph is* $u_d(1-o(1))$*?*

4.5 5-regular graphs

In order to study the brush number for 5-regular graphs yielded by the degree-greedy algorithm, we cannot consider phase 1 only as before; we need 3 brushes to clean vertices of degree 4 but also 1 brush to clean vertices of degree 3. Thus two phases must be considered.

In phase 1, $y_1(x) = y_2(x) = y_3(x) = 0$ and we have the following system of differential equations

$$\begin{aligned}\frac{dy_5}{dx} &= \frac{-20y_5(x)}{8y_4(x)+5y_5(x)}\\ \frac{dy_4}{dx} &= \frac{-8y_4(x)+15y_5(x)}{8y_4(x)+5y_5(x)}\end{aligned}$$

with the initial conditions $y_5(0) = 1$ and $y_4(0) = 0$. The numerical solution (see Figure 4 (a)) *suggests* that the phase finishes at time $t_1 = 0.1733n$. The number of brushes needed in this phase is asymptotic to

$$\begin{aligned}u_5^1 &= (1+o(1))\left(3n\int_0^{t_1/n}\frac{\tau_1}{\tau_1+\beta_1}dx + n\int_0^{t_1/n}\frac{\beta_1}{\tau_1+\beta_1}dx\right)\\ &= (1+o(1))\left(t_1+2n\int_0^{t_1/n}\frac{\tau_1}{\tau_1+\beta_1}dx\right) \approx 0.3180n\,.\end{aligned}$$

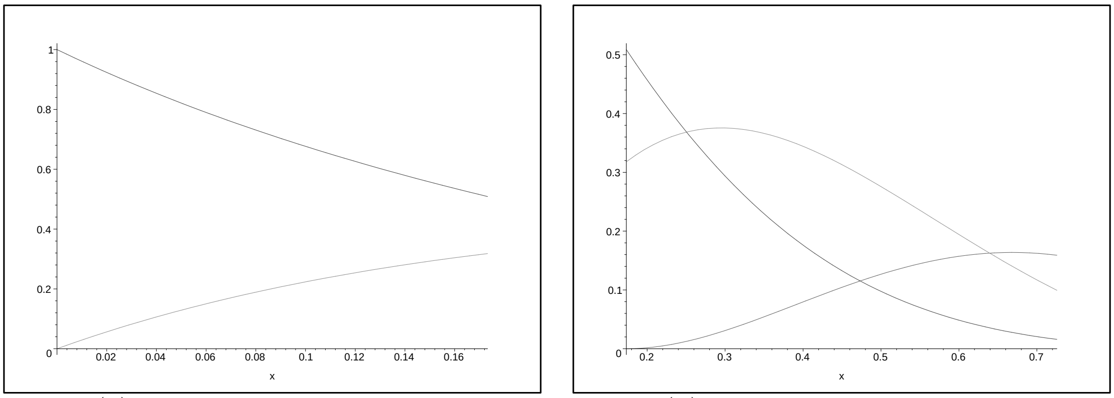
In phase 2, $z_1(x) = z_2(x) = 0$ and we have another system of differential equations

$$\begin{aligned}\frac{dz_5}{dx} &= \frac{-15z_5(x)}{6z_3(x)+4z_4(x)+5z_5(x)}\\ \frac{dz_4}{dx} &= \frac{-3(4z_4-5z_5(x))}{6z_3(x)+4z_4(x)+5z_5(x)}\\ \frac{dz_3}{dx} &= \frac{-6z_3(x)+8z_4(x)-5z_5(x)}{6z_3(x)+4z_4(x)+5z_5(x)}\end{aligned}$$

with the initial conditions $z_5(t_1/n) = y_5(t_1/n) = 0.5088$, $z_4(t_1/n) = y_4(t_1/n) = 0.3180$ and $z_3(t_1/n) = 0$. The numerical solution (see Figure 4 (b)) *suggests* that the phase finishes (approximately) at time $t_2 = 0.7257n$. The number of brushes needed in this phase is asymptotic to (the numerical solution)

$$u_5^2 = (1+o(1))n\int_{t_1/n}^{t_2/n}\frac{\tau_2}{\tau_2+\beta_2}dx \approx 0.3259n\,.$$

Finally, we get an asymptotically almost sure upper bound of $u_5 = u_5^1 + u_5^2 \approx 0.6439n$.



(a) 5-regular graph, phase 1 (b) 5-regular graph, phase 2

Figure 4: Solution to the differential equations.

4.6 d-regular graphs of higher order

d	l_d/n	u_d/n
3	0.0922	0.500
4	0.220	0.334
5	0.365	0.644
6	0.521	0.684
7	0.686	0.949
8	0.858	1.06
9	1.03	1.31
10	1.21	1.45
11	1.39	1.69
12	1.58	1.85

d	l_d/n	u_d/n
13	1.77	2.08
14	1.96	2.25
15	2.16	2.49
16	2.35	2.67
17	2.55	2.90
18	2.75	3.08
19	2.95	3.32
20	3.16	3.51
21	3.36	3.74
22	3.56	3.93

d	l_d/n	u_d/n
23	3.77	4.16
24	3.98	4.36
25	4.18	4.59
26	4.39	4.80
27	4.60	5.03
28	4.81	5.23
29	5.02	5.46
30	5.23	5.67
31	5.44	5.90
32	5.66	6.11

d	l_d/n	u_d/n
99	20.6	21.5
100	20.8	21.7
149	32.1	33.2
150	32.4	33.5
199	43.8	45.1
200	44.1	45.3
249	55.6	57.0
250	55.9	57.3
299	67.5	69.0
300	67.7	69.3

Table 1: Approximate upper and lower bounds on the brush number.

Note that the lower bound for $d = 4$ (see Section 3.1) will be considerably lower than the lower bound of $n/2+2$ for $d = 3$, whereas the upper bound we have been discussing is the same degree-greedy algorithm in all cases. However, the upper bound is also sensitive to the parity of d. For the 4-regular case, vertices of degree 2 are processed 'for free' and so one only really worries about degree 3 vertices and there are fewer of those processed than degree 2 vertices when $d = 3$. But it seems that the parity of d does not greatly affect the value of u_d/n for d big enough (see Figure 5 and Table 1).

In Figure 5, the values of l_d/dn (see Section 3.1 for more details about the lower bound) and u_d/dn have been presented for all d-values up to 100, although we have only listed the first 30 and a few more values for higher d in Table 1. The computations presented in the paper were performed by using Maple$^{\text{TM}}$ [16]. The worksheets can be found at the following address: "`http://www.mathstat.dal.ca/~pralat/`".

In [15] the following open question was asked "Does $\lim_{d\to\infty} u_d/dn$ exist?" (Open Prob-

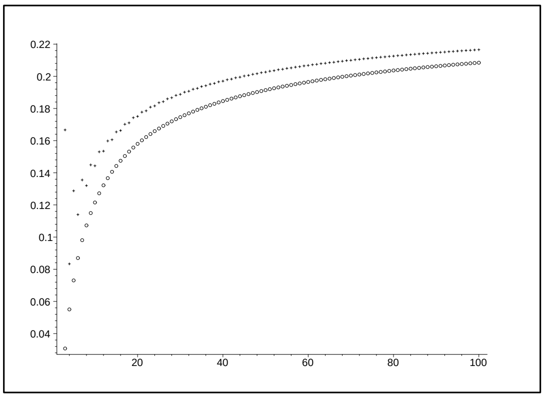


Figure 5: A graph of u_d/dn and l_d/dn versus d (from 3 to 100).

lem 3) and it was conjectured that there is a constant c such that the brush number is asymptotically cdn (Open Problem 4). The next theorem settles both questions.

Theorem 4.3 *The brush number of a random d-regular graph is asymptotically almost surely $\frac{n}{4}(d + o(d))$. Moreover, $\lim_{d\to\infty} u_d/dn = 1/4$, that is, for large d, the degree-greedy algorithm a.a.s. achieves the optimal number of brushes up to a lower order term.*

Proof: The first part of the theorem follows from Corollary 3.3 (or Corollary 3.6) and Corollary 3.8, which show that if $G \in \mathcal{G}_{n,d}$ then a.a.s.

$$\frac{dn}{4}\left(1 - \frac{2\sqrt{\ln 2}}{\sqrt{d}}\right) \le b(G) \le \frac{n(d+1)}{4}.$$

The upper bound here can in fact be slightly improved, as shown in Theorem 4.4 below.

It remains to estimate the performance of the degree-greedy algorithm. Let $d > 2$ be an integer, and let $G \in \mathcal{G}_{n,d}$, as before. It follows from Lemmas 3.4 and 3.5 that a.a.s. for all $m \in \{0, 1, \dots, n-1\}$ and all sets $X \subseteq V$ with $|X| = m$,

$$|E(G[V \setminus X])| \quad \le \quad \frac{(n-m)^2 d}{2n} + \frac{1}{2} 2\sqrt{d}(n-m)\,,$$

since the number of edges inside $G[V \setminus X]$ is $|E(V \setminus X, V \setminus X)|/2$. So the average degree of $G[V \setminus X]$ (and thus the minimum degree as well) is at most

$$\xi_m = \min\left\{\frac{(n-m)d}{n} + 2\sqrt{d}, d\right\}.$$

Thus, using (1) we get that a.a.s. the number of brushes used by the degree-greedy algorithm is at most

$$\sum_{m=0}^{n-1} \max\{2\xi_m - d, 0\} \le \frac{dn}{4} + O(\sqrt{d}n)\,.$$

It follows, by Corollary 3.3, that for large d the greedy algorithm achieves, a.a.s., essentially the optimum number of brushes. This completes the proof of the theorem. ∎

The numerical values of the upper bound following from the degree-greedy algorithm suggest that the brush number of a random d-regular graph is a.a.s. smaller than $dn/4$ for every $d \geq 3$. This fact can be proved combining the basic idea in the proof of Theorem 3.7 with some known properties of random d-regular graphs. Indeed, the bound in Theorem 3.7 holds for every d-regular graph, and for a random d-regular graph G one can slightly improve the result as follows. It is known (see [20]) that, for the purpose of proving statements a.a.s., such a random graph can be viewed as the multigraph formed from the union of a Hamilton cycle and a random $(d-2)$-regular graph G' on the same vertex set. (The probability of multiple edges being created is bounded away from 1, and the resulting graph, conditional upon no multiple edges, is contiguous to a random d-regular graph. Indeed, Molloy and Reed [17] exploited this fact in a way related to our argument here.) This is equivalent to taking a fixed Hamilton cycle, together with a random $(d-2)$-regular graph G' and permuting its vertices randomly by a permutation π. Therefore, if we clean this multigraph according to the order of the Hamilton cycle, which we denote by $1, 2, \ldots, n$, the edges of G' will be cleaned according to a random permutation. We can thus apply the estimate proved in Corollary 3.8 and conclude that the *expected* number of brushes needed is at most the bound given in that corollary for $(d-2)$-regular graph, plus 2 additional brushes needed to be placed in the first vertex in order to start the process; one of them will keep going along the Hamilton cycle, cleaning all its edges, and the other one will clean the edge $1, n$ and stay in vertex n until the end of the process. This implies that when G is generated by taking a Hamilton cycle, and a random $(d-2)$-regular G' permuted randomly on the cycle, the expected number of brushes when cleaning along the cycle is at most $2 + \frac{n}{4}(d - 1 - \frac{1}{d-1})$ when d is even, and at most $2 + \frac{n}{4}(d-1)$ when d is odd.

However, this is only a bound for the expectation, whereas we need to get an estimate that holds a.a.s. This can be done using a standard martingale together with the fact that if we change the permutation π by a single transposition, the number of brushes needed when cleaning along the Hamilton cycle changes by at most $O(d)$ (see, e.g., [3] for a similar argument). Alternatively, since in the random pairing corresponding to G', the number of brushes changes by at most $O(1)$ if two pairs are 'switched', [20, Theorem 2.19] immediately implies that a.a.s. the number of brushes required does not deviate from the expectation by more than $O(w(n)\sqrt{n})$, where $w(n)$ is any function tending to infinity with n. We have thus proved the following.

Theorem 4.4 *Let G be a random d-regular graph on n vertices, where $d \geq 3$. Then, a.a.s., if d is even*

$$b(G) \leq \frac{n}{4}\left(d - 1 - \frac{1}{d-1}\right)(1 + o(1))$$

and if d is odd then

$$b(G) \leq \frac{n}{4}(d-1)(1 + o(1)),$$

where the $o(1)$ term tends to zero as n tends to infinity.

Note that the numerical bounds obtained using the degree-greedy algorithm appearing in Table 1 are a little stronger than the general one obtained here.

We also note that the estimates in Theorem 4.4 can be further improved, by introducing greedy steps into the proof. Instead of simply cleaning along the cycle, one may swap the order of cleaning vertices if such a swap will save a brush, for example if a vertex has more dirty edges than the next one around the cycle. We do not elaborate on this since, although simple arguments like this will give improvements that can be described for general d, it seems likely that carrying the argument as far as possible one would arrive at the degree-greedy algorithm in any case.

4.7 Variants

We conclude with a few additional open problems.

Open Problem 4.5 *What is the brush number for the binomial random graphs $G(n,p)$? What is a lower/upper bound? How about other random graph models, for example models that give power law degree distribution or d-regular graphs generated by the d-process?*

It is not difficult to show that $b(G) = (1+o(1))pn^2/4$ for $G \in G(n,p)$ and $p > \omega(n)/n$ where $\omega(n)$ is any function tending to infinity. Indeed, in order to get an upper bound it is enough to use Theorem 3.7 since the number of edges is well concentrated around $p\binom{n}{2}$. To get a lower bound, one can show that the expected number of sets of size $\lfloor n/2 \rfloor$ with less than $(1 - 1/\omega^{1/3}(n))pn^2/4$ edges to its complement is tending to zero as n tends to infinity. The problem of determining the behavior of the brush number for sparser random graphs seems more difficult.

Another version of the cleaning process was introduced in [13]. In this version, when a vertex is cleaned multiple brushes are allowed to traverse each dirty edge. Thus, the brush number $B(G)$ of this generalized version is at most the classic one $b(G)$. Using the degree-greedy algorithm to clean a random d-regular graph for d even, no brush 'gets stuck' in the first $\lfloor (d-1)/2 \rfloor$ phases, there is no point to introduce more brushes in the initial configuration, and vertices in the last phases are cleaned 'for free'. So the upper bound obtained is the same as before. For d odd, it is clear that one can save some brushes at phase $(d-1)/2$ but the following is still open.

Open Problem 4.6

- *Is it true that for $G \in \mathcal{G}_{n,d}$, d even, $b(G) - B(G) = o(n)$ a.a.s.?*
- *Is it true that for $G \in \mathcal{G}_{n,d}$, d odd, $b(G) - B(G) = \Theta(n)$ a.a.s.? How far apart are they?*

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